Curves, surfaces, and syzygies

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Abstract. This article surveys recent work with Sederberg, Chen, Goldman, Zhang, Schenck, Busé and D'Andrea on how syzygies can be used to implicitize rational curves and surfaces. There are also non-technical discussions of local complete intersections, regularity, and saturation.

Introduction

The purpose of this paper is to survey some recent work on the use of syzygies to give determinantal formulas for the equations of parametrized curves and surfaces. The paper is organized into five sections as follows, where the parentheses indicate the joint authors involved.

1. Curves (with Sederberg and Chen [**8**])
2. Surfaces without Base Points (with Goldman and Zhang [**6**])
3. Base Points (with Schenck [**5**])
4. Saturation and Regularity
5. Surfaces with Base Points (with Busé and D'Andrea [**2**])

One of my goals is to illustrate how the geometric modeling community is asking interesting and nontrivial questions which involve some surprisingly sophisticated commutative algebra. Sections 1 and 2 are based on [**4**] while Sections 3 and 5 report on subsequent developments. Section 4 is devoted to a discussion of the concepts of saturation and regularity.

1. Curves

For curves, the goal is to find the implicit equation of a parametrized curve in the projective plane $\mathbb{P}^2$. This means that we want to implicitize a parametrization of the form

$$\begin{aligned}&\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^2\\ &\phi(s,t) = \big(a(s,t), b(s,t), c(s,t)\big),\end{aligned} \tag{1.1}$$

where a, b, c are homogeneous of degree n and $\gcd(a,b,c) = 1$. In this paper, we will work over the complex numbers $\mathbb{C}$. Thus a, b, c lie in the polynomial ring $\mathbb{C}[s,t]$,

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and the gcd condition implies that we have no base points to worry about. This is one of the really nice features of the curve case.

In practice, implicitization of curves and surfaces can be done by any of the three following methods:

- Gröbner Bases
- Resultants
- Syzygies

This paper will concentrate on the third of these methods.

Moving Lines. A *moving line* in $\mathbb{P}^2$ is an equation of the form

$$A(s,t)x + B(s,t)y + C(s,t)z = 0,$$

for A, B, C are homogeneous of the same degree. We say that the moving line *follows* ϕ from (1.1) if

$$A(s,t)a(s,t) + B(s,t)b(s,t) + C(s,t)c(s,t) \equiv 0,$$

where $\equiv 0$ means vanishes identically. In algebraic geometry, one says that (A, B, C) is a *syzygy* on (a, b, c). This is written

$$(A, B, C) \in \mathrm{Syz}(a, b, c),$$

where $\mathrm{Syz}(a, b, c)$ is the *syzygy module* of (a, b, c). Note that $\mathrm{Syz}(a, b, c)$ is a module over the ring $\mathbb{C}[s, t]$. We also let

$$\mathrm{Syz}(a, b, c)_k$$

denote the set of syzygies (A, B, C) where A, B, C are homogeneous of degree k. Thus $\mathrm{Syz}(a, b, c)_k$ is a vector space over $\mathbb{C}$. We say that elements of $\mathrm{Syz}(a, b, c)_k$ are *moving lines of degree k* that follow ϕ.

There is one degree which is especially important.

CLAIM 1.1. The moving lines of degree $n-1$ that follow ϕ determine the implicit equation of the parametrization ϕ.

To see why this is true, let $R = \mathbb{C}[s, t]$ and let R_k denote the vector space of homogeneous polynomials in R of degree k. Then $\mathrm{Syz}(a, b, c)_{n-1}$ is the kernel of the map

$$\underbrace{R^3_{n-1}}_{\dim\ 3n} \xrightarrow{(a,b,c)} \underbrace{R_{2n-1}}_{\dim\ 2n} \tag{1.2}$$

given by dot product with (a, b, c). Later we will show that this map has maximal rank. Assuming this, we can find n linearly independent moving lines of degree $n-1$, say (A_i, B_i, C_i), $i = 0, \ldots, n-1$. Since each A_i, B_i, C_i is a polynomial in s, t, we can write this moving line as

$$A_i x + B_i y + C_i z = \sum_{j=0}^{n-1} L_{ij}(x, y, z) s^j t^{n-1-j},$$

where $L_{ij}(x, y, z)$ is a linear form in the homogeneous coordinates x, y, z of $\mathbb{P}^2$. Then we have the following result proved in [**8**].

THEOREM 1.2. *The implicit equation of ϕ is $F = 0$, where*

$$F^h = \det(L_{ij}(x, y, z))$$

and h is the generic degree of $\phi : \mathbb{P}^1 \to \mathbb{P}^2$.

Note that $(L_{ij}(x,y,z))$ is an $n \times n$ matrix of linear forms, so that its determinant has degree n in x, y, z. This is exactly the degree that we would expect in this case.

The Hilbert Syzygy Theorem. The next step is to look a little more deeply into the commutative algebra involved in this situation. Let $I = \langle a, b, c\rangle \subset R = \mathbb{C}[s,t]$. Then I is an ideal of R and we have an exact sequence

$$0 \longrightarrow \mathrm{Syz}(a,b,c) \longrightarrow R(-n)^3 \xrightarrow{(a,b,c)} I \longrightarrow 0. \tag{1.3}$$

This is standard notation in commutative algebra. Since $R(-n)^3 \to I$ sends (A,B,C) to $Aa+Bb+Cc$, the exactness of (1.3) simply restates the known facts that I is generated by a,b,c and that $\mathrm{Syz}(a,b,c)$ is the kernel of the map $R(-n)^3 \to I$, i.e., $\mathrm{Syz}(a,b,c)$ is the syzygy module.

The notation $R(-n)$ in (1.3) means that we are shifting degrees by $-n$ to compensate for the fact that multiplication by a,b,c shifts degrees by $+n$. Thus $R(-n)_k = R_{k-n}$, so that in (1.3), $(A,B,C) \in R(-n)^3_k$ means that A,B,C have degree $k-n$ and hence $Aa+Bb+Cc$ has degree k. It follows that dot product with (a,b,c) maps $R(-n)^3_k$ to I_k, i.e., this map preseves degrees. This is why notation like $R(-n)$ is standard in commutative algebra.

The *Hilbert Syzygy Theorem* describes the structure of free resolutions of homogeneous ideals in polynomial rings. In the case of two variables, the Syzygy Theorem implies that the syzygy module $\mathrm{Syz}(a,b,c)$ in (1.3) is *free*, meaning that every element of the module can be expressed uniquely as a sum of basis elements multipled by elements of R. Furthermore, using the *Hilbert polynomial*, one can show that

$$\mathrm{Syz}(a,b,c) \simeq R(-n-\mu_1) \oplus R(-n-\mu_2), \quad \mu_1+\mu_2 = n. \tag{1.4}$$

The details of this argument can be found in [**8**]. In more down-to-earth terms, the above isomorphism means that if we set $\mu = \mu_1 \le \mu_2 = n-\mu$, then there are syzygies $p, q \in \mathrm{Syz}(a,b,c)$ such that

$$\mathrm{Syz}(a,b,c) = R\underbrace{p}_{\deg\ \mu} \oplus R\underbrace{q}_{\deg\ n-\mu}.$$

We call p, q a *μ-basis* of the parametrization (1.1).

