Algebraic Surface Design with Hermite Interpolation

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This paper presents an efficient algorithm, called Hermite interpolation, for constructing low-degree algebraic surfaces, which contain, with C^1 or tangent plane continuity, any given collection of points and algebraic space curves having derivative information. Positional as well as derivative constraints on an implicitly defined algebraic surface are translated into a homogeneous linear system, where the unknowns are the coefficients of the polynomial defining the algebraic surface. Computational details of the Hermite interpolation algorithm are presented along with several illustrative applications of the interpolation technique to construction of joining or blending surfaces for solid models as well as fleshing surfaces for curved wire frame models. A heuristic approach to interactive shape control of implicit algebraic surfaces is also given, and open problems in algebraic surface design are discussed.

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1. INTRODUCTION

The need for efficient construction and manipulation of geometric curves and surfaces arise in several computer applications of which a few are computer-aided design for manufacturing, computer graphics, medical image processing, pattern recognition, robotics, and computer vision. In this paper, we present solutions to the problem of C^1 data interpolation using the lowest degree, implicitly represented algebraic surfaces in three-dimensional real space $\mathbb{R}^3$ and exhibit its use in solid model design. An algebraic surface S in $\mathbb{R}^3$ is implicitly defined by a single polynomial equation $f(x, y, z) = 0$, where

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coefficients of f are over $\mathbb{R}$. The class of algebraic surfaces have the advantage of closure under several geometric operations (intersections, union, offset, etc.) often required in a solid modeling system. Furthermore, we choose the implicit equation representation for interpolation design since it captures all elements in the class of algebraic surfaces. This is in contrast to the rational parametric equation representation wherein only a subset of algebraic surfaces can be defined by the trio of x, y, and z given explicitly as rational functions of two parameters. In essence, all rational parametric surfaces can be represented in implicit form, although the reverse is not true [22].

Past research in surface design has been largely dominated by the theory of parametrically represented surfaces, such as parametric Bézier surfaces, Coons patches, and B-spline surfaces [6, 7, 9, 11, 13]. While parametric surfaces have been successfully used in modeling complex physical objects, the flexibility has come at the cost of a very high algebraic degree of the surfaces. The algebraic degree of a surface is the maximum number of intersections between the surface and a line, counting both real and complex intersections and at infinity. This degree is the same as the degree of the defining polynomial in the implicit representation.[1] On the other hand, a degree n parametric surface can be of algebraic degree n^2 while a bidegree n parametric surface can be of algebraic degree up to $2n^2$. See Figure 1 for a degree hierarchy of typically used surfaces in geometric design. Designing with surfaces of low algebraic degree is important since the computational efficiency of further surface operations is directly related to this degree. For instance, the intersection computation of two surfaces of algebraic degree n can yield an intersection curve of algebraic degree n^2 having as many as n^4 point singularities. The algebraic degree of the *offset* of a surface of algebraic degree n can be as high as n^3. Bajaj and Kim's paper has examples [5] where many bounds on surface manipulations have been effectively computed. Observing gaps in the degree hierarchy of algebraic surfaces typically used in geometric design, we naturally ask whether algebraic surfaces of low degree (up to degree 5) are sufficient for all design applications? Note, that the gaps are even larger than first apparent in Figure 1, if we remember that the rational parametric surfaces of a fixed algebraic degree n (with n greater than two), are a strict subset of the class of all degree n algebraic surfaces. Our research has originated from such questions, and in the hope of filling the gaps in the hierarchy with flexible design surfaces possessing the lowest possible algebraic degrees.

In recent years there have been several noteworthy results on generating tangent plane continuous surface fits with low-degree algebraic surfaces. Dahmen [8] presents an algorithm that constructs a mesh of smooth piecewise quadratic surface patches for a class of polyhedra that admit a certain transversal system of planes. Sederberg [19, 20] discusses techniques for free-form algebraic surface design, paying special attention to cubic surfaces.

[1] Hence the term, algebraic degree.

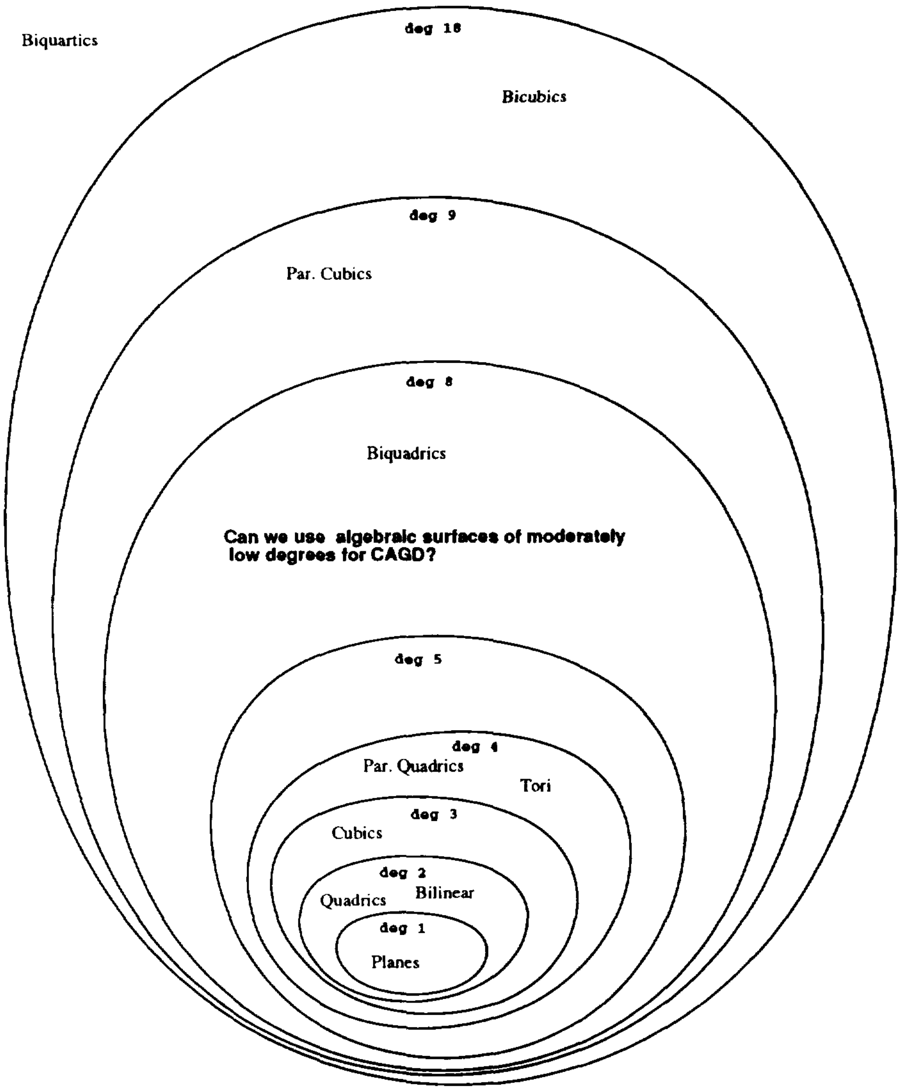

Fig. 1. A degree hierarchy of algebraic surfaces.

Quartic surfaces are used by Middleditch et al. [16] and Hoffmann et al. [15] to blend two primary quadric surfaces. In his Ph.D. thesis, Warren [24] characterized the family of algebraic surfaces that meet a given surface with a specified order of geometric continuity and applied the theory to surface blending with low-degree algebraic surfaces [25]. A good exposition of exact and least squares fitting of implicit algebraic surfaces through given data points is given by Pratt [18]. Sederberg [21] discusses C^0 interpolation of data points and curves using implicitly defined algebraic surfaces. In this paper, we extend the results of Pratt [18] and Sederberg [21] by devising an efficient algorithm, called *Hermite interpolation*, which characterizes, in terms of the nullspace of a matrix, the structure of a family of all algebraic surfaces that contain, with C^1 or tangent plane continuity, a given collection of data points and space curves (defined implicitly as the common intersection of algebraic surfaces or in rational parametric form) possibly having associated normal directions. The Hermite interpolation algorithm efficiently computes the family of algebraic surfaces of the lowest degree, which C^1 interpolates a specified collection of points, curves, and derivative information. We apply this algorithm to several examples of *joining* and *blending* primary surfaces of solid models and exhibit lower algebraic degree solutions than presented for similar examples by Hoffmann and Hopcroft [15], Middleditch and Sears [16], Owen and Rockwood [17], and Warren [25].

The rest of the paper is organized as follows: Section 2 introduces Hermite interpolation and provides some fundamental definitions and a key theorem used in the interpolation algorithm. Sections 3 and 4 present details of the Hermite interpolation algorithm for algebraic surfaces. In Section 5, we prove that the Hermite interpolation algorithm computes all the algebraic surfaces that interpolate given geometric data with C^1 continuity. An alternate formulation of tangent plane continuity, namely G^1 continuity, is also shown to be equivalent. Section 6 considers computational details of the algorithm, and in Section 7, several examples are presented to construct low-degree Hermite interpolating surfaces for *joining* and *blending* primary surfaces of solid models as well as for *fleshing* curved wire frame models of physical objects. Section 8 presents a heuristic method to interactively and intuitively select desirable instance surfaces from a family of algebraic surfaces computed by the Hermite interpolation algorithm. Finally, conclusions and open problems in algebraic surface design are discussed in Section 9.

2. HERMITE INTERPOLATION FOR ALGEBRAIC SURFACES

The primary objective of this work is to construct or approximate physical objects using meshes of algebraic surface patches. For aesthetic or functional reasons, it is usually required that the surface patches meet with geometric continuity. In many applications, C^1 or tangent plane continuity is sufficient. In his thesis, Warren [24] investigated algebraic structures of all surfaces meeting a given algebraic surface smoothly at a point or along a curve on that surface. He applied ideal theory to characterize the class of such surfaces in terms of polynomial expressions.

In this section, we present an algorithm, called *Hermite interpolation*, which algorithmically characterizes the class of all algebraic surfaces of a fixed degree which satisfy given geometric specifications. Input to this algorithm is a combination of points and algebraic space curves that are expressed either implicitly or parametrically. The points and space curves may have associated first derivative information in the form of normal vectors that define tangent planes at the points and space curves. Given an algebraic surface $S: f(x, y, z) = 0$ of degree n, the Hermite interpolation algorithm constructs a homogeneous linear system $\mathbf{M_I x} = \mathbf{0}$, $\mathbf{M_I} \in \mathbb{R}^{n_i \times n_v}$, $\mathbf{x} \in \mathbb{R}^{n_v}$ of n_i equations and n_v unknowns where the unknowns $\mathbf{x}$ are $n_v \left(= \binom{n+3}{3} \right)$ coefficients of S.[2] Only when the rank r of $\mathbf{M_I}$ is less than the number of the coefficients n_v does there exist a nontrivial solution to the system. All the vectors except $\mathbf{0}$ in the nullspace of $\mathbf{M_I}$ form a family of algebraic surfaces of degree n, satisfying the given input specifications, whose coefficients are expressed by homogeneous combinations of $q(= n_v - r)$-free parameters where q is the dimension of the nullspace.

As a result, the Hermite interpolation algorithm characterizes the family of algebraic surfaces with specified geometric properties in terms of the nullspace of a matrix. The algorithm is also useful in proving the existence or nonexistence of algebraic surfaces of degree n satisfying the input specifications since, when the rank of $\mathbf{M_I}$ is n_v, there is only the trivial solution $\mathbf{0}$ that does not correspond to an algebraic surface.