The existence of a μ-basis has some strong consequences. For example, it implies that every syzygy of degree $n-1$ can be written uniquely as

$$\underbrace{h_1}_{\deg\ n-\mu-1} p + \underbrace{h_2}_{\deg\ \mu-1} q. \tag{1.5}$$

Since there are $n-\mu$ (resp. μ) linearly independent choices for h_1 (resp. h_2), it follows that there are precisely n linearly independent moving lines of degree $n-1$ that follow ϕ. Thus (1.2) has maximal rank, as claimed earlier.

Another interesting aspect of (1.5) is that if we let h_1 (resp. h_2) range over all monomials of degree $n-\mu-1$ (resp. $\mu-1$), then the matrix $(L_{ij}(x,y,z))$ becomes the Sylvester matrix of p and q. Thus we get the following corollary of Theorem 1.2.

COROLLARY 1.3. *If p, q is a μ-basis of the parametrization ϕ, then*

$$\mathit{Res}(p,q) = F^h,$$

where $F = 0$ is the implicit equation of the curve and h is the generic degree of ϕ.

As explained in [**8**], it is also possible to express F^h as a $(n-\mu)\times(n-\mu)$ determinant with

$$n-2\mu \text{ linear rows built from } p, \text{ and}$$
$$\mu \text{ quadratic rows built from the Bezoutian of } p,q.$$

More generally, expressing implicit equations and resultants as "mixed" determinants of the above type is an active area of research.

Regularity. We should also mention that the existence of a μ-basis tells us about the *regularity* of the ideal $I=\langle a,b,c\rangle$. Here, the regularity of I, denoted $\mathrm{reg}(I)$, means the following. Since $a,b,c\in R=\mathbb{C}[s,t]$ have no common zeros, then setting $t=1$ gives polynomials $\tilde{a}(s)=a(s,1),\tilde{b}(s)=b(s,1),\tilde{c}(s)=c(s,1)$ in $\mathbb{C}[s]$ with no common zeros. By the Nullstellensatz, it follows that $\tilde{a},\tilde{b},\tilde{c}$ generate the unit ideal of $\mathbb{C}[s]$, i.e., $\langle\tilde{a},\tilde{b},\tilde{c}\rangle=\mathbb{C}[s]$.

In the homogeneous case, $I=\langle a,b,c\rangle$ can't equal $R=\mathbb{C}[s,t]$ since elements of I have degree at least n. However, it is true that $I_k=R_k$ for k sufficiently large (this follows from the projective Nullstellensatz). But what does "sufficiently large" mean? For ideals without base points, this is exactly what regularity tells us. In other words, $\mathrm{reg}(I)$ is the smallest integer k_0 such that $I_k=R_k$ for all $k\geq k_0$.

Using (1.3) and (1.4), one can show that the regularity of $I=\langle a,b,c\rangle$ is

$$\mathrm{reg}(I)=2n-\mu-1.$$

Thus the μ-basis determines the regularity in our situation.

In general, the regularity of a homogeneous ideal I is a subtle number $\mathrm{reg}(I)$ computed from the minimal free resolution of the ideal. The intuition is that the regularity of I measures how big k needs to be in order for I_k to behave nicely. In Section 3, we will explain what "behave nicely" means when I has finitely many base points.

Some History. In 1997, Sederberg and Chen conjectured the existence of μ-bases and asked if I had any ideas for how to prove their conjecture. I worked out an elementary proof (which appears in [**8**]), but I had a nagging suspicion that something more was involved. To my embarrassment, it was over six months before I realized that the Hilbert Syzygy Theorem was the answer.

As indicated above, the Syzygy Theorem does a wonderful job of revealing the underlying structure of what's going on. This led me to believe that I wasn't the first person to look at this case. Checking the literature led me to an 1887 paper of Franz Meyer, where he proves the existence of a μ-basis p,q. More generally, he conjectured that for a collection of m homogeneous polynomials $a_1,\ldots,a_m\in R_n=\mathbb{C}[s,t]_n$ without common factors, the syzygy module $\mathrm{Syz}(a_1,\ldots,a_m)$ should be a free module with $m-1$ generators of degrees $\mu_1,\ldots,\mu_{m-1}$ which sum to n. He tried very hard to prove the case $m=4$ but failed.

In 1890, just three years after Meyer's paper, Hilbert published his amazing paper which proves the Syzygy Theorem and defines Hilbert polynomials. This paper is a cornerstone of modern commutative algebra. And the very first application given by Hilbert is to prove Meyer's conjecture! (References to the papers of Meyer and Hilbert can be found in [**8**].)

For curves, the moral of the story is that commutative algebra provides precisely the tools needed to understand syzygies and how they relate to the implicit equation. As we will soon see, surfaces are more complicated.

2. Surfaces

We now consider surface parametrizations in $\mathbb{P}^3$. We will begin with the tensor product case, where the parametrization is given by

$$\begin{aligned} &\phi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3 \\ &\phi(s,t,u,v) = \big(a(s,t,u,v), b(s,t,u,v), c(s,t,u,v), d(s,t,u,v)\big). \end{aligned} \tag{2.1}$$

Here $a, b, c, d \in R = \mathbb{C}[s, u; t, v]$ are homogeneous polynomials of bidegree (m, n) and $\gcd(a, b, c, d) = 1$. Unlike the curve case, the gcd condition still allows for the possibility of finitely many *base points*, which are points of $\mathbb{P}^1 \times \mathbb{P}^1$ where a, b, c, d vanish simultaneously. The possible presence of base points is why we use the broken arrow $\dashrightarrow$ in (2.1); it means that ϕ might not be defined on all of $\mathbb{P}^1 \times \mathbb{P}^1$.

In this section, we will assume that the map ϕ of (2.1) has no base points and is generically one-to-one. Thus we can write ϕ as

$$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow S \subset \mathbb{P}^3,$$

where S is the image of ϕ. Our goal is to compute the implicit equation

$$F = 0$$

of the surface S. The degree of F is $2mn$ since the generic degree is 1.

Moving Planes and Quadrics. In analogy with the moving lines used in the study of curves, a 4-tuple $(A, B, C, D) \in R^4$ of homogeneous polynomials of the same bidegee gives a *moving plane*

$$Ax + By + Cz + Dw = 0$$

in $\mathbb{P}^3$, and this moving plane *follows* ϕ if

$$\begin{aligned} &A(s,t,u,v)\, a(s,t,u,v) + B(s,t,u,v)\, b(s,t,u,v) \\ &+ C(s,t,u,v)\, c(s,t,u,v) + D(s,t,u,v)\, d(s,t,u,v) \equiv 0. \end{aligned}$$

The set of all moving planes that follow ϕ is the syzygy module $\mathrm{Syz}(a, b, c, d)$.

Similarly, a *moving quadric* is an equation

$$Ax^2 + Bxy + \cdots + Jw^2 = 0$$

where $(A, \ldots, J) \in R^{10}$ are homogeneous of the same bidegree, and a moving quadric *follows* ϕ if

$$A(s,t,u,v)\, a(s,t,u,v)^2 + \cdots + J(s,t,u,v)\, d(s,t,u,v)^2 \equiv 0.$$

The moving quadrics that follow ϕ form the syzygy module $\mathrm{Syz}(a^2, ab, \ldots, d^2)$.

Given a bidegree (k, l), we will let

$$R_{k,l} \ \text{ resp. } \ \mathrm{Syz}(a,b,c,d)_{k,l} \ \text{ resp. } \ \mathrm{Syz}(a^2, ab, \ldots, d^2)_{k,l}$$

denote the vector spaces of polynomials resp. moving planes that follow ϕ resp. moving quadrics that follow ϕ of this bidgree.

As in the curve case, there is one bidegree which is especially interesting. First, the moving planes of bidegree $(m-1, n-1)$ that follow ϕ are the kernel of

$$MP : \underbrace{R^4_{m-1,n-1}}_{\dim\, 4mn} \xrightarrow{(a,b,c,d)} \underbrace{R_{2m-1,2n-1}}_{\dim\, 4mn}. \tag{2.2}$$

If this map has maximal rank, then $\mathrm{Syz}(a, b, c, d)_{m-1,n-1} = \{0\}$, i.e., there are no moving planes of bidegree $(m-1, n-1)$ that follow ϕ.

Second, moving quadrics of bidegree $(m-1,n-1)$ are the kernel of

$$MQ : \underbrace{R^{10}_{m-1,n-1}}_{\dim\, 10mn} \xrightarrow{(a^2,ab,\ldots,d^2)} \underbrace{R_{3m-1,3n-1}}_{\dim\, 9mn}. \tag{2.3}$$

If this map has maximal rank, then $\mathrm{Syz}(a^2,ab,\ldots,d^2)_{m-1,n-1}$ has dimension mn, i.e., there are mn linearly independent moving quadrics of bidegree $(m-1,n-1)$ that follow ϕ.

Let's assume that MQ has maximal rank. This gives mn moving quadrics of degree $(m-1,n-1)$ that follow ϕ, say $Q_1,\ldots,Q_{mn}$. Let $u=v=1$ and write

$$\begin{aligned} Q_i &= A_i x^2 + \cdots + J_i w^2 \\ &= \Big(\sum_{j,k} A_{i,jk} s^j t^k\Big) x^2 + \cdots + \Big(\sum_{j,k} J_{i,jk} s^j t^k\Big) w^2 \\ &= \sum_{j,k} (\underbrace{A_{i,jk} x^2 + \cdots + J_{i,jk} w^2}_{Q_{i,jk}(x,y,z,w)}) s^j t^k. \end{aligned}$$

For each $0 \le j \le m-1$ and $0 \le k \le n-1$, this gives a quadric polynomial $Q_{i,jk}(x,y,z,w)$. So for a fixed i, we get mn quadrics, and since i ranges from 1 to mn, we get a square matrix of quadrics

$$M = (Q_{i,jk}(x,y,z,w)).$$

Here is the first main result of [**6**].

THEOREM 2.1. *If $\phi : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^3$ has no base points, is generically one-to-one, and MP has maximal rank, then the implicit equation of the surface $S \subset \mathbb{P}^3$ paramatrized by ϕ is*

$$F = \det(M),$$

where M is the matrix described above.

The Role of Commutative Algebra. We won't prove Theorem 2.1 in detail, but we will explain how commutative algebra is used in the argument. The proof begins by changing coordinates in $\mathbb{P}^3$ if necessary so that a,b,c have no base points. Then consider the matrix

$$MQ' : \underbrace{R^{9}_{m-1,n-1}}_{\dim\, 9mn} \xrightarrow{(a^2,ab,\ldots,cd)} \underbrace{R_{3m-1,3n-1}}_{\dim\, 9mn}$$

given by $a^2,ab,\ldots,cd$. If we can show that $\det(MQ') \neq 0$, then MQ will have maximal rank, which will in turn enable us to construct M. Furthermore, as explained in [**6**], $\det(MQ') \neq 0$ enables us to prove that $\det(M)$ is not identically 0. It follows that $\det(M)$ is a polynomial of degree $2mn$ and vanishes on the surface S (since the moving quadrics used to construct M all follow the parametrization). This proves that $\det(M) = 0$ is the implicit equation of S.

Hence, to complete the proof, we only need to show that $\det(MQ') \neq 0$. Let us sketch two proofs.

First Proof of $\det(MQ') \neq 0$. Suppose we could prove that

$$\det(MQ') = \det(MP)^3 \,\mathrm{Res}(a,b,c). \tag{2.4}$$

We are assuming $\det(MP) \neq 0$, and $\mathrm{Res}(a,b,c) \neq 0$ since a,b,c have no base points. Then (2.4) immediately implies that $\det(MQ') \neq 0$. The formula (2.4) was conjectured in Goldman and Zhang in [**6**] and proved by D'Andrea in [**9**]. Thus (2.4) gives a very quick proof that $\det(MQ') \neq 0$.

Second Proof of $\det(MQ') \neq 0$. If $\det(MQ') = 0$, then the columns of MQ' are linearly dependent. This gives a relation of the form

$$Aa^2 + Bab + \cdots + Icd = 0,$$

where $A, \ldots, I$ have bidegree $(m-1, n-1)$. We can write this as

$$0 = (Aa + Bb + Cc + Dd)a + (Eb + Fc + Gd)b + (Hc + Id)c. \tag{2.5}$$

This is a syzygy on a, b, c of degree $(2m-1, 2n-1)$. I remember Ron Goldman asking me if (2.5) implies that

$$Hc + Id = -h_1 a - h_3 b \tag{2.6}$$

for some polynomials h_1 and h_3 of bidegree $(m-1, n-1)$. If this is true, then (2.6) gives a nontrivial syzygy of bidegree $(m-1, n-1)$ among a, b, c, d, which contradicts our assumption that MP has maximal rank.