2.1 Preliminaries

We give brief definitions of certain terms we need and also state a form of Bezout theorem. For detailed and additional definitions, refer to Abhyankar [1] and Walker [23]. For any multivariate polynomial f, partial derivatives are written by subscripting, for example, $f_x = \partial f/\partial x$, $f_{xy} = \partial^2 f/(\partial x\, \partial y)$, and so on. An *algebraic surface of degree* n in $\mathbb{R}^3$ is implicitly defined by a single polynomial equation $f(x, y, z) = \sum_{i+j+k \le n} c_{ijk} x^i y^j z^k = 0$ where the coefficients c_{ijk} of f are real numbers. The *normal* or *gradient* of $f(x, y, z) = 0$ is the vector function $\nabla f = (f_x, f_y, f_z)$. A point $\mathbf{p} = (x_0, y_0, z_0)$ on a surface is a *regular* point if the gradient at $\mathbf{p}$ is not null. Otherwise, the point is *singular*. An algebraic surface $f(x, y, z) = 0$ is *irreducible* if $f(x, y, z)$ does not factor over the field of complex numbers. An *algebraic space curve* is defined by the common intersection of two or more algebraic surfaces. It is not known if a complete algebraic space curve can be always determined by the intersection of only two surfaces. In geometric design, therefore, we often restrict our consideration to a specific curve segment that is contained in the intersection of two algebraic surfaces. A *rational parametric space curve* is represented by the triple $G(s) = (x = G_1(s),\ y = G_2(s),\ z = G_3(s))$, where G_1, G_2, and G_3 are rational functions in s. The *degree of an algebraic surface* is the number of intersections between the surface and a line, properly counting complex, infinite, and multiple intersections. This degree is also the same as the degree of the defining polynomial. The *degree of an algebraic space curve* is the number of intersections between the curve and a plane, properly counting complex, infinite, and multiple intersections. The degree of an algebraic curve segment given as the intersection curve of two algebraic surfaces is also no larger than the product of the degrees of the two surfaces. Furthermore, the degree of a rational parametric curve is the same as the maximum degree of the numerator and denominator polynomials in the defining triple of rational functions.

[2] An algebraic surface of degree n has $\binom{n+3}{3}$ terms.

The following definitions are pertinent to our Hermite interpolation algorithm:

Definition 2.1. Let $\mathbf{p} = (p_x, p_y, p_z)$ be a point with an associated normal vector $\mathbf{n} = (n_x, n_y, n_z)$ in $\mathbb{R}^3$. An algebraic surface $S: f(x, y, z) = 0$ is said to contain $\mathbf{p}$ with C^1 or tangent plane continuity if

(1) $f(\mathbf{p}) = f(p_x, p_y, p_z) = 0$ (containment condition), and
(2) $\nabla f(\mathbf{p})$ is not zero and $\nabla f(\mathbf{p}) = \alpha \mathbf{n}$ for some nonzero α (tangency condition).

Definition 2.2. Let C be an algebraic space curve with an associated varying normal vector $\mathbf{n}(x, y, z) = (n_x(x, y, z), n_y(x, y, z), n_z(x, y, z))$, defined for all points on C. An algebraic surface $S: f(x, y, z) = 0$ is said to contain C with C^1 or tangent plane continuity if

(1) $f(\mathbf{p}) = 0$ for all points $\mathbf{p}$ of C (containment condition), and
(2) $\nabla f(\mathbf{p})$ is not identically zero and $\nabla f(\mathbf{p}) = \alpha \mathbf{n}(\mathbf{p})$ for some α and for all points $\mathbf{p}$ of C (tangency condition).

Definition 2.3. An algebraic surface $S: f(x, y, z) = 0$ is said to *Hermite interpolate* a given collection of points and space curves with associated normal vectors if S contains all the points and space curves with C^1 continuity.

The following is one form of *Bezout theorem*, the oldest theorem of algebraic geometry. As will be seen, this theorem plays an important role in proving the correctness of the Hermite interpolation algorithm.

THEOREM 2.1. (*Bezout*) *An algebraic curve C of degree d intersects an algebraic surface S of degree n in exactly nd points, properly counting complex, infinite, and multiple intersections, or C intersects S infinitely often, that is, a component of C lies entirely on S.*

3. INTERPOLATION OF POINTS WITH NORMALS

3.1 Containment

From the containment condition of Definition 2.1, it follows that any algebraic surface $S: f(x, y, z) = 0$, whose coefficients satisfy the linear equation $f(\mathbf{p}) = 0$, will contain the point $\mathbf{p}$. For a set of k data points, this yields k homogeneous linear equations. Since division of $f(x, y, z) = 0$ by a nonzero number does not change the surface the polynomial $f(x, y, z)$ represents, an algebraic surface of degree n has, in fact, $F = \binom{n+3}{3} - 1$ degrees of freedom. Interpolation of all the points is achieved by selecting an algebraic surface of degree n such that $F \geq r$, where r $(\leq k)$ is the rank of a system of k homogeneous linear equations. Similar approaches for constructing algebraic surfaces that interpolate points are discussed by Pratt [18].

3.2 Containment with Tangency

A point $\mathbf{p} = (p_x, p_y, p_z)$ with a normal vector $\mathbf{n} = (n_x, n_y, n_z)$ determines a unique plane $P: n_x x + n_y y + n_z z - (n_x p_x + n_y p_y + n_c p_z) = 0$ at the point $\mathbf{p}$. An algebraic surface $S: f(x, y, z) = 0$ of degree n that Hermite interpolates the point $\mathbf{p}$ can be constructed by setting up a linear system of equations as follows:

For each point $\mathbf{p}$ with a normal vector $\mathbf{n} = (n_x, n_y, n_z)$:

(1) *Containment condition*. Use the linear equation $f(\mathbf{p}) = 0$ in the unknown coefficients of S.

(2) *Tangency condition*. Select one of the following:

(a) If $n_x \neq 0$, use the equations $n_x f_y(\mathbf{p}) - n_y f_x(\mathbf{p}) = 0$ and $n_x f_z(\mathbf{p}) - n_z f_x(\mathbf{p}) = 0$.

(b) If $n_y \neq 0$, use the equations $n_y f_x(\mathbf{p}) - n_x f_y(\mathbf{p}) = 0$ and $n_y f_z(\mathbf{p}) - n_z f_y(\mathbf{p}) = 0$.

(c) If $n_z \neq 0$, use the equations $n_z f_x(\mathbf{p}) - n_x f_z(\mathbf{p}) = 0$ and $n_y f_z(\mathbf{p}) - n_z f_y(\mathbf{p}) = 0$.

(3) Next, ensure that the coefficients of $f(x, y, z) = 0$, satisfying the above three linear equations, additionally satisfy the constraints $\nabla f(\mathbf{p}) \neq 0$, since nontangency at $\mathbf{p}$ may occur if S turns out to be singular at $\mathbf{p}$.

The proof of correctness of the above algorithm follows from the following lemma.

LEMMA 3.1. *The equations of the above algorithm satisfy Definition* 2.1 *of point containment and tangency.*

PROOF. The first linear equation $f(\mathbf{p}) = 0$ satisfies containment by definition. We now show that the remaining equations satisfy $\nabla f(\mathbf{p}) = \alpha \cdot \mathbf{n}$ for a nonzero α. Since $\mathbf{n}$ is not a null vector, without loss of generality, we may assume that $n_x \neq 0$ in step 2 above. Other cases of $n_y \neq 0$ or $n_z \neq 0$ can be handled analogously. Now let $\alpha = f_x / n_x$, assuming $n_x \neq 0$. Then $f_x = \alpha \cdot n_x$, and substituting it in the selected linear equation $n_x f_y - n_y f_x = 0$ yields $f_y = \alpha \cdot n_y$, and substituting it again in the other selected linear equation $n_x f_z - n_z f_x = 0$ yields $f_z = \alpha \cdot n_z$. Hence $\nabla f(\mathbf{p}) = \alpha \cdot \mathbf{n}$. Finally, note that $f_x = 0$ for $n_x \neq 0$, in the selected linear equations of step 2(a), would cause $\nabla f(\mathbf{p}) = 0$, which we ensured would not happen in step 3 of the algorithm. Hence $f_x \neq 0$ and so $\alpha \neq 0$, and the lemma is proved. □

4. INTERPOLATION OF CURVES WITH NORMALS

The varying normal vector associated with a space curve C can be defined implicitly by the triple $\mathbf{n}(x, y, z) = (n_x(x, y, z), n_y(x, y, z), n_z(x, y, z))$ where n_x, n_y, and n_z are polynomials of maximum degree m and defined for all points $\mathbf{p} = (x, y, z)$ along the curve C. For the special case of a rational curve, which we shall treat separately in Subsections 4.1.2 and 4.2.2, the varying normal vector can be also defined parametrically as $\mathbf{n}(s) = (x = n_x(s), y = n_y(s), z = n_z(s))$, with n_x, n_y, and n_z now rational functions in s.

4.1 Containment

4.1.1 *Algebraic Curves: Implicit Definition*. Let $C : (f_1(x, y, z) = 0, f_2(x, y, z) = 0)$ implicitly define an algebraic space curve of degree d. The irreducibility of the curve is not a restriction, since reducible curves can be handled by treating each irreducible curve component separately. For precise definitions of irreducible components of an algebraic curve, see Walker [23]. The containment condition (as well as the tangency condition) requires the interpolating surface to be zero at a finite number of points on the curve. To ensure containment of a specific irreducible component requires choosing this finite number of points on that component. The precise number, derived from Bezout theorem, is a linear function of the degree of that curve component.

The situation is more complicated in the real setting, if we wish to achieve separate containment of one of possibly several connected real ovals of a single irreducible component of the space curve. There is a nontrivial problem of specifying a single isolated real oval of a curve. Arnon et al. [2] derive a solution in terms of a decomposition of space into cylindrical cells that separates out the various components of any real curve (or any real algebraic or semialgebraic set).

An interpolating surface $S: f(x, y, z) = 0$ of degree n for containment of an irreducible curve component C, is computed as follows:

(1) Choose a set L_c of $nd + 1$ points on C, $L_c = \{\mathbf{p}_i = (x_i, y_i, z_i) \mid i = 1, \cdots, nd + 1\}$. The set L_c may be computed, for example, by tracing the intersection of $f_1 = f_2 = 0$ [3]. Thus, alternatively, an algebraic curve may be given as a list of points.

(2) Next, set up $nd + 1$ homogeneous linear equations $f(\mathbf{p}_i) = 0$, for all $\mathbf{p}_i \in L_c$. Any nontrivial solution of this linear system will represent an algebraic surface that interpolates the entire curve C.

The proof of correctness of the above algorithm is captured in the following Lemma.

LEMMA 4.1. *To satisfy the containment condition of an algebraic curve C of degree d by an algebraic surface S of degree n, it suffices to satisfy the containment condition of $nd + 1$ points of C by S.*

PROOF. This is essentially a restatement of Bezout theorem in Section 2.1. Making S contain $nd + 1$ points of C ensures that S must intersect C infinitely often, and hence S must contain the entire curve. □

Recall that $S: f(x, y, z) = 0$ of degree n has $F = \binom{n+3}{3} - 1$ degrees of freedom. Let r be the rank of the system of $nd + 1$ linear equations. There are nontrivial solutions to this homogeneous system if and only if $F > r$ and a unique nontrivial solution when $F = r$. Again, an interpolating surface can be obtained by choosing a degree n such that $F \geq r$.

4.1.2 *Rational Curves: Parametric Definition.* When a curve is given in rational parametric form, its equations can be used directly to produce a linear system for interpolation, instead of first computing $nd + 1$ points on the curve. Let $C: (x = G_1(t), y = G_2(t), z = G_3(t))$ be a rational curve of degree d. An interpolating surface $S: f(x, y, z) = 0$ of degree n that contains C is computed as follows:

(1) Substitute $(x = G_1(t), y = G_2(t), z = G_3(t))$ into the equation $f(x, y, z) = 0$.

(2) Simplify and rationalize the expression from step 1 to obtain the numerator $Q(t) = 0$, where Q is a polynomial in t of degree at most nd with coefficients that are homogeneous linear expressions in the coefficients of f. For Q to be identically zero, each of its coefficients must be zero, and hence we obtain a system of at most $nd + 1$ linear equations, where the unknowns are the coefficients of f. Any nontrivial solution of this linear system will represent a surface S that interpolates C.

LEMMA 4.2. *The containment condition is satisfied by step* 2 *of the above algorithm.*

PROOF. Obvious. □

4.2 Containment with Tangency

In order to Hermite interpolate an algebraic curve C with a normal vector $\mathbf{n}$ by an algebraic surface S, we again need to solve a homogeneous linear system, whose equations stem from both the containment condition and the tangency conditions of Definition 2.2.