So how do we prove that (2.5) implies (2.6)? The solution (which fortunately didn't take me six months to figure out) uses an object in commutative algebra known as the *Koszul complex*. The basic idea is that some obvious syzygies on a, b, c are given by

$$\begin{aligned} c \cdot a + 0 \cdot b + (-a) \cdot c &= 0 \\ b \cdot a + (-a) \cdot b + 0 \cdot c &= 0 \\ 0 \cdot a + c \cdot b + (-b) \cdot c &= 0. \end{aligned}$$

Furthermore, if we multiply the first equation by h_1, the second by h_2, and the third by h_3, then we get the *Koszul syzygy* $Aa + Bb + Cc = 0$, where

$$\begin{aligned} A &= \quad h_1 c + h_2 b \\ B &= -h_2 a + h_3 c \\ C &= -h_1 a - h_3 b. \end{aligned} \tag{2.7}$$

So a natural question is whether all syzygies on a, b, c are Koszul syzygies.

If we are in the triangular case and $a, b, c \in \mathbb{C}[s, t, u]$ are homogeneous of the same degree, then having no base points implies (by standard arguments in commutative algebra) that the entire Koszul complex is exact, so that in particular, all syzygies on a, b, c are Koszul. But in the tensor product case, even if $a, b, c \in \mathbb{C}[s, u; t, v]$ have no base points, it is no longer true that the Koszul complex is exact. In general, commutative algebra works best in the triangular case, where one deals with ordinary homogeneous polynomials.

The solution to this difficulty in the tensor product case is to realize that while the Koszul complex is not exact in all bidegrees, it is exact in some. This is proved using sheaf cohomology, and the result is that every syzygy of bidegree $(2m-1, 2n-1)$, including (2.5), is Koszul. Hence (2.6) follows, which completes the second proof of $\det(MQ') \neq 0$.

We should also mention that in [**9**], D'Andrea has generalized Theorem 2.1 to the case when ϕ is not generically one-to-one. More precisely, he proves that if a parametrization ϕ as in (2.1) has no base points and the matrix MP has maximal rank, then the implicit equation $F = 0$ of the surface satisfies

$$F^h = \det(M),$$

where h is the generic degree of ϕ and M is the matrix of Theorem 2.1.

Triangular Surfaces. We next describe briefly what happens when we switch from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$. This means that we have

$$\begin{aligned} &\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3 \\ &\phi(s,t,u) = \big(a(s,t,u), b(s,t,u), c(s,t,u), d(s,t,u)\big), \end{aligned} \tag{2.8}$$

where $a, b, c, d \in \mathbb{C}[s,t,u]$ are homogeneous of degree n and $\gcd(a,b,c,d) = 1$. As in the tensor product case, we will assume that ϕ has no base points. Then the image $S \subset \mathbb{P}^3$ is a surface defined by an equation

$$F = 0$$

of degree n^2. In this situation, one can define moving planes and quadrics that follow ϕ, and as in the curve case, the moving planes and quadrics of degree $n-1$ are those of interest. In this degree, we get matrices

$$MP : R_{n-1}^4 \xrightarrow{(a,b,c,d)} R_{2n-1} \tag{2.9}$$

and

$$MQ : R_{n-1}^{10} \xrightarrow{(a^2,ab,\dots,d^2)} R_{3n-1} \tag{2.10}$$

similar to (2.2) and (2.3) whose kernels give the moving planes and quadrics of degree $n-1$ that follow ϕ. One surprise is that in the triangular case, the kernel of MP has dimension at least n, which means that there are always at least n linearly independent moving planes of degree $n-1$ that follow ϕ.

In the case when MP and MQ both have maximal rank, we get

- n linearly independent moving planes of degree $n-1$.
- $(n^2+7n)/2$ linearly independent moving quadrics of degree $n-1$.

However, each moving plane gives four moving quadrics by multiplying by x, y, z, w. Thus, in the second bullet, we have

$$(n^2+7n)/2 - 4n = (n^2-n)/2$$

linearly independent moving quadrics of degree $n-1$ which don't come from moving planes. Using these moving quadrics, we can construct a $(n^2+n)/2 \times (n^2+n)/2$ matrix M built from

- n rows coming from moving planes of degree $n-1$.
- $(n^2-n)/2$ rows coming from moving quadrics of degree $n-1$.

Then we can describe the implicit equation $F = 0$ of the surface as follows.

THEOREM 2.2. *Assume ϕ as in* (2.8) *has no base points and has precisely n linearly independent moving planes of degree $n-1$ that follow ϕ. Then:*

$$F^h = \det(M),$$

where h is the generic degree of ϕ.

This is proved in [**6**] when ϕ is generically one-to-one. The general case is due to D'Andrea in [**9**].

Regularity. In the triangular case, Theorem 2.2 has an interesting relation to the regularity of the ideals $I = \langle a,b,c,d\rangle$ and $I^2 = \langle a^2, ab, \dots, d^2\rangle$ of $R = \mathbb{C}[s,t,u]$. Since we are assuming that a, b, c, d don't vanish simultaneously, it follows that both I and I^2 have no base points. As explained in the last section, this implies that regularity has the following meaning for these ideals:

$$\operatorname{reg}(I) = \text{the smallest integer } k_0 \text{ such that } I_k = R_k \text{ for } k \geq k_0$$
$$\operatorname{reg}(I^2) = \text{the smallest integer } k_0 \text{ such that } (I^2)_k = R_k \text{ for } k \geq k_0.$$

We can relate this to the proof of Theorem 2.2 as follows. Similar to what we did in Theorem 2.1, the key step of the proof is to show that

$$MP \text{ has maximal rank} \Rightarrow MQ \text{ has maximal rank}, \tag{2.11}$$

where MP and MQ are defined by (2.9) and (2.10).

Since the image of MP is I_{2n-1}, it follows that

$$MP \text{ has maximal rank} \iff I_{2n-1} = R_{2n-1} \iff 2n-1 \geq \operatorname{reg}(I).$$

Similarly, the image of MQ is $(I^2)_{3n-1}$, so that

$$MQ \text{ has maximal rank} \iff (I^2)_{3n-1} = R_{3n-1} \iff 3n-1 \geq \operatorname{reg}(I^2).$$

It follows that (2.11) is equivalent to the regularity result

$$\operatorname{reg}(I) \leq 2n-1 \Rightarrow \operatorname{reg}(I^2) \leq 3n-1.$$

when $I = \langle a, b, c, d\rangle$ has no base points. In general, one area of research in commutative algebra concerns how the regularity of an ideal relates to the regularity of its powers. See, for example, [**3**].

As already noted, Section 3 will explain what regularity means for triangular surfaces when base points are present. Then, in Section 5, we will use regularity results for such ideals to prove a version of Theorem 2.2 for triangular surfaces with certain kinds of base points.

A final comment is that the above discussion is special to the case of homogeneous ideals in $\mathbb{C}[s,t,u]$. What about the bihomogeneous ideal $I = \langle a,b,c,d\rangle \subset \mathbb{C}[s,u;t,v]$ that we get from a tensor product parametrization such as (2.1)? What does regularity mean in this case? The answer is that the study of regularity for $\mathbb{P}^1 \times \mathbb{P}^1$ is just beginning. Some preliminary results, such as the forthcoming work of Hoffman and Wang [**11**], indicate that regularity may be a useful tool in studying tensor product surfaces.