4.2.1 *Algebraic Curves with Normals: Implicit Definition.* As before, let $C: (f_1(x, y, z) = 0,\ f_2(x, y, z) = 0)$ implicitly define an irreducible algebraic space curve of degree d, together with an associated normal vector defined implicitly by the triple $\mathbf{n}(x, y, z) = (n_x(x, y, z), n_y(x, y, z), n_z(x, y, z))$ where n_x, n_y, and n_z are polynomials of maximum degree m and defined for all points $\mathbf{p} = (x, y, z)$ along the curve C. A Hermite-interpolating surface $S: f(x, y, z) = 0$ of degree n, which contains C with C^1 continuity, is then computed as follows:

(1) Choose a set L_c of $nd + 1$ points on C, $L_c = \{\mathbf{p}_i = (x_i, y_i, z_i) \mid i = 1, \cdots, nd + 1\}$. The set L_c may be computed, as before, by tracing the intersection of $f_1 = f_2 = 0$.
(2) Construct a list L_t of $(n - 1 + m)d + 1$ point-normal pairs on C, $L_t = \{[(x_i, y_i, z_i), (n_{xi}, n_{yi}, n_{zi})] \mid i = 1, \cdots, (n - 1 + m)d + 1\}$, where $(n_{xi}, n_{yi}, n_{zi}) = \mathbf{n}(x_i, y_i, z_i)$ for all i. Thus, alternatively, an algebraic curve C and its associated normal vector $\mathbf{n}$ may (either or both) be given as a list of points or point-normal pairs.
(3) *Containment condition.* Next, set up $nd + 1$ homogeneous linear equations $f(\mathbf{p}_i) = 0$, for $\mathbf{p}_i \in L_c$, $i = 1, \cdots, nd + 1$.
(4) *Tangency condition*
    (a) Compute $\mathbf{t}(x, y, z) = \nabla f_1(x, y, z) \times \nabla f_2(x, y, z)$. Note $\mathbf{t} = (t_x, t_y, t_z)$ is the tangent vector to C.
    (b) Select one of the following:
        (i) If $t_x \neq 0$, use the equation $f_y \cdot n_z - n_y \cdot f_z = 0$.
        (ii) If $t_y \neq 0$, use the equation $f_x \cdot n_z - n_x \cdot f_z = 0$.
        (iii) If $t_z \neq 0$, use the equation $f_x \cdot n_y - n_x \cdot f_y = 0$.

    Substitute each point-normal pair in L_t into the above-selected equation to yield $(n - 1 + m)d + 1$ additional homogeneous linear equations in the coefficients of $f(x, y, z)$.

(5) In total, we obtain a homogeneous system of $(2n - 1 + m)d + 2$ linear equations. Any nontrivial solution of the homogeneous linear system, for which, additionally, ∇f is not identically zero for all points of C (that is, the surface S is not singular at all points along the curve C), will represent a surface that Hermite interpolates C.

The proof of correctness of the above algorithm follows from Lemma 4.1 and the following lemma, which shows why the selected equation of step 4(b), evaluated at $(n - 1 + m)d + 1$ point-normal pairs, is sufficient.

LEMMA 4.3. *To satisfy the tangency condition of an algebraic curve C of degree d with a normal vector $\mathbf{n}$ of degree m, by an algebraic surface S of degree n, it suffices to satisfy the tangency condition at $(n - 1 + m)d + 1$ points of C by S as in step 4 of the above algorithm.*

PROOF. In step 4(b) above, assume without loss of generality that $t_x \neq 0$. Then the selected equation

$$f_y \cdot n_z - n_y \cdot f_z = 0. \tag{1}$$

We first show that even if Eq. (1) is evaluated at only $(n - 1 + m)d + 1$ points of C in step 4(b) above, it holds for all points on C. Equation (1) defines an algebraic surface $H: h(x, y, z) = 0$ of degree $(n - 1 + m)$, which intersects C of degree d, at most, $(n - 1 + m)d$ points. Invoking the Bezout theorem, it follows that C must lie entirely on the surface H if Eq. (1) is evaluated at $(n - 1 + m)d + 1$ points of C. Hence Eq. (1) is valid along the entire curve C.

We now show that step 4 of the above algorithm satisfies the tangency condition as specified in Definition 2.2. Since $\mathbf{t}$ of step 4(a) is a tangent vector at all points of C, and the surface $S: f = 0$ contains C, the gradient vector ∇f is orthogonal to $\mathbf{t}$, which yields the equation:

$$f_x \cdot t_x + f_y \cdot t_y + f_z \cdot t_z = 0 \tag{2}$$

valid for all points of C. Next, from the definition of a normal vector of a space curve,

$$n_x \cdot t_x + n_y \cdot t_y + n_z \cdot t_z = 0 \tag{3}$$

valid for all points of C. Now it is impossible that both $n_y(x, y, z)$ and $n_z(x, y, z)$ are identically zero along C, since if they were, then Eq. (3) would imply that $n_x \cdot t_x = 0$, and as we assumed that $t_x \neq 0$, would in turn imply that also $n_x = 0$ along C, which would contradict the earlier assumption that $\mathbf{n}$ is not identically zero. Hence, at least, one of n_y and n_z must also be nonzero. Without loss of generality, let $n_y \neq 0$. Also, let $\alpha(x, y, z) = f_y / n_y$. Then, we get

$$f_y = \alpha \cdot n_y, \tag{4}$$

and substituting it into Eq. (1) yields

$$f_z = \alpha \cdot n_z \tag{5}$$

for all points on C. From Eqs. (2), (4), and (5) we obtain

$$f_x \cdot t_x + \alpha \cdot n_y \cdot t_y + \alpha \cdot n_z \cdot t_z = 0. \qquad (6)$$

By multiplying α to Eq. (3) and subtracting Eq. (6) from it, we obtain

$$f_x \cdot t_x = \alpha \cdot n_x \cdot t_x, \qquad (7)$$

and since $t_x \neq 0$, we finally obtain

$$f_x = \alpha \cdot n_x \qquad (8)$$

valid at all points of C. Hence Eqs. (4), (5), and (8) together imply that $\nabla f(x, y, z) = \alpha \cdot \mathbf{n}$ for all points C and some nonzero α.[3] Hence, the tangency condition of Definition 2.2 is met. □

4.2.2 *Rational Curves with Normals: Parametric Definition.* When both a space curve and its associated normal vector are given in rational parametric form, their equations can be used directly to produce a linear system for interpolation, instead of first computing points and point-normal pairs of the curve. Let $C:(x = G_1(s), y = G_2(s), z = G_3(s))$ be a rational curve of degree d with a normal vector $\mathbf{n}(s) = (n_x(s), n_y(s), n_z(s))$ of degree m. A Hermite interpolating surface $S: f(x, y, z) = 0$ of degree n, which contains C with C^1 continuity, is computed as follows:

(1) *Containment condition*. Substitute $(x = G_1(s), y = G_2(s), z = G_3(s))$ into the equation $f(x, y, z) = 0$. This results in, at most, $nd + 1$ homogeneous linear equations as in Subsection 4.1.2.

(2) *Tangency condition*
   (a) Compute $\nabla f(s) = \nabla f(G_1(s), G_2(s), G_3(s))$ and $\mathbf{t}(s) = (dx/ds, dy/ds, dz/ds)$. Note that $\mathbf{t} = (t_x, t_y, t_z)$ is the tangent vector to C.
   (b) Select one of the following:
      (i) If $t_x \neq 0$, use the equation $f_y(s) \cdot n_z(s) - n_y(s) \cdot f_z(s) = 0$.
      (ii) If $t_y \neq 0$, use the equation $f_x(s) \cdot n_z(s) - n_x(s) \cdot f_z(s) = 0$.
      (iii) If $t_z \neq 0$, use the equation $f_x(s) \cdot n_y(s) - n_x(s) \cdot f_y(s) = 0$.

   In each case, the numerator of the simplified rational polynomial is set to zero. This yields at most $(n - 1)d + m + 1$ additional homogeneous linear equations in the coefficients of the surface $S: f(x, y, z) = 0$.

(3) In total, we obtain a homogeneous system of at most $(2n - 1)d + m + 2$ linear equations. Any nontrivial solution of the linear system, for which additionally ∇f is not identically zero for all points of C (that is, the surface S is not singular along the curve C), will represent a surface that Hermite interpolates C.

[3] From Equation (6) we see that $\alpha(x, y, z)$ must not be identically zero along C, for otherwise, $\nabla f = (0, 0, 0)$ for points along C, and would contradict the fact that we chose a nontrivial solution for the surface $S: f = 0$ where ∇f is not identically zero.

The proof of correctness of the above algorithm follows from Lemma 4.2 and the following lemma.

LEMMA 4.4. *If we choose a nontrivial solution for which the resulting Hermite interpolating surface S is not singular along the entire curve C, step 2 guarantees that the tangency condition of Definition 2.2 is met.*

PROOF. The proof is very similar to that of Lemma 4.3 with minor modifications and is omitted. □

5. GEOMETRIC CONTINUITY

In the Hermite interpolation algorithm, tangent plane continuity between two surfaces is achieved by making the tangent planes of the two surfaces identical at a point or at all points along a common curve of intersection. This definition of continuity agrees with several other definitions of G^1 *geometric continuity* given for parametric and implicit algebraic surfaces. De Rose [10] gave a definition of higher orders of geometric continuity between parametric surfaces where two surfaces F_1 and F_2 meet with *order k geometric continuity* or G^k *continuity* along a curve C if and only if there exist reparameterizations F_1' and F_2' of F_1 and F_2, respectively, such that all partial derivatives of F_1' and F_2' up to degree k agree along C.

Warren [24] formulated an intuitive definition of G^k continuity between implicit surfaces as following:

Definition 5.1. Two algebraic surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ meet with G^k rescaling continuity at a point p or along an algebraic curve C if and only if there exists two polynomials $a(x, y, z)$ and $b(x, y, z)$, not identically zero at p or along C, such that all derivatives of $af - bg$ up to degree k vanish at p or along C.

This formulation is more general than just making all the partials of $f(x, y, z) = 0$ and $g(x, y, z) = 0$ agree at a point or along a curve. For example, [24], consider the intersection of the cone $f(x, y, z) = xy - (x + y - z)^2 = 0$ and the plane $g(x, y, z) = x = 0$ along the line defined by two planes $x = 0$ and $y = z$. It is not hard to see that these two surfaces meet smoothly along the line since the normals to $f(x, y, z) = 0$ at each point on the line are scalar multiples of those to $g(x, y, z) = 0$. But, this scale factor is a function of z. Situations like this are thus corrected by allowing multiplication by rescaling polynomials, not identically zero along an intersection curve. Note that multiplication of a surface of polynomials nonzero along a curve does not change the geometry of the surface in the neighborhood of the curve. Garrity et al. [12] showed that both definitions of geometric continuity for a parametric and an implicit surface are equivalent by introducing the concept of a manifold that describes an intrinsic and local property of a surface.

The definition for G^0 rescaling continuity corresponds to the containment definition in Section 2.1. The following lemma shows that the C^1 continuity definition in Section 2.1 agrees with the G^1 rescaling continuity definition.

LEMMA 5.1. *G^1 rescaling continuity between $f(x, y, z) = 0$ and $g(x, y, z) = 0$ at a common point p or along a common curve C corresponds to $f(x, y, z)$*

$= 0$ and $g(x, y, z) = 0$ having common tangent planes at p or along every point of C.

PROOF. The requirement for G^1-rescaling continuity is that there exist $a(x, y, z)$ and $b(x, y, z)$, not identically zero at p or along C, such that

$$\frac{\partial(af - bg)}{\partial x} = a_x f + af_x - b_x g - bg_x$$
$$= 0 \text{ at } p \text{ or along } C,$$
$$\frac{\partial(af - bg)}{\partial y} = a_y f + af_y - b_y g - bg_y$$
$$= 0 \text{ at } p \text{ or along } C,$$
$$\frac{\partial(af - bg)}{\partial z} = a_z f + af_z - b_z g - bg_z$$
$$= 0 \text{ at } p \text{ or along } C.$$

Since p or C is contained in both f and g (that is, $f = g = 0$ at p or along C), the requirement becomes

$$af_x = bg_x$$
$$af_y = bg_y$$
$$af_z = bg_z,$$

which means $(f_x, f_y, f_z) = b/a(g_x, g_y, g_z)$ at p or along C. Hence, f and g are required to have common tangent planes at p or along C. □

The correctness proofs in Section 3 and Section 4 imply that Hermite interpolation finds all the algebraic surfaces that have common tangent planes at a point or a curve. It also yields the following theorem.

THEOREM 5.1. *Hermite interpolation finds all the algebraic surfaces F that meet a surface H at a point p or along a curve C on H with G^1 rescaling continuity.*

A family of algebraic surfaces F as in the above theorem can be constructed in the Hermite interpolation framework of Section 4 as follows. Given a surface H and a point p or curve C on H, defined implicitly or parametrically, the input to the Hermite interpolation algorithm is the point p or the curve C and the normal vector to p or C obtained directly from the ∇H, evaluated at p or along C. The algorithm then yields a solution for the coefficients of the family of algebraic surfaces that meet H at p or along C with C^1, tangent plane, or G^1 rescaling continuity. Several examples of this are provided in the next section.

6. COMPUTATIONAL ASPECTS OF HERMITE INTERPOLATION

The basic mechanics of Hermite interpolation for algebraic surfaces, as presented in the algorithms of Section 3 and Section 4, are

(1) geometric properties of a surface to be designed are described in terms of a combination of points, curves, and possibly associated normal vectors;

(2) these properties are translated into a homogeneous linear system of equations with extra surface constraints; and
(3) nontrivial solutions of the linear system are computed.

In this section we discuss some computational aspects of Hermite interpolation and give several examples of algebraic surface design with Hermite interpolation.

6.1 On Computing Nontrivial Solutions

As explained before, the Hermite interpolation algorithm converts geometric properties of a surface into a homogeneous linear system:

$$\mathbf{M_I x} = \mathbf{0}(\mathbf{M_I} \in \mathbb{R}^{n_i \times n_v}, \mathbf{x} \in \mathbb{R}^{n_v}),$$

where n_i is the number of equations generated, n_v is the number of unknown coefficients of a surface of degree n $\left(n_v = \binom{n+3}{3}\right)$, $\mathbf{M_I}$ is a matrix for the linear equations, and $\mathbf{x}$ is a vector whose elements are unknown coefficients of a surface.

In order to solve the linear system in a computationally stable manner, we compute the *singular value decomposition* (SVD) of $\mathbf{M_I}$ [14]. Hence, $\mathbf{M_I}$ is decomposed as $\mathbf{M_I} = U \Sigma V^T$ where $U \in \mathbb{R}^{n_i \times n_i}$ and $V \in \mathbb{R}^{n_v \times n_v}$ are orthonormal matrices, and $\Sigma = diag(\sigma_1, \sigma_2, \cdots, \sigma_s) \in \mathbb{R}^{n_i \times n_v}$ is a diagonal matrix with diagonal elements $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_s \geq 0$ ($s = min\{n_i, n_v\}$). It can be proved that the rank r of $\mathbf{M_I}$ is the number of the positive diagonal elements of Σ, and that the last $n_v - r$ columns of V span the nullspace of $\mathbf{M_I}$. Hence, the nontrivial solutions of the homogeneous linear system are compactly expressed as $\{\mathbf{x}(\neq \mathbf{0}) \in \mathbb{R}^{n_v} | \mathbf{x} = \sum_{i=1}^{n_v - r} w_i \cdot \mathbf{v}_{r+i}$, *where* $w_i \in \mathbb{R}$, *and* $\mathbf{v}_j$ *is the jth column of* $V\}$, or $\mathbf{x} = V_{n_v - r}\mathbf{w}$ where $V_{n_v - r} \in \mathbb{R}^{n_v \times (n_v - r)}$ is made of the last $n_v - r$ columns of V, and $\mathbf{w}$ is a $(n_v - r)$ vector for free parameters.

Example 6.1. *Computation of Nontrivial Solutions*. Let $C: (2t/1 + t^2, 1 - t^2/1 + t^2, 0)$, and $\mathbf{n}(t) = (4t/1 + t^2, 2 - 2t^2/1 + t^2, 0)$, which is from the intersection of a sphere $x^2 + y^2 + z^2 - 1 = 0$ with the plane $z = 0$. To find a surface of degree 2 that Hermite interpolates C, we let $f(x, y, z) = c_1 x^2 + c_2 y^2 + c_3 z^2 + c_4 xy + c_5 yz + c_6 zx + c_7 x + c_8 y + c_9 z + c_{10}$. From the containment condition, we get 5 equations, $c_{10} - c_8 + c_2 = 0$, $2c_7 - 2c_4 = 0$, $2c_{10} - 2c_2 + 4c_1 = 0$, $2c_7 + 2c_4 = 0$, $c_{10} + c_8 + c_2 = 0$, and from the tangency condition, we also get 5 equations, $-2c_9 + 2c_5 = 0$, $-4c_6 = 0$, $-4c_5 = 0$, $4c_6 = 0$, $2c_9 + 2c_5 = 0$. In matrix form,

$$\mathbf{M_I x} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \end{pmatrix} = \mathbf{0}.$$

The Σ in the SVD of $\mathbf{M}_I$ is $diag(5.657, 4.899, 4.899, 2.828, 2.828, 2.828, 2.0, 1.414, 0.0, 0.0)$.[4]

Hence, we see that the rank of $\mathbf{M}_I$ is 8, and the null space of $\mathbf{M}_I$ is

$$\mathbf{x} = w_1\mathbf{v}_9 + w_2\mathbf{v}_{10} = w_1 \begin{pmatrix} 0.0 \\ 0.0 \\ 1.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{pmatrix} + w_2 \begin{pmatrix} 0.57735 \\ 0.57735 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ -0.57735 \end{pmatrix}.$$

The nontrivial solutions are obtained by making sure that the free parameters w_1 and w_2 do not vanish simultaneously. Hence, the Hermite interpolating surface is $f(x, y, z) = 0.57735w_2x^2 + 0.57735w_2y^2 + w_1z^2 - 0.57735w_2 = 0$, which has one degree of freedom in controlling its coefficients. The surface $f(x, y, z) = 0$ can be made to contain a point, say, (1, 0, 1). That is, $f(1, 0, 1) = 0.57735w_2 + w_1 - 0.57735w_2 = w_1 = 0$. So, the circular cylinder $f(x, y, z) = 0.57735w_2(x^2 + y^2 - 1) = 0$ is an appropriate Hermite interpolating surface. □

6.2 Bounding the Degree of Surfaces

The total number of linear equations generated for a possible algebraic surface of degree n to Hermite interpolate k points with fixed constant normal directions and also to contain, with C^1 continuity, l space curves of degree d with assigned normal directions, varying as a polynomial of degree m, is $3k + (2n - 1 + m)dl + 2l$. This number becomes $3k + (2n - 1)dl + ml + 2l$ when all the space curves and associated normal vectors are defined parametrically.

For a given configuration of points, curves, and normal vectors, the above interpolation scheme allows one to both upper- and lower-bound the degree of Hermite interpolating surfaces.

(1) *Lower bound*. Let $r(n)$ be the rank of a homogeneous system of linear equations, obtained from the given geometric configuration and surface degree n. The rank tells us the exact number of independent constraints on the coefficients of the desired algebraic surface of degree n. Dependencies arise from spatial interrelationships of the given points and curves. From the rank, we can conclude that there exists no algebraic surface of a degree less than or equal to n_0 where n_0 is the largest n such that $F(n) < r(n)$ with $F(n) = \binom{n+3}{3} - 1$.

(2) *Upper bound*. Alternatively, the smallest n can be chosen such that $F(n) \geq r(n)$. The nontrivial solutions of the linear system represents a

[4] The subroutine dsvdc of LINPACK was used to compute the SVD of $\mathbf{M}_I$.

$(F(n) - r(n) + 1)$-parameter family (with $F(n) - r(n)$ degrees of freedom) of algebraic surfaces of degree n that interpolate the given geometric data. We select suitable surfaces from this family, which additionally satisfy our nonsingularity and irreducibility constraints.[5]

One way to apply the Hermite interpolation technique to computation of a lowest degree algebraic surface, which has given geometric properties, is to search through the degrees, i.e., from $n = 1, 2, 3, \cdots$ for an interpolating surface. In Example 7.7 in Section 7, we give an example that shows how the geometric dependency between given curves and normals affects the algebraic dependency between linear equations generated by the Hermite interpolation algorithm. However, since the dependencies between linear equations do depend on the specific spatial interrelationships of given geometric objects, it is, in general, very difficult to bound the degree of interpolating surfaces a priori. For example, it is possible to design input data, made of an arbitrary number of degree four curves with normal directions, that can be interpolated by a quadric surface.

We now enumerate some results in which we lower-bound the degrees of some Hermite interpolating surfaces.

(1) Two skewed lines in space with constant direction normals cannot be Hermite interpolated with nondegenerate quadric surfaces. The only quadric that satisfies both containment and tangency conditions reduces into two planes.
(2) Two lines in space with constant direction normals can be Hermite interpolated with a quadric surface if and only if the lines are parallel or intersect at a point, and the normals are not orthogonal to the plane containing them. The quadric is a cylinder when the lines are parallel and a cone when the lines intersect.
(3) The minimum degree of an algebraic surface, which Hermite interpolates two lines in space, one with a constant direction normal, the other with a linearly varying normal is three.
(4) Two lines with linearly varying normals can be Hermite interpolated by a quadric in only some special cases. In general, a surface of at least degree three is needed. When quadric surface interpolation is possible, the quadric is either a hyperboloid of one sheet (the two lines may be parallel, intersecting, or skewed) or a hyperbolic paraboloid (the two lines can only be intersecting or skewed).

7. EXAMPLES

In this section we exhibit the method of Hermite interpolation by constructing lowest degree Hermite interpolating surfaces for joining and blending

[5] However, some of these interpolating surfaces still might not be suitable for the design application they were intended to benefit. These problems arise when the given points or curves are smoothly interpolated, but, lie on separate real components of the same nonsingular, irreducible algebraic surface.

primary surfaces of solid models as well as for fleshing curved wire frame models of physical objects.[6]

Example 7.1. (*JOINING* 1) *A cubic surface for smoothly joining two elliptic cylinders*. Consider computing a lowest degree surface that can smoothly join two truncated elliptic cylinders $CY\,L_1:(y+1)^2+(z^2/4)-1=0$ for $x\le -2$ and $CY\,L_2:25z^2+36y^2-96xy+64x^2-100=0$ for $3x+4y\ge 0$. Here, we illustrate the Hermite interpolation technique, which not only computes the *unique* cubic interpolating surface but also proves that degree three is the *lowest* for an algebraic surface to satisfy the smooth-join requirement for this configuration. We take an ellipse $C_1:(-2,-2t^2/1+t^2,4t/1+t^2)$ on $CY\,L_1$ with the associated rational normal $\mathbf{n}_1(t):(0,2-2t^2/1+t^2,2t/1+t^2)$ and another ellipse $C_2:(-4+4t^2/5+5t^2,3-3t^2/5+5t^2,4t/1+t^2)$ on $CY\,L_2$ with the associated rational normal $\mathbf{n}_2(t):(-80+80t^2/1+t^2,120-120t^2/1+t^2,200t/1+t^2)$. Both C_1 and C_2's normals are respectively chosen in the same directions as the gradients of their corresponding surfaces $CY\,L_1$ and $CY\,L_2$. This ensures that any Hermite interpolating surface for C_1 and C_2 will also meet $CY\,L_1$ and $CY\,L_2$ smoothly along these curves. A degree-two algebraic surface does not suffice for Hermite interpolation, since the rank of the constructed linear system is greater than 9, which is the degrees of freedom of a quadric surface. (Note that a quadric surface has 10 coefficients.) Next, as a possible Hermite interpolant, consider a degree-three algebraic surface with 20 coefficients. Applying the Hermite interpolation algorithm of Subsection 4.2.2 to the curves results in 26 linear equations (28 equations are supposed to be generated, but 2 of the 28 are degenerate). The rank of this linear system is 19, and thus there is a unique cubic Hermite interpolating surface, which is $f(x,y,z)=r_1(2yz^2-xz^2-5z^2+8y^3-4xy^2-4y^2+8x^2y+24xy-8y-4x^3-11x^2+4x+20)$. See Figure 2.