3. Base Points

Now suppose that a triangular parametrization

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$$
$$\phi(s,t,u) = \big(a(s,t,u), b(s,t,u), c(s,t,u), d(s,t,u)\big)$$

has base points. As in (2.8), we assume that $a,b,c,d \in \mathbb{C}[s,t,u]$ are homogeneous of the same degree and $\gcd(a,b,c,d) = 1$. Hence there are at most finitely many base points in $\mathbb{P}^2$ where a,b,c,d all vanish simultaneously. We will work with triangular surfaces since commutative algebra works best in this case.

The goal of this section is to explain why certain base points called *local complete intersections* are especially nice. We also discuss regularity and saturations.

Local Complete Intersection Base Points. Let $I \subset R = \mathbb{C}[s,t,u]$ and assume $\mathbf{V}(I) \subset \mathbb{P}^2$ is finite. Elements of $\mathbf{V}(I)$ are called the *base points* of I. Then I is a *local complete intersection* (LCI) if for every base point $p \in \mathbf{V}(I)$, I can be generated by two elements in a neighborhood of p.

EXAMPLE 3.1. The ideal $I = \langle s^2u, stu, t^2u, t^3\rangle$ has base points $(1,0,0), (0,0,1)$. This ideal has four generators, but if we work locally near the base points, then fewer generators are needed:

$$\begin{aligned} s = 1 \quad &\Rightarrow \quad I = \langle u, tu, t^2u, t^3\rangle = \langle u, t^3\rangle \text{ near } (1,0,0)\\ u = 1 \quad &\Rightarrow \quad I = \langle s^2, st, t^2, t^3\rangle = \langle s^2, st, t^2\rangle \text{ near } (0,0,1). \end{aligned}$$

Thus I can be generated by two elements in a neighborhood of $(1,0,0)$, so that this base point is LCI. However, $(0,0,1)$ is *not* LCI because I is minimally generated by three elements near this base point. Since not all base points are LCI, we see that I is not LCI. □

Syzygies of LCI Base Points. Suppose that $a, b, c \in R = \mathbb{C}[s,t,u]$ are homogeneous of degree n with $\gcd(a,b,c) = 1$, and suppose that $I = \langle a, b, c\rangle$ has base points (necessarily finite in number). What can we say about syzygies in this situation? As in (2.7), a *Kozsul syzygy* is a syzygy of the form

$$\begin{aligned} A &= h_1c + h_2b\\ B &= -h_2a + h_3c\\ C &= -h_1a - h_3b. \end{aligned}$$

The observation is that a Koszul syzygy vanishes at the base points (because a, b, c vanish at the base points by definition). This leads to the question:

(3.1) Is every syzygy vanishing at the base points a Koszul syzygy?

To see the relevance of this question, observe that in (2.5), we had the syzygy

(3.2) $$0 = (Aa + Bb + Cc + Dd)a + (Eb + Fc + Gd)b + (Hc + Id)c.$$

Assuming that a, b, c, d and a, b, c have the same base points, it follows that this syzygy vanishes at the base points. Hence, if the answer to (3.1) is "yes", then (3.2) is a Koszul syzygy, which as in (2.6) gives the equation

$$Hc + Id = -h_1a - h_3b$$

used in the proof of Theorem 2.1. So if we want to adapt the proof to the case when base points are present, then question (3.1) arises naturally.

Because of multiplicities, we need a careful definition of what it means to vanish at the basepoints.

DEFINITION 3.2. *A syzygy $Aa + Bb + Cc = 0$* ***vanishes at the basepoints*** *of $I = \langle a, b, c\rangle$ if A, B, C locally lie in I.*

In Section 4, we will explain how the phrase "locally lie in" is related to the *saturation* of the ideal I.

It is easy to show that any Koszul syzygy vanishes at the basepoints. Then (3.1) asks if this necessary condition is also sufficient. Here is a result proved in [**5**].

THEOREM 3.3. *Let $I = \langle a, b, c\rangle \subset R = \mathbb{C}[s,t,u]$ with $\mathbf{V}(I) \subset \mathbb{P}^2$ finite. Then the following are equivalent:*

(1) *I is LCI.*
(2) *Every syzygy of a, b, c that vanishes at the base points of I is Koszul.*

For me, the interesting feature of this theorem is that it is a result in pure commutative algebra yet its underlying idea was suggested by questions raised by geometric modelers.

Multiplicities and LCI Base Points. Base points of parametrized surfaces are interesting because of their effect on the degree of the surface. For example, suppose that

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$$

is given by homogeneous polynomials $a, b, c, d \in R = \mathbb{C}[s,t,u]$ of degree n with $\gcd(a,b,c,d) = 1$. If ϕ is generically one-to-one, then its image $S \subset \mathbb{P}^3$ has degree

$$n^2 - \sum_{p \in \mathbf{V}(I)} e(I,p),$$

where $e(I,p)$ is the *multiplicity* of I at p. Recall that $e(I,p)$ is defined as follows. One localizes I at p to get an ideal I_p in the local ring R_p (see Chapter 4 of [**7**] for a discussion of local rings). Then

$$e(I,p) = \dim R_p/\langle f, g\rangle$$

for generic linear combinations f, g of the generators of I_p.

Another important invariant of $I_p \subset R_p$ is its *degree*, which is defined to be

$$\deg(I,p) = \dim R_p/I_p. \tag{3.3}$$

Since $\langle f, g\rangle \subset I_p$, it follows that we always have the inequality

$$e(I,p) \geq \deg(I,p). \tag{3.4}$$

Here is an example to show that the inequality can be strict.

EXAMPLE 3.4. Suppose that $I = \langle s^2, st, t^2\rangle \subset \mathbb{C}[s,t,u]$. The only base point is $p = (0,0,1)$, and localizing at p is (essentially) done by setting $u = 1$. This gives $R_p/I_p \simeq \mathbb{C}[s,t]/\langle s^2, st, t^2\rangle$. A basis of R_p/I_p is given by $1, s, t$, so that

$$\deg(I,p) = 3.$$

To compute $e(I,p)$, one can show that we can use $f = s^2$ and $g = t^2$ in this case. Then a basis of $R_p/\langle f,g\rangle = R_p/\langle s^2, t^2\rangle$ is given by $1, s, t, st$, so that

$$e(I,p) = 4.$$

Thus $e(I,p) > \deg(I,p)$ in this case. □

The interesting observation is that when we compute $e(I,p)$ as the dimension of $R_p/\langle f,g\rangle$, the ideal $\langle f,g\rangle$ is LCI at p since it is generated by 2 elements. So the multiplicity is computed using the best approximation of I_p by an ideal which is LCI at p. This means that whenever you consider the multiplicity of a base point, there is an LCI ideal lurking in the background.