Example 7.2. (*JOINING* 2) *A quartic surface for smoothly joining three circular cylinders*. Consider computing a lowest degree surface that smoothly joins three truncated orthogonal circular cylinders $CYL_1: x^2+y^2-1=0$ for $z\ge 2$, $CYL_2: y^2+z^2-1=0$ for $x\ge 2$, and $CYL_3: z^2+x^2-1=0$ for $y\ge 2$.

Warren [25] found a degree-five surface for joining these cylinders. After applying the Hermite interpolation algorithm, we find out that the minimum degree for such joining surfaces is 4, and we get a 2-parameter (one degree of freedom) family of algebraic surfaces.

As before, we take a circle $C_1:(2t/1+t^2,1-t^2/1+t^2,2)$ on CYL_1 with the associated rational normal $\mathbf{n}_1(t):(4t/1+t^2,2-2t^2/1+t^2,0)$, the circle

[6] The solutions of all the examples in this subsection were obtained using MACSYMA in which the Gaussian elimination algorithm applied. The reason was to express solutions more clearly; however, the singular value decomposition algorithm was used in our implementation. Of course, the solution spaces are the same whichever method to be used in computing the nullspace, although the bases that span the vector subspace are different.

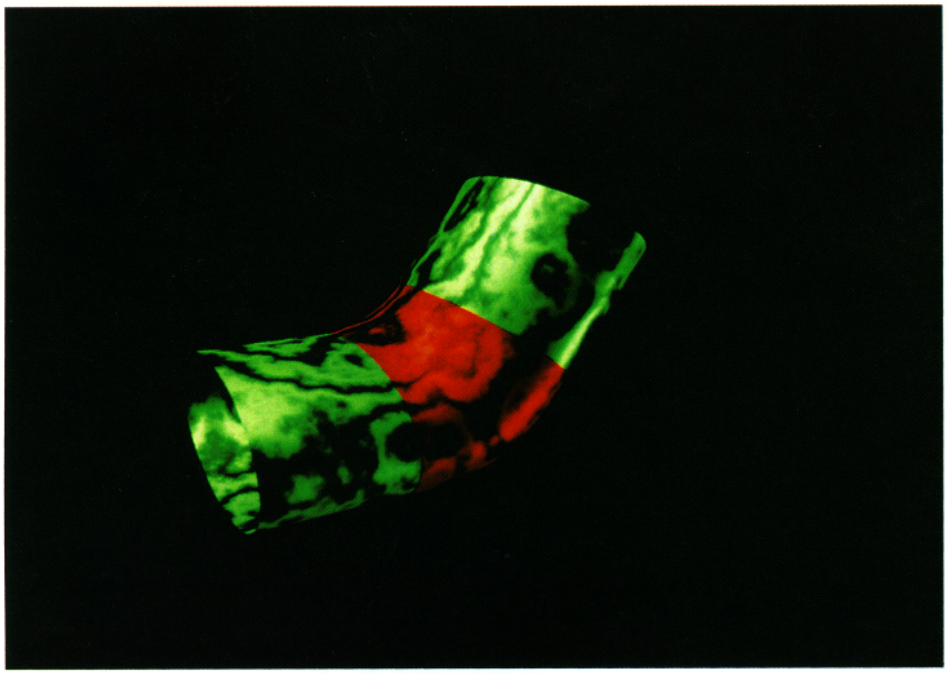

Fig. 2. Smooth joining of two cylinders with a cubic surface.

$C_2 : (2, 2t/1 + t^2, 1 - t^2/1 + t^2)$ on CYL_2 with the associated rational normal $\mathbf{n}_2(t)$: $(0, 4t/1 + t^2, 2 - 2t^2/1 + t^2)$, and the circle $C_3 : (2t/1 + t^2, 2, 1 - t^2/1 + t^2)$ on CYL_3 with the associated rational normal $\mathbf{n}_3(t) : (4t/1 + t^2, 0, 2 - 2t^2/1 + t^2)$. Again, all C_1, C_2, and C_3's normals are respectively chosen in the same direction as the gradients of their corresponding surfaces CYL_1, CYL_2, and CYL_3. This ensures that any Hermite interpolating surface for C_1, C_2, and C_3 will also meet CYL_1, CYL_2, and CYL_3 smoothly along these curves. A degree-three algebraic surface does not suffice for Hermite interpolation, since the rank of the resulting linear system is greater than 19. Next, as a possible Hermite interpolant, consider a degree-four algebraic surface with 35 coefficients, and 34 degrees of freedom. Applying the Hermite interpolation algorithm to the curves results in 52 equations. The rank of this linear system is 33, and thus there is a 2-parameter family of quartic Hermite interpolating surfaces, which is $f(x, y, z) = r_1 z^4 + (r_2 + 10r_1/12) yz^3 + (r_2 + 10r_1/12) xz^3 - (r_2 + 10r_1/3) z^3 + 2r_1 y^2 z^2 + (r_2 + 10r_1/12) xyz^2 - (r_2 + 10r_1/3) yz^2 + 2r_1 x^2 z^2 - (r_2 + 10r_1/3) xz^2 + r_2 z^2 + (r_2 + 10r_1/12) y^3 z + (r_2 + 10r_1/12) xy^2 z - (r_2 + 10r_1/3) y^2 z + (r_2 + 10r_1/12) x^2 yz - (r_2 + 10r_1/3) xyz + (r_2 + 10r_1/4) yz + (r_2 + 10r_1/12) x^3 z - (r_2 + 10r_1/3) x^2 z + (r_2 + 10r_1/4) xz + (r_2 + 10r_1/3) z + r_1 y^4 + (r_2 + 10r_1/12) xy^3 - (r_2 + 10r_1/3) y^3 + 2r_1 x^2 y^2 - (r_2 + 10r_1/3) xy^2 + r_2 y^2 + (r_2 + 10r_1/12) x^3 y - (r_2 + 10r_1/3) x^2 y + (r_2 + 10r_1/4) xy + (r_2 + 10r_1/3) y + r_1 x^4 - (r_2 + 10r_1/3) x^3 + r_2 x^2 + (r_2 + 10r_1/3) x + (5r_1 - 7r_2/3)$.

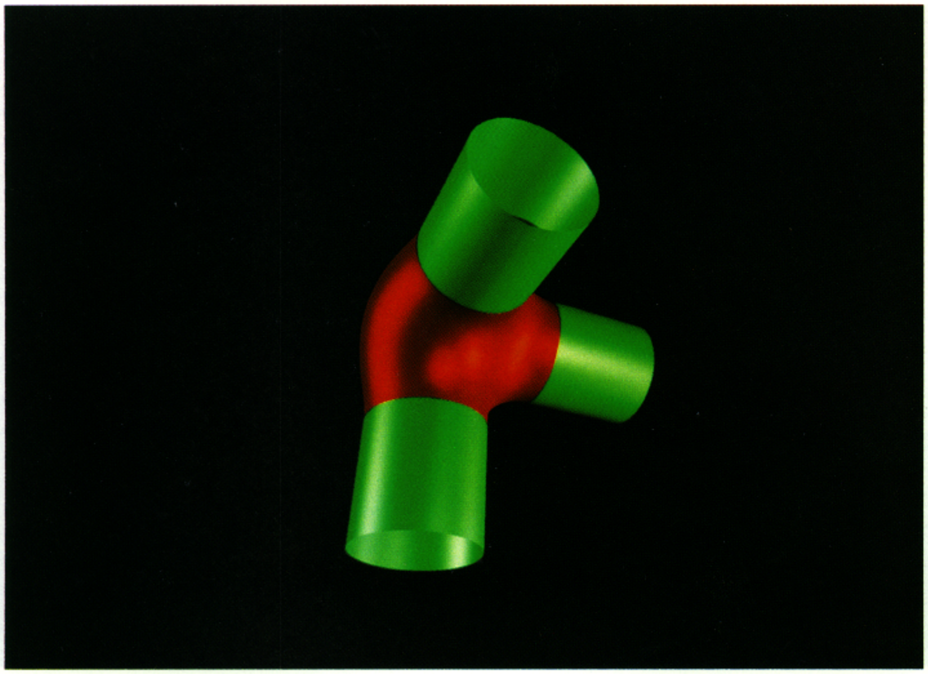

Fig. 3. Smooth joining of three cylinders with a quartic surface

An instance of this family $(r_1=1, r_2=10)$ is shown in Figure 3. It should be noted that every surface in the computed family is not always appropriate for geometric modeling. The quartic surface in Figure 4 is one used in Figure 3. On the other hand, the surface in Figure 5, which is not useful for geometric modeling, is also in the same family with $r_1=1$ and $r_2=-1$.

Example 7.3. (JOINING 3) A quartic surface for smoothly joining four circular cylinders. In this example, we compute a lowest degree surface that smoothly joins four truncated parallel circular cylinders defined by $CY\ L_1: y^2+z^2-1=0$ for $x \geq 2$, $CY\ L_2: y^2+z^2-1=0$ for $x \leq -2$, $CY\ L_3: (y-4)^2+z^2-1=0$ for $x \geq 2$, and $CY\ L_4: (y-4)^2+z^2-1=0$ for $x \leq -2$.

The Hermite interpolation technique indicates that the minimum degree for such a joining surface is 4, and computes a 2-parameter (one degree of freedom) family of algebraic surfaces which is $f(x,y,z)=(r_1/14)z^4 + r_1/7)y^2z^2 - (4r_1/7)yz^2 + r_1z^2 + (r_1/14)y^4 - (4r_1/7)y^3 + r_1y^2 + (4/7)r_1y + 14r_2+15r_1/224)x^4 - (14r_2+15r_1/28)x^2 + r_2$. An instance of this family $(r_1=392, r_2=-868)$ is shown in Figure 6.

Example 7.4. (BLENDING 1) Hyperboloid patches for blending two perpendicular cylinders. The case of two circular cylinders is a common test case for blending algorithms. Various different ways have been given, (for example, see [15, 17, 25]) for computing a suitable surface that smooths or blends the intersection of two equal radius cylinders, $CY\ L_1: x^2+z^2-1=0$ and

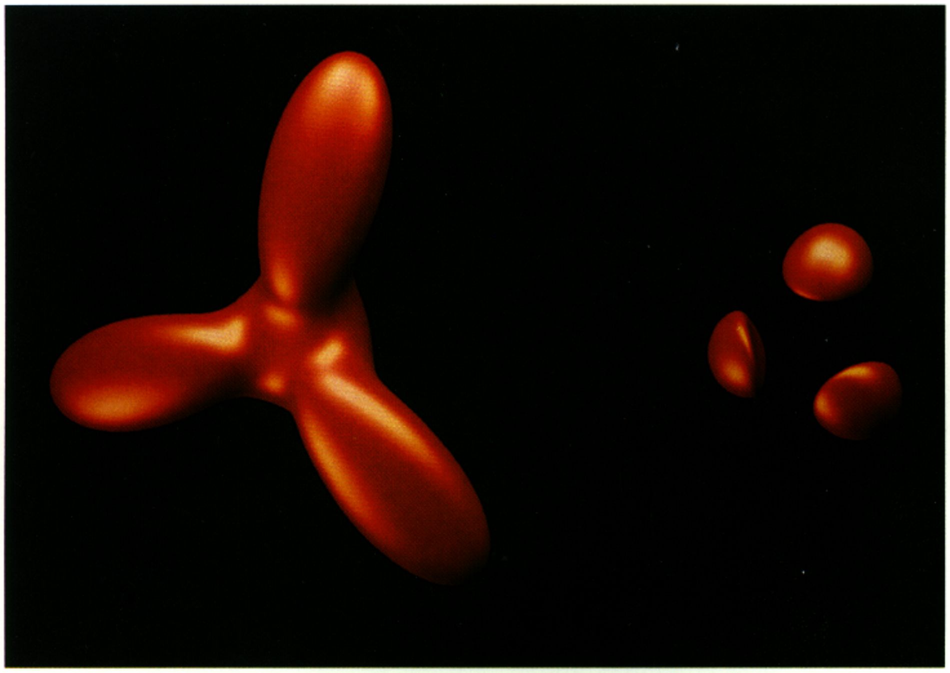

Fig. 4. A "good" quartic surface. Fig. 5. A "bad" quartic surface.

$CYL_2 : y^2 + z^2 - 1 = 0$. We consider an ellipse C_1 on CYL_1 (it is the intersection with the plane 3x + y = 0), defined parametrically, $C_1 : (2t/1 + t^2, -6t/1 + t^2, 1 - t^2/1 + t^2)$ with the associated rational normal $\mathbf{n}_1(t) = (4t/1 + t^2, 0, 2 - 2t^2/1 + t^2)$, and the ellipse C_2 on CYL_2 defined implicitly, $C_2 : (y^2 + z^2 - 1 = 0, x + 3y = 0)$ with the associated normal $\mathbf{n}_2(x, y, z) = (0, 2y, 2z)$. As a possible Hermite interpolant, we consider a degree-two algebraic surface. Applying the method of Subsection 4.2.2 to C_1 results in eight equations, five from the containment condition and three from the tangency condition. (Five equations are supposed to be generated, but two of these turn out to be degenerate.) For C_2, we use the method of Subsection 4.2.1 and first compute $L_c = \{(0, 0, 1), (-3, 1, 0), (3, -1, 0), (-2.4, 0.8, -0.6), (2.4, -0.8, -0.6)\}$ and $L_t = \{[(0, 0, 1), (0, 0, 2)], [(-3, 1, 0), (0, 2, 0)], [(3, -1, 0), (0, -2, 0)], [(-2.4, 0.8, -0.6), (0, 1.6, -1.2)], [(2.4, -0.8, -0.6), (0, -1.6, -1.2)]\}$. From these lists, we get ten equations, five from the containment condition and another five from the tangency condition. Hence, overall the linear system consists of 10 unknowns and 18 equations. The rank of this system is 9, and hence we get the unique surface solution $f_1(x, y, z) = r_1(x^2 + y^2 - 8z^2 + 6xy + 8 = 0)$. This quadric satisfies both the nonsingularity and irreducibility constraints. It is a hyperboloid of one sheet and the lowest degree surface which blends, together with a symmetric hyperboloid $f_2(x, y, z) = r_1(x^2 + y^2 - 8z^2 - 6xy + 8 = 0)$, the intersection of the two cylinders. See Figure 7.

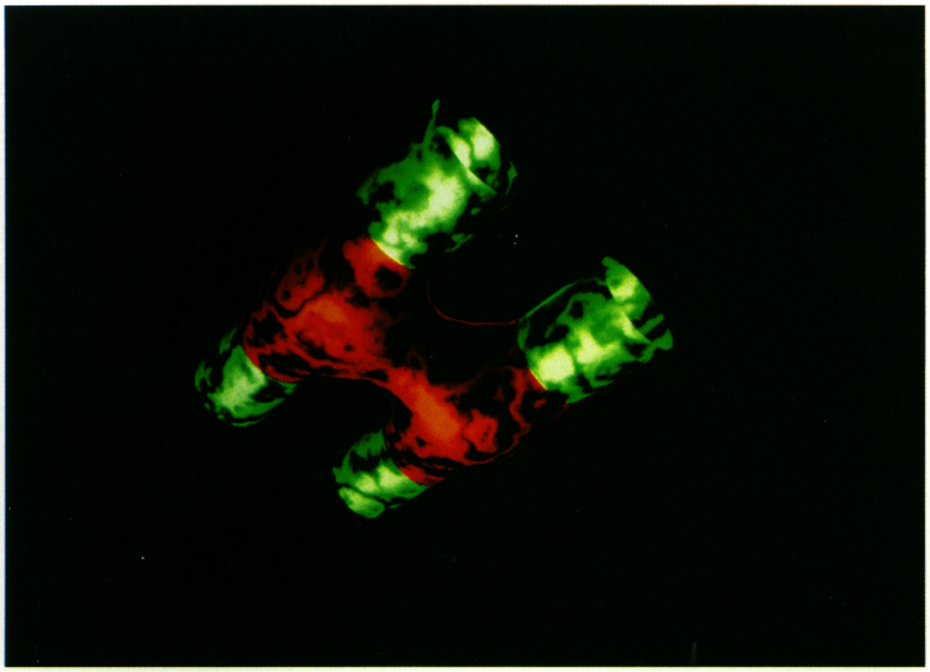

Fig. 6. Smooth joining of four cylinders with a quartic surface.

Example 7.5. (BLENDING 2) A quartic surface for blending two elliptic cylinders. In this example, we compute a lowest degree surface that blends two perpendicular elliptic cylinders. We have seen a quadric blending of the circular cylinders in Example 7.4. Here, we try a quartic blending surface by taking different types of input curves.

Input to Hermite interpolation is defined by $CYL_1: y^2+4z^2-4=0$ for $x \geq 1$, $CYL_2: y^2+4z^2-4=0$ for $x \leq -1$, $CYL_3: 9x^2+y^2-9=0$ for $z \geq 1$, and $CYL_4: 9x^2+y^2-9=0$ for $z \leq -1$.

The Hermite interpolation algorithm proves that 4 is the minimum degree for such a blending surface and that it generates a linear system with 72 equations of rank 33. The 2-parameter (one degree of freedom) family of algebraic surfaces is $f(x, y, z) = r_1 z^4 - (8r_2 + 81r_1/72)y^2z^2 - (8r_2 + 81r_1/8)x^2z^2 + (8r_2 + 65r_1/8)z^2 - (8r_2 + 99r_1/288)y^4 - (8r_2 + 81r_1/32)x^2y^2 + (104r_2 + 1053r_1/288)y^2 + (81r_1/16)x^4 + r_2x^2 - (16r_2 + 65r_1/16)$. An instance of this family $(r_1=1, r_2=2)$ is shown in Figure 8.

Example 7.6. (FLESHING 1) A quartic surface for fleshing a circular wire frame. Consider a wire frame a solid model consisting of two circles, $C_1: (x^2+y^2+z^2-25=0, x=0)$, and $C_2: (x^2+y^2+z^2-25=0, y=0)$. Each curve is associated with a "normal" direction, which is chosen in the same direction as the gradients of the sphere. That is, $\mathbf{n}_1(x, y, z) = (0, 2y, 2z)$, and $\mathbf{n}_2(x, y, z) = (2x, 0, 2z)$. The wire frame has four faces to be fleshed,

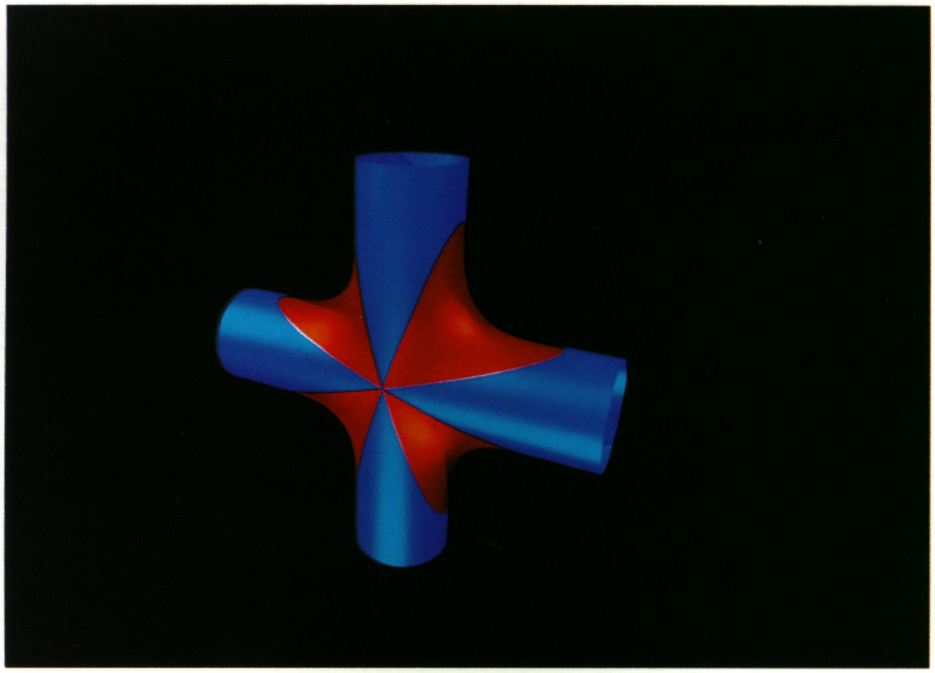

Fig. 7. Smooth blending of two cylinders with a quadric surface.

$face_1 = (x \geq 0, y \geq 0)$, $face_2 = (x \geq 0, y \leq 0)$, $face_3 = (x \leq 0, y \leq 0)$, and *face,* $= (x \leq 0, y \geq 0)$.

In Figure 9, *face,* and ***face,*** are filled with the patches taken from the sphere $x^2 + y^2 + z^2 - 25 = 0$ (green patches).

To smoothly flesh the remaining faces requires degree-four surface patches. Applying the **Hermite** interpolation method in either Subsection 4.2.2 or Subsection 4.2.1 to C_1 and C_2 results in **11-parameter (10 independent)** family of quartic interpolating surfaces, which is $f(x,y,z) = r_1 z^4 + (r_2 y + r_6 x + 5r_4)z^3 + (r_3 y^2 + (r_7 x + 5r_8)y + r_{10} x^2 + 5r_{11} x - 25r_9 - 25r_1)z^2 + (r_2 y^3 + (r_6 x + 5r_4)y^2 + (r_2 x^2 - 25r_2)y + r_6 x^3 + 5r_4 x^2 - 25r_6 x - 125r_4)z$ $(r_3 - r_1)y^4 + (r_7 x + 5r_8)y^3 + (r_5 x^2 + 5r_{11} x - 25r_9 - 25r_3 + 25r_1)y^2 + (r7x^3 + 5r_8 x^2 - 25r_7 x - 125r_8)\ y + r_{10} - r_1)\ x_4 + 5r_{11} x^3 + (-25r_9 - 25r_{10} + 25r_1)x^2 - 125r_{11} x + 625r_q$. An instance $f(x,y,z) = -1250 - x^4 - y^4 - x^2z^2 - y^2z^2 + 50z^2 + 75y^2 + 75x^2$ of this family is used to flesh $face_2$ and *face,* in Figure 9 (red patches).

Example 7.7. ***(FLESHING*** **2)** ***A quintic surface for fleshing a triangular wire frame*****. Figure 10 shows an instance of a 5-parameter family of quintic algebraic surfaces which fleshes the triangular wire frame, made of three conic curves with quadratic normals (both curves and normals are in quadratic rational parametric form) where the three curves meet pairwise, and the normals coincide at the three intersection points. We are currently working**

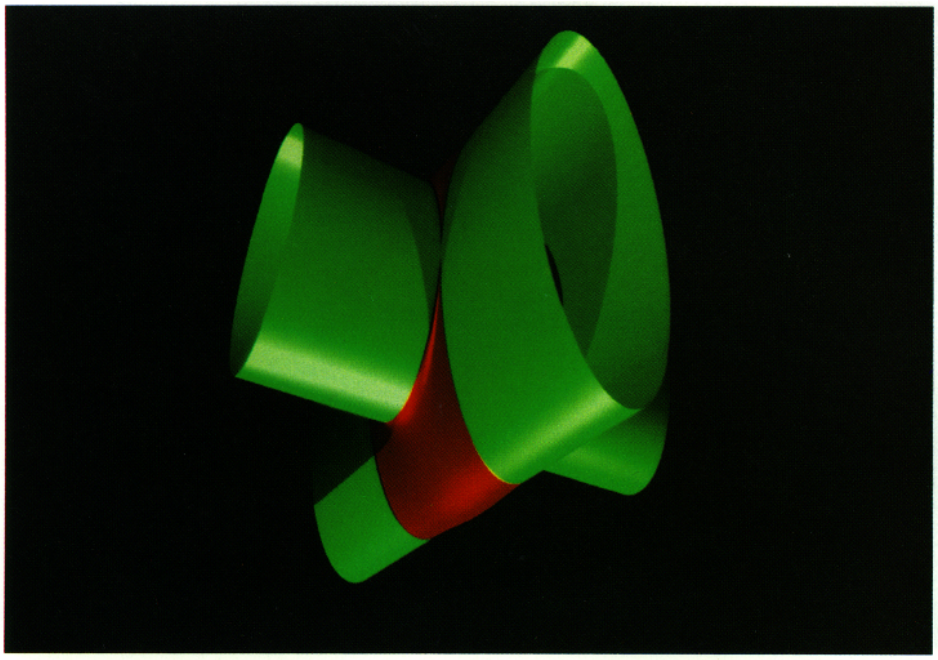

Fig. 8. Smooth blending of two cylinders with a quartic surface.

on a polyhedron-smoothing scheme in which this kind of triples of quadratic curve-normal pairs are automatically generated.