In particular, if I is LCI, then we can let $\langle f,g\rangle = I_p$ for each p, so that

$$e(I,p) = \deg(I,p) \quad \text{for all } p \in \mathbf{V}(I). \tag{3.5}$$

In fact, I is LCI if and only if (3.5) is true.

Moving Planes and LCI Base Points. We next discuss how base points affect the number of moving planes that follow the parametrization. The goal is to show that LCI base points arise naturally when one tries naively to extend Theorem 2.2 to the case when base points are present.

Assume that we have $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ given by a, b, c, d of degree n. When ϕ has no base points, Theorem 2.2 computes the implicit equation of the image of ϕ using a matrix M with

$$\begin{gathered} n \text{ rows coming from moving planes of degree } n-1 \\ (n^2-n)/2 \text{ rows coming from moving quadrics of degree } n-1. \end{gathered}$$

Now introduce base points. The hope is that when a base point drops the implicit degree by 1, one row of M should switch from quadratic to linear. Thus each base point should give a new moving plane that follows ϕ.

To make this intuition more precise, we need to study carefully how base points affect the number of moving planes that follow ϕ. Given an integer ℓ, the moving planes of degree ℓ are given by the kernel of the matrix

$$MP_\ell : R_\ell^4 \xrightarrow{(a,b,c,d)} R_{\ell+n}.$$

(hence MP_{n-1} is the matrix MP of (2.9)). The image of this map is $I_{\ell+n}$, so that the number of moving planes of degree ℓ that follow ϕ is determined by the size $I_{\ell+n}$. A Hilbert polynomial calculation implies that

$$\dim R_{\ell+n} = \dim I_{\ell+n} + \deg(I), \tag{3.6}$$

for $\ell \gg 0$, where

$$\deg(I) = \sum_{p \in \mathbf{V}(I)} \deg(I, p)$$

is the *degree* of I and $\deg(I,p) = \dim R_p/I_p$ is from (3.3).

Now suppose that (3.6) holds when $\ell = n-1$. We will see in the next section that this is equivalent to assuming that $\operatorname{reg}(I) \leq 2n-1$. Then we have the following.

PROPOSITION 3.5. *Equation* (3.6) *holds when* $\ell = n-1$ *if and only if*

$$\dim \operatorname{Syz}(a,b,c,d)_{n-1} = n + \deg(I).$$

PROOF. Since MP_{n-1} has kernel $\operatorname{Syz}(a,b,c,d)_{n-1}$ and image I_{2n-1}, we have

$$\dim \operatorname{Syz}(a,b,c,d)_{n-1} = 4\dim R_{n-1} - \dim I_{2n-1}.$$

If (3.6) holds with $\ell = n-1$, then $\dim R_{2n-1} = \dim I_{2n-1} + \deg(I)$. Thus

$$\dim \operatorname{Syz}(a,b,c,d)_{n-1} = 4\dim R_{n-1} - (\dim R_{2n-1} - \deg(I)) = n + \deg(I),$$

where we have used $\dim R_k = \binom{k+2}{2}$. The converse is equally easy. $\square$

It follows that if (3.6) holds for $\ell = n-1$, then the number of moving planes that follow ϕ increases from n to $n + \deg(I)$. If we use all of these moving planes to construct a new matrix M, then $\deg(I)$ quadratic rows shift to linear rows, so that the degree of $\det(M)$ will drop by

$$\deg(I) = \sum_{p \in \mathbf{V}(I)} \deg(I, p).$$

However, the degree of the implicit equation drops by the sum of the multiplicities

$$\sum_{p \in \mathbf{V}(I)} e(I, p).$$

It follows that if $\det(M) = 0$ is to be the implicit equation of the surface, then these drops must match, i.e., we must have

$$\sum_{p\in \mathbf{V}(I)} e(I,p) = \sum_{p\in \mathbf{V}(I)} \deg(I,p).$$

By (3.4), this means

$$e(I,p) = \deg(I,p) \quad \text{for all } p \in \mathbf{V}(I),$$

which by (3.5) happens if and only if I is LCI. So our naive strategy of extending Theorem 2.2 can only hope to succeed when the base points are LCI! As we will explain in Section 5, this strategy can be made rigorous in certain cases. But first, we need to learn about regularity and saturation.

4. Saturation and Regularity

In Section 3, we noted that (3.6) is true for ℓ sufficiently large. Here, we will show that the meaning of "sufficiently large" is closed related to the regularity of the ideal I. But before we can understand this, we need to discuss saturation.

Saturation. Given a homogeneous ideal $I \subset R = \mathbb{C}[s,t,u]$ with any number of generators, its *saturation* is defined to be

$$\mathrm{sat}(I) = \{f \in R \mid \text{there is } k \geq 0 \text{ such that } s^k f, t^k f, u^k f \in I\}.$$

One can show that $\mathrm{sat}(I)$ is a homogeneous ideal of R. Furthermore:

- $I \subset \mathrm{sat}(I)$. This follows by using $k = 0$ in the above definition. Below we will give an example to show that $\mathrm{sat}(I)$ can be strictly bigger than I.
- I and $\mathrm{sat}(I)$ give the same ideal on every affine piece of $\mathbb{P}^2$. For example, suppose we dehomogenize by setting $s = 1$. If $f \in \mathrm{sat}(I)$, then $s^k f \in I$ for some $k \geq 0$. Since $s^k f$ and f have the same dehomogenization when $s = 1$, it follows that I and $\mathrm{sat}(I)$ dehomogenize to the same ideal in $\mathbb{C}[t,u]$. The affine pieces where $t = 1$ and $u = 1$ are handled similarly.
- It follows that I and $\mathrm{sat}(I)$ have the same base points which have the same degree and the same multiplicity. (In more technical language, I and $\mathrm{sat}(I)$ define the same *subscheme* of $\mathbb{P}^2$.)

One important observation is that $\mathrm{sat}(I)$ is the *largest* ideal that gives the same ideal as I on every affine piece of $\mathbb{P}^2$. This follows from the above definition and justifies the statement that $\mathrm{sat}(I)$ consists of all polynomials that locally are in I.

Here is an example of a saturation.

EXAMPLE 4.1. Let $I = \langle s^5, t^5, su^4, st^2u^2\rangle \subset R = \mathbb{C}[s,t,u]$. This is generated by polynomials of degree 5. Note that

$$\begin{aligned} s \cdot s^5 &\in I \\ s \cdot u^5 = u \cdot su^4 &\in I \\ s \cdot t^5 &\in I. \end{aligned}$$

It follows that $s \in \mathrm{sat}(I)$, so that $\mathrm{sat}(I)$ is strictly bigger than I. Using the `saturate` command of Macaulay 2 [**10**], one can show that

$$\mathrm{sat}(I) = \langle s, t^5\rangle.$$

The ideal $\langle s, t^5\rangle$ clearly has the single base point $(0,0,1)$ which is LCI of degree 5 and multiplicity 5. By the third of the above bullets, the same is true for I. □

Conditions Imposed by Base Points. We next interpret saturation in terms of conditions imposed by base points. To see what this phrase means, consider

$$R_k \longrightarrow \bigoplus_{p\in \mathbf{V}(I)} R_p/I_p, \tag{4.1}$$

where $I_p \subset R_p$ is as in Section 3. This map is given by dehomogenization followed by the map to the quotient ring. Note that the dimension of the right-hand side is $\sum_{p\in\mathbf{V}(I)} \dim R_p/I_p = \sum_{p\in\mathbf{V}(I)} \deg(I,p) = \deg(I)$. This is the degree of I defined in Section 3

The phrase "conditions imposed by base points" refers to the kernel of (4.1) and thus describes those polynomials of degree k which locally belong to I. But this is the saturation! Hence we have the following result.

PROPOSITION 4.2. *The conditions imposed in degree k by the base points of I describe* $\mathrm{sat}(I)_k$. *Thus*

$$I_k = \mathrm{sat}(I)_k \iff \begin{array}{l}\textit{the conditions imposed in degree } k \textit{ by}\\ \textit{the basepoints of } I \textit{ describe } I_k \textit{ exactly.}\end{array}$$

The next question to ask is whether the base point conditions are independent. This leads to the following definition.

DEFINITION 4.3. *We say that the conditions imposed by the base points of I are* ***independent in degree*** *k if the map* (4.1) *is onto.*

We then get the following basic result.

PROPOSITION 4.4. *The conditions imposed by the base points of I are independent in degree k if and only if*

$$\dim \mathrm{sat}(I)_k + \deg(I) = \dim R_k.$$

PROOF. Let $W \subset \bigoplus_{p\in\mathbf{V}(I)} R_p/I_p$ be the image of (4.1). Since the kernel is $\mathrm{sat}(I)_k$, the dimension theorem from linear algebra implies that

$$\dim \mathrm{sat}(I)_k + \dim W = \dim R_k.$$

The result follows immediately since $\deg(I) = \dim \bigoplus_{p\in\mathbf{V}(I)} R_p/I_p$. □

Here is an example.

EXAMPLE 4.5. Consider $I = \langle s^2u, stu, t^2u, t^3\rangle \subset \mathbb{C}[s,t,u]$ from Example 3.1. In this case, the map (4.1) becomes

$$R_k \longrightarrow \mathbb{C}[t,u]/\langle u,t^3\rangle \oplus \mathbb{C}[s,t]/\langle s^2, st, t^2\rangle,$$

where $f \in R_k$ is sent to $(f(1,t,u), f(s,t,1))$. Notice also that $\deg(I) = 6$. Since R_1 has dimension 3, it follows that the base points do not impose independent conditions in degree 1. However, one can easily check that the base point conditions are independent in degrees 2 and higher. For later purposes, we note that I is saturated, i.e., $I = \mathrm{sat}(I)$. This is easily checked using Macaulay 2. □

Regularity. We are now ready to discuss regularity. We begin with the special case of a homogeneous ideal $I \subset R$ with no base points. In this case, $\mathrm{sat}(I) = R$ since an empty base point locus means that locally I generates the whole ring. But now recall that when I has no base points, the regularity $\mathrm{reg}(I)$ of I is the smallest integer such that $I_k = R_k$ for $k \geq \mathrm{reg}(I)$. Since $R_k = \mathrm{sat}(I)_k$, we get the following nice result.

PROPOSITION 4.6 (Regularity without base points). *Let $I \subset \mathbb{C}[s,t,u]$ be a homogenous ideal with no base points. Then, for any integer $k \geq 0$, we have*

$$\operatorname{reg}(I) \leq k \iff I_k = R_k \iff I_k = \operatorname{sat}(I)_k.$$

One useful consequence of this proposition is that if we know a single degree k such that $I_k = R_k$, then we automatically have $I_\ell = R_\ell$ for all $\ell \geq k$.

Now let $I \subset R$ be a homogeneous ideal with finitely many base points. What does regularity mean in this case? The definition of regularity given in [**1**] can be stated in various ways using either sheaf cohomology or minimal free resolutions. Fortunately, for the ideals of interest to us, regularity can be formulated as follows.

PROPOSITION 4.7 (Regularity with base points). *Let $I \subset \mathbb{C}[s,t,u]$ be a homogenous ideal with a finite positive number of base points. Then, for any integer $k \geq 0$, we have*

$$\begin{aligned}\operatorname{reg}(I) \leq k &\iff I_k = \operatorname{sat}(I)_k \text{ and } \dim \operatorname{sat}(I)_{k-1} + \deg(I) = \dim R_{k-1}\\ &\iff \textit{the conditions imposed by the base points of } I \textit{ describe}\\ &\qquad I_k \textit{ in degree } k \textit{ and are independent in degree } k-1.\end{aligned}$$

The second equivalance of the proposition uses Propositions 4.2 and 4.4. One consequence is that if we know one degree k where the conditions imposed by the base points of I describe I_k and are independent in degree $k-1$, then the same is true for all larger degrees. Here is an example of Proposition 4.7.

EXAMPLE 4.8. Consider $I = \langle s^2u, stu, t^2u, t^3\rangle \subset \mathbb{C}[s,t,u]$. In Example 4.5, we noted that this ideal was saturated and that the base point conditions were independent in degrees 2 and higher. By Proposition 4.7, it follows that $\operatorname{reg}(I) = 3$. This can be confirmed using the `regularity` command in Macaulay 2. □

When dealing with a triangular surface with base points, Proposition 4.7 can be simplified as follows.

THEOREM 4.9 (Regularity for triangular surfaces with base points). *Consider an ideal $I = \langle a,b,c,d\rangle \subset \mathbb{C}[s,t,u]$, where a,b,c,d have degree n and* $\gcd(a,b,c,d) = 1$. *Also assume that $n \geq 2$ and that a,b,c,d are linearly independent. Then, if $k \geq 2n-2$, we have*

$$\begin{aligned}\operatorname{reg}(I) \leq k &\iff \dim I_k + \deg(I) = \dim R_k\\ &\iff \textit{the conditions imposed by the basepoints of } I \textit{ are}\\ &\qquad \textit{independent in degree } k \textit{ and describe } I_k \textit{ exactly.}\end{aligned}$$

This is proved in Appendix B of [**2**]. Here is a corollary of Theorem 4.9 relevant to the discussion at the end of Section 3.

COROLLARY 4.10. *Let I be as in Theorem 4.9. Then:*

$$\begin{aligned}\operatorname{reg}(I) \leq 2n-1 &\iff \dim I_{2n-1} + \deg(I) = \dim R_{2n-1}\\ &\iff (3.6) \textit{ holds for } \ell = n-1\\ &\iff \dim \operatorname{Syz}(a,b,c,d)_{n-1} = n + \deg(I).\end{aligned}$$

PROOF. Set $k = 2n-1$ in Theorem 4.9 and use Proposition 3.5. □

Thus the naive idea of extending Theorem 2.2 discussed in Section 3 leads naturally to the notion of regularity. We will say more about this in Section 5.

The intuition behind regularity is that I_k "behaves nicely" when $k \geq \operatorname{reg}(I)$. We now have a better idea of what this means!

5. Surfaces with Base Points

In this final section we will show that in certain cases, the methods of Section 2 can be extended to the triangular surface case when base points are present. Details can be found in [**2**].

Base Point Conditions. Let $I = \langle a, b, c, d \rangle \subset R = \mathbb{C}[s, t, u]$ give the rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$. Assume the following *base point conditions*:

BP1: a, b, c, d are homogeneous of degree n and linearly independent over $\mathbb{C}$.
BP2: $\gcd(a, b, c, d) = 1$ and I is LCI.
BP3: $\dim \mathrm{Syz}(a, b, c, d)_{n-1} = n + \deg(I)$.
BP4: $d \in \mathrm{sat}(\langle a, b, c \rangle)$.
BP5: $\mathrm{Syz}(a, b, c)_{n-1} = \{0\}$.

These conditions give the following result proved in [**2**].

Theorem 5.1. *Assume the base point conditions BP1–BP5. Then there is a $(n^2 + n)/2 \times (n^2 + n)/2$ matrix M with*

$$n + \deg(I) \textit{ rows coming from moving planes of degree } n - 1$$

$$(n^2 - n)/2 - \deg(I) \textit{ rows coming from moving quadrics of degree } n - 1,$$

such that the implicit equation $F = 0$ of the image of ϕ satisfies

$$F^h = \det(M),$$

where h is the generic degree of ϕ.

Rather than give the proof, we will instead explain what the five base point conditions mean.

Explain BP1. This is fairly obvious. Notice that if a, b, c, d are linearly dependent, then the image of ϕ lies in a plane.

Explain BP2. The gcd condition implies that $\mathbf{V}(I)$ is finite. To see why we need LCI base points, note that the determinant of the matrix M of Theorem 5.1 has degree $n^2 - \deg(I)$, while F^h has degree $n^2 - \sum_{p \in \mathbf{V}(I)} e(I, p)$. These are equal since the base points are LCI.

Explain BP3. This condition is needed to ensure that M has the correct number of linear rows. Also, by Corollary 4.10, BP3 implies that $\mathrm{reg}(I) \le 2n - 1$. In the proof of Theorem 5.1, we use $\mathrm{reg}(I)$, together with a result of Chandler [**3**], to bound the regularity of I^2. This is needed to understand how the base points affect the number of moving quadrics that follow ϕ.

Explain BP4. We need $d \in \mathrm{sat}(\langle a, b, c \rangle)$ so that I and $\langle a, b, c \rangle$ have the same LCI base points. Then we can use Theorem 3.3 to show that (2.5) implies (2.6) just as in the proof of Theorem 2.1. We can always arrange $d \in \mathrm{sat}(\langle a, b, c \rangle)$ by a suitable change of coordinates on $\mathbb{P}^3$. So this assumption is harmless.

Explain BP5. An important step in the proof is to show that $\det(M)$ doesn't vanish identically. This is done by showing that the coefficient of $w^{n^2 - \deg(I)}$ is nonzero. However, a syzygy of degree $n - 1$ on a, b, c gives a row of M with no w, which means that $w^{n^2 - \deg(I)}$ does not appear $\det(M)$. Hence this messes up the proof that $\det(M)$ is nonzero. One can sometimes avoid this problem by changing coordinates on $\mathbb{P}^3$. However, there are examples where $\mathrm{Syz}(a, b, c)_{n-1} \neq \{0\}$ no matter which coordinates we use on $\mathbb{P}^3$. Hence we are stuck with this assumption.

Here are two examples of Theorem 5.1 in action.

EXAMPLE 5.2. Let ϕ be given by $a = s^3$, $b = t^2u$, $c = s^2t + u^3$, and $d = stu$. One can check that the base point conditions are satisfied, and the only base point is $(0, 1, 0)$, which is LCI of multiplicity 2. Since $n = 3$ and $\deg(I) = 2$, we have $(n^2 + n)/2 = 6$. Thus M is a 6×6 matrix (5 linear rows, one quadratic row) and $\det(M)$ has degree $n^2 - \deg(I) = 7$.

One can compute that the five moving planes of degree 2 that follow ϕ are

$$\begin{aligned} P_1 &= s^2\, w - tu\, x \\ P_2 &= st\, w - s^2\, y \\ P_3 &= t^2\, w - st\, y \\ P_4 &= tu\, w - su\, y \\ P_5 &= u^2\, w + t^2\, x - st\, z, \end{aligned}$$

and that the single moving quadric that follows ϕ is

$$Q = su\, w^2 - u^2\, xy.$$

If we let the columns of M correspond to the coefficients of $s^2, st, su, t^2, tu, u^2$, then we get the matrix

$$M = \begin{pmatrix} w & 0 & 0 & 0 & -x & 0 \\ -y & w & 0 & 0 & 0 & 0 \\ 0 & -z & 0 & x & 0 & w \\ 0 & -y & 0 & w & 0 & 0 \\ 0 & 0 & -y & 0 & w & 0 \\ 0 & 0 & w^2 & 0 & 0 & -xy \end{pmatrix}.$$

Thus $\det(M) = w^7 - x^2y^3zw + x^3y^4$ is the implicit equation of the surface. □

EXAMPLE 5.3. Now suppose that $I = \langle s^5, t^5, su^4, st^2u^2 \rangle \subset R = \mathbb{C}[s, t, u]$. In Example 4.1, we saw that I has a single LCI base point of multiplicity 5. Thus the corresponding surface in $\mathbb{P}^3$ has degree $5^2 - 5 = 20$. Also notice that $(n^2+n)/2 = 15$, meaning that M needs to be a 15×15 matrix.

However, using Macaulay 2, the regularity of I is 10, yet condition BP3 means $\operatorname{reg}(I) \le 2 \cdot 5 - 1 = 9$. In fact, one can compute that

$$\dim \operatorname{Syz}(a, b, c, d)_4 = 11 \neq 10 = n + \deg(I).$$

If we use all moving planes of degree 4 in M, we get 11 linear rows and 4 quadratic rows, so that $\det(M)$ has degree 19. This is clearly wrong and shows that the method fails in this case. □

A version of Theorem 5.1 should hold for $\mathbb{P}^1 \times \mathbb{P}^1$. We don't know how to prove this since we don't have a good theory of regularity (though this may change once [**11**] appears).

Smaller Matrices. There is a version of Theorem 5.1 that uses $n - 2$ in place of $n - 1$, with a matrix of size $(n^2 - n)/2 \times (n^2 - n)/2$. The idea is that one uses slightly different base point conditions. More precisely, the third and fifth base point conditions are modified as follows:

BP3: $\dim \operatorname{Syz