Consider the problem of smooth fleshing of the three curves with a degree n algebraic surface $f(x, y, z) = 0$. In order for f to Hermite interpolate a curve that is a quadratic rational parametric curve, a linear system of $2(2n-1)+2+2=4n+2$ is generated whose rank is uniformly observed to be $4n$. Since there are 3 such curves, $12n$ is the maximum rank of the linear system.

On the other hand, we observe some geometric dependency between the curves, which leads to algebraic dependency. First, since the curves intersect pairwise, there must be three rank deficiencies between the equations from containment conditions.[7] Secondly, at each intersection point, two incident curves automatically determine the normal at the point. This means satisfying containment conditions for the three curves and guarantees that any interpolating surface has normals at the three points as specified. This fact implies that, for each curve, there are two rank deficiencies between the linear equations for containment condition and the equations for its tangency

[7] If we always choose the intersection points for the list L_c of each curve in the algorithm in Section 4.2.1, three equations are generated twice.

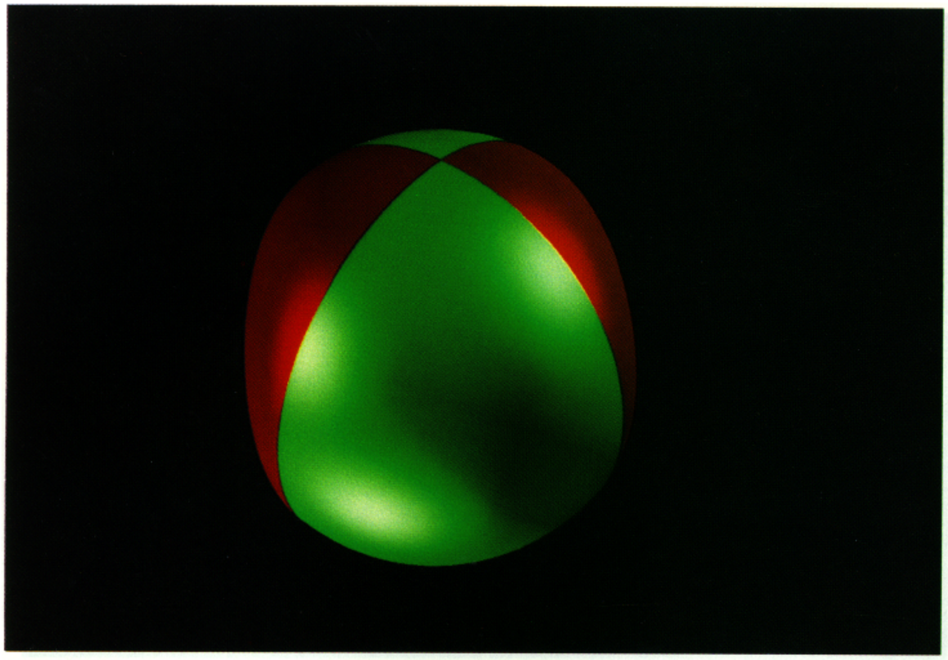

Fig. 9. Smooth fleshing of a wire frame with quadric and quartic surfaces

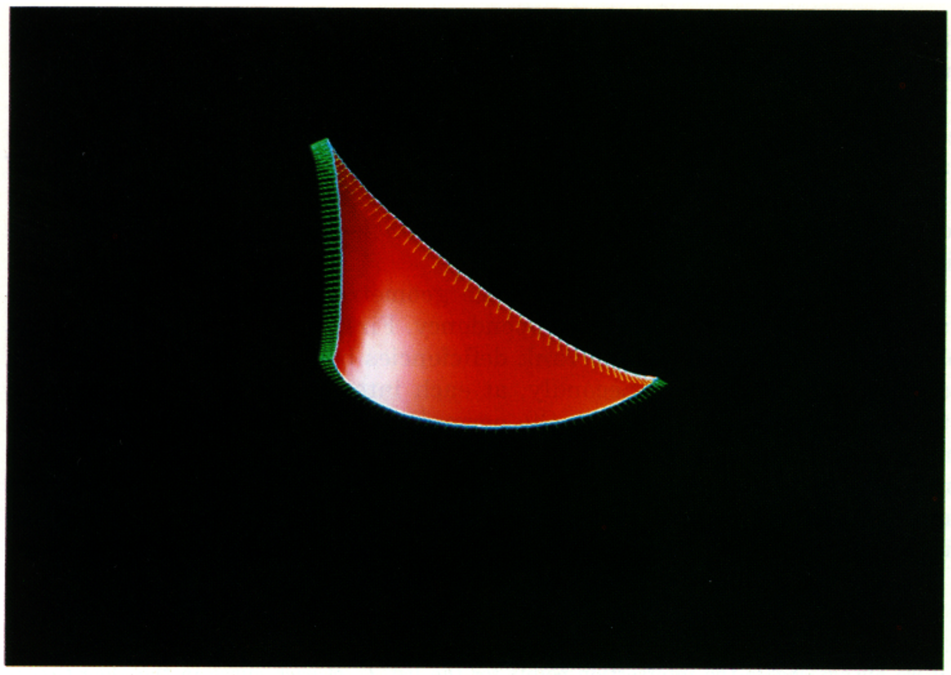

Fig. 10. Smooth fleshing of a triangular wire frame with a quintic surface.

condition.[8] Hence, six additional rank deficiencies with the previous three indicate that $12n - 9$ is the maximum possible rank of the linear system.

Since $f(x, y, z) = 0$ of degree n has $\binom{n+3}{3}$ coefficients, and the rank of the linear system should be less than the number of coefficients in order for a nontrivial surface to exist, we see that 5 is the minimum degree. In the quintic surface case, there are 56 coefficients, and the rank is 51, which results in a family of interpolating surfaces with at least 4 degrees of freedom.

In the next section, we present an interactive technique of controlling the shape of a suitable surface instance selected from a large family of interpolating surfaces.

8. INTERACTIVE SHAPE CONTROL OF HERMITE INTERPOLATING SURFACES

As mentioned before, the result of Hermite interpolation is a q-parameter family of algebraic surfaces $f(x, y, z) = 0$ of a given degree that satisfy given geometric properties. The equation of the family has the generic form

$$f(x, y, z) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} c_{ijk} \cdot x^i y^j z^k = 0, \tag{9}$$

where each c_{ijk} is a homogeneous linear combination of q parameters $r_1, r_2, \cdots, r_q$.

Bajaj et al. [4] proposed least squares approximation to select an initial instance surface from the family obtained from Hermite interpolation. Even though we can get some geometric intuition from least squares approximation, we may want to change the shape of the computed surface interactively by modifying the values of the free parameters. However, since the computed surface $f(x, y, z) = 0$ is a polynomial in the standard power basis, its coefficients are algebraic, not geometric. That is, they contain little intuitive geometric information; hence they do not provide a convenient tool with which the shape of an algebraic surface can be controlled intuitively.

Sederberg [19] presented an idea in which free-form piecewise algebraic surface patches defined in trivariate barycentric coordinates using a reference tetrahedron and a regular lattice of control points imposed on the tetrahedron. The coefficients of a surface defined in this way are assigned to the control points, and there is a meaningful relationship between the coefficients and the shape of the surface.

The essence of his idea is to consider an algebraic surface $f(x, y, z) = 0$ as the zero contour of the trivariate function $w = f(x, y, z)$. Note that the surface equation of the family of Hermite interpolating algebraic surfaces contains q free variables r_i in its coefficients. A specific portion of a surface can be selected for shape control by defining a tetrahedron that encloses that

[8] Again, for each curve, we can choose point-normal pairs at the two end points. The resulting two linear equations should be linearly dependent on the equations for containment.

portion. Given a tetrahedron, the polynomial $f(x, y, z)$ in power basis can be symbolically converted into a polynomial $F(s, t, u)$ in barycentric coordinates, defined with respect to the tetrahedron.

Let a tetrahedron be specified by the four noncoplanar vertices $P_{n00}, P_{0n0}, P_{00n}$, and P_{000}. Then, the coordinates $P = (x, y, z)$ of a point inside the tetrahedron are related to the barycentric coordinates (s, t, u) by $P = sP_{n00} + tP_{0n0} + uP_{00n} + (1 - s - t - u)P_{000}$, with $s, u, t, (1 - s - t - u) > 0$. Control points on the tetrahedron are defined by $P_{ijk} = (i/n)P_{n00} + (j/n)P_{0n0} + (k/n)P_{00n} + (n - i - j - k/n)P_{000}$ for nonnegative integers i, j, k such that $i + j + k \le n$. Each control point is associated with a weight w_{ijk}, which is a linear combination of r_i, $i = 1, 2, \cdots, q$. All these together define the q-parameter algebraic surface family in barycentric coordinates,

$$F(s, t, u) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} w_{ijk} \cdot \binom{n}{i, j, k} \cdot s^i t^j u^k (1 - s - t - u)^{n-i-j-k} = 0. \tag{10}$$

Example 8.1. *Conversion from Power to Bernstein*. Consider, as a simple example, a quadric surface that Hermite interpolates a line $LN: (1 - t, t, 0)$ with a normal $(0, 0, 1)$. The Hermite interpolation algorithm returns a 5-parameter family $f(x, y, z) = 0$ of algebraic surfaces, as in (9), with $n = 2$, where $c_{200} = r_1$, $c_{110} = 2r_1$, $c_{101} = r_4$, $c_{100} = -2r_1$, $c_{020} = r_1$, $c_{011} = r_5$, $c_{010} = -2r_1$, $c_{002} = r_3$, $c_{001} = r_2$, and $c_{000} = r_1$. For a given tetrahedron with vertices $P_{n00} = (2, 0, 0)$, $P_{0n0} = (0, 2, 0)$, $P_{00n} = (0, 0, 2)$, and $P_{000} = (0, 0, 0)$, the surface $f(x, y, z) = 0$ is transformed to $F(s, t, u) = 0$, as in (10), with $n = 2$, where $w_{000} = r_1$, $w_{001} = r_1 + r_2$, $w_{002} = r_1 + 2r_2 + 4r_3$, $w_{010} = -r_1$, $w_{011} = -r_1 + r_2 + 2r_5$, $w_{020} = r_1$, $w_{100} = -r_1$, $w_{101} = -r_1 + r_2 + 2r_4$, $w_{110} = r_1$, and $w_{200} = r_1$.

Since the weights w_{ijk} of $F(s, t, u) = 0$ for a q-parameter family of algebraic surfaces have only q degrees of freedom, they cannot be selected or modified independently. For example, suppose $w_1 = r_1 + r_2 + r_3 + 2r_4 - 1$, $w_2 = r_1 + r_2 + r_4 + 5$, and $w_3 = r_3 + r_4$. From these, we can derive the linear relation $w_1 - w_2 - w_3 - 6 = 0$ between the weights, and then an invariant $\Delta w_1 - \Delta w_2 - \Delta w_3 = 0$, which must be satisfied each time some of the weights are modified. (For notational simplicity, we assume the weights are indexed by a single number instead of a triple.)

In general, using Gaussian elimination, we can derive a *system of invariant equations*

$$\begin{aligned} I_1(\Delta w_1, \Delta w_2, \cdots, \Delta w_c) &= 0 \\ I_2(\Delta w_1, \Delta w_2, \cdots, \Delta w_c) &= 0 \\ &\vdots \\ I_l(\Delta w_1, \Delta w_2, \cdots, \Delta w_c) &= 0 \end{aligned}$$

for the linear expressions of the weights

$$w_1(r_1, r_2, \cdots, r_p) = w_1$$
$$w_2(r_1, r_2, \cdots, r_p) = w_2$$
$$\vdots$$
$$w_c(r_1, r_2, \cdots, r_p) = w_c.$$

Changing the weights can now be considered as moving from a weight vector $W = (w_1, w_2, \cdots, w_c)$ to another $W' = (w'_1, w'_2, \cdots, w'_c)$, with the constraint that $\Delta W = W' - W$ is a solution of the system of invariant equations.

Example 8.2. *Shape control of a family of quadric surfaces.* The invariant system for the family of algebraic surfaces in Example 8.1 is $\Delta w_{010} + \Delta w_{000} = 0$, $\Delta w_{020} - \Delta w_{000} = 0$, $\Delta w_{100} + \Delta w_{000} = 0$, $\Delta w_{110} - \Delta w_{000} = 0$, $\Delta w_{200} - \Delta w_{000} = 0$. Figure 11 (upper left) shows an instance from the family where $w_{000} = -4$, $w_{001} = 4$, $w_{002} = 8$, $w_{010} = 4$, $w_{011} = 14$, $w_{020} = -4$, $w_{100} = 4$, $w_{101} = 12$, $w_{110} = -4$, and $w_{200} = -4$.

Now, suppose we want to pull the surface patch toward the control points P_{002} (the leftmost vertex in the figure). This can be achieved by decreasing the value of w_{002}, say, $\Delta w_{002} = -7$.

Other Δw_{ijk} can be arbitrarily chosen as long as they satisfy the equations in the invariant system. Let $\Delta w_{000} = \Delta w_{200} = \Delta w_{110} = \Delta w_{020} = -1$, $\Delta w_{100} = \Delta w_{010} = 1$, $\Delta w_{001} = -4$, $\Delta w_{101} = \Delta w_{011} = -2$. The new instance surface is shown in Figure 11 (upper right).

Example 8.3. *Shape control of a family of quartic surfaces.* Figure 11 (bottom left) illustrates three different instances of the family computed in Example 7.2, corresponding to the three different values of w_{000} for $P_{000} = (0,0,0)$. As a weight w_{000} increases from a negative value, the surface approaches to P_{000}. The surface passes through P_{000} when $w_{000} = 0$, and gets separated into three irreducible components as w_{000} becomes positive. (See also Figure 4 and 5.)

Sometimes, we may want to see how the shape of a surface changes as a specific weight is modified. However, if a weight, say, w_1 is modified, then this modification affects other weights as related in the invariant system. Usually, the linear system of invariant equations is underdetermined, yielding an infinite number of choices of Δw_i $(i = 2, 3, \cdots, c)$. Then, how can we select the other weights such that their effects to w_1 are minimized?

One possible heuristic is to minimize the 2-norm of $(\Delta w_2, \cdots, \Delta w_c)$, and hence the 2-norm $\|\Delta W\|_2 = (\Delta w_1^2 + \Delta w_2^2 + \cdots + \Delta w_c^2)^{1/2}$ of ΔW. For $\Delta w_1 = d$, we know that the linear system

$$I_1(d, \Delta w_2, \cdots, \Delta w_c) = 0$$
$$I_2(d, \Delta w_2, \cdots, \Delta w_c) = 0$$
$$\vdots$$
$$I_l(d, \Delta w_2, \cdots, \Delta w_c) = 0$$

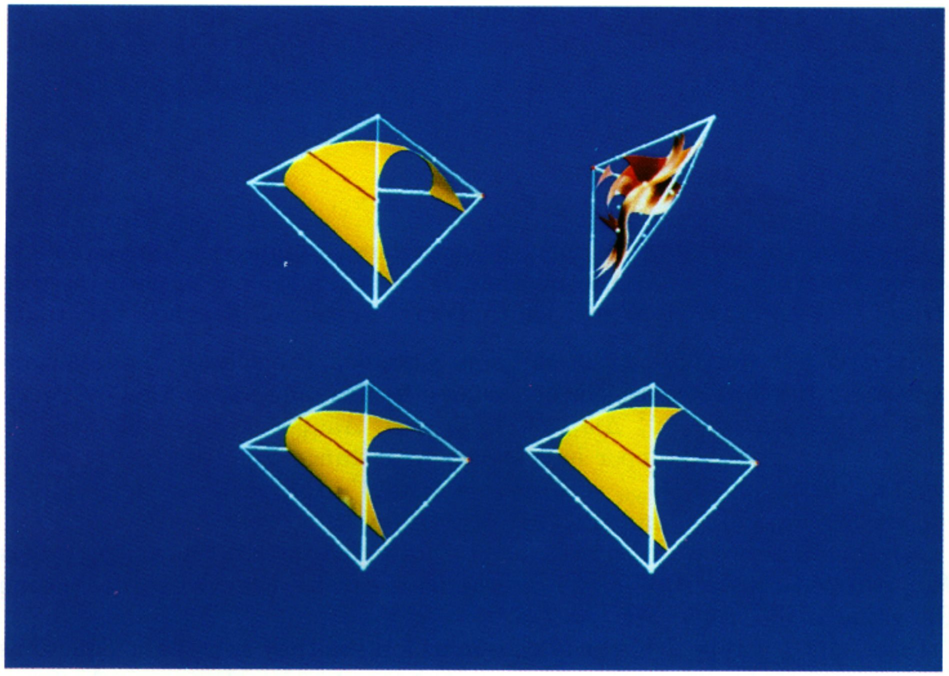

Fig. 11. Interactive shape control using **barycentric** coordinates.

has a solution $\Delta W^0 = (d, \Delta w_2^0, \cdots, \Delta w_c^0)$ where Δw_i^0's are expressed linearly through another set of free parameters $p_1, p_2, \cdots, p_s$. Hence, $\|\Delta W^0\|_2^2$ is a quadratic function $Q(p_1, p_2, \cdots, p_s)$ of the new parameters.

Since Q is quadratic, $Q(p_1, p_2, \cdots, p_s)$ is minimized at the solution of the linear system $\nabla Q(p_1, p_2, \cdots, p_s) = \mathbf{0}$. If the minimum of Q occurs at a point $(p_1^0, p_2^0, \cdots, p_s^0)$, then $\Delta W^0 = (d, \Delta w_2^0, \cdots, \Delta w_c^0)$ corresponding to the point defines the desired change of weights $w_2, \cdots, w_c$ having the minimum effect, in the least squares sense, on the shape of the surface. The instance surface corresponding to the new weights $W' = W + \Delta W^0$ will then reflect predominantly the effect of the change of w_1 by $\Delta w_1 = d$.

Example 8.4. Heuristic approach to shape control using 2-norm. Consider the surface in Example 8.2 again. This time we wish to pull the patch more toward P_{002}, and hence set $\Delta w_{002} = -15$. From the invariant system in which Δw_{002} is replaced by -15, $\Delta w_{000} = \Delta w_{020} = \Delta w_{110} = \Delta w_{200} = p_1$, $\Delta w_{010} = \Delta w_{100} = -p_1$, $\Delta w_{001} = p_2$, $\Delta w_{101} = p_3$, $\Delta w_{011} = p_4$, and we obtain the quadratic function $Q(p_1, p_2, p_3, p_4) = 225 + 6p_1^2 + p_2^2 + p_3^2 + p_4^2$. Q has the global minimum at $p_1 = p_2 = p_3 = p_4 = 0$. Hence, the influence of the change of all the weights other than w_{002} is minimized by setting to zero their Δw, that is, not changing them at all.

This new instance is shown in Figure 11 (bottom right). Note that the overall shape of the new surface patch, other than close to P_{002}, has not changed as much as the surface patch in Figure 11 (upper right), even though w_{002} has decreased by a larger amount.

9. CONCLUDING REMARKS

In this paper we presented the Hermite interpolation algorithm for algebraic surfaces. With the algorithm, it was possible to characterize the class of algebraic surfaces of a fixed degree that have given positional and tangential properties, in terms of the nullspace of a matrix. The rank of the matrix, produced by the algorithm, was used in proving existence or nonexistence of algebraic surfaces of a given degree. We considered computational aspects of the algorithm and illustrated the usefulness of the algorithm from several examples. We also proposed a surface control scheme in which the shape of a specific portion of an algebraic surface is controlled with geometric intuition in the barycentric coordinate system.

We have implemented the Hermite interpolation algorithm, as presented in Sections 3 and 4. The program takes as input any collection of points and curves, with/without associated normals. Both implicit and rational parametric representations of the space curves and normals are allowed. The homogeneous linear system of equations generated by the algorithm is decomposed using the singular value decomposition method. As a result, the rank of the system and the null space (that is, the solution of the system) are computed. The nontrivial solutions (and hence the coefficients of the corresponding surfaces), if any, are expressed in terms of linear combinations of free parameters and represent a family of interpolation surfaces.

Even though the change of polynomial representations from power to Bernstein bases yields additional geometry to the coefficients of the polynomial representation of the algebraic surface, for large-parameter (four and greater) family of surfaces, much depends on the initial assignment of weight values. One possibility that we are currently exploring for solution instantiation from large-parameter families is the use of least squares approximation for implicit algebraic surfaces to compute these initial vector of coefficients. Additional data points, curves, and even simple surfaces are inserted by the designer in near proximity to the desired surface, and the free parameters of the interpolating solution family are chosen to minimize the square of the distance error. This overall problem of Hermite interpolation and least squares approximation amounts to solving an optimization problem of the form:

$$\begin{aligned} \text{minimize} \quad & \mathbf{x}^T \mathbf{M}_A{}^T \mathbf{M}_A \mathbf{x} \\ \text{subject to} \quad & \mathbf{M}_I \mathbf{x} = \mathbf{0} \\ & \mathbf{x}^T \mathbf{x} = 1, \end{aligned}$$

where $\mathbf{M}_A$ and $\mathbf{M}_I$ are, respectively, matrices for least squares approximation and Hermite interpolation. The solution can be obtained by computing eigenvectors of some educed matrix. Details are provided by Bajaj et al. [4].

The triangular quintic surface in Example 7.7 was produced by applying least squares approximation to the resulting 5-parameter family of interpolating algebraic surfaces and some additional geometric data. We are currently using this method to instantiate 5-parameter triangular quintic surface patches for construction of a compact implicit model of a human head, reconstructed from NMR imaging data.

There still remain several open problems that need to be solved before the entire class of algebraic surfaces can be effectively used for computer-aided geometric design. First, as illustrated in Example 7.2, some of the irreducible interpolating algebraic surfaces produced directly from Hermite interpolation may not be suitable for the geometric design application they were initially intended to benefit. More precisely, the input data when interpolated may lie on several separate real components of the algebraic surface. One heuristic, which we currently used is to interactively include additional points and curves to effectively bridge the gap between separate real components. However, the question remains open for a priori generating *polynomial constraints* on the coefficients of the interpolating surfaces, which would ensure that all given points and curves lie on the same continuous real-surface component.

Secondly, polynomial constraints are again required to ensure that a specific algebraic surface patch has no singularities or self intersections. Having some singularities is not always an unfavorable feature in the design of algebraic surfaces. For example, the blending surface in Example 7.5 is singular at the four points where the four cylinders contact each other. (In fact, the only solution for satisfying the normals along the four ellipses is to have singular points.) While singularities along the boundary of a patch may be allowable, no singularity or self intersection would seem tolerable in the interior of the patch. Hence, deriving uniform polynomial constraints, imposed on a family of interpolating surfaces, which ensure nonsingularity of the specific portion of algebraic surface, would be highly desirable.

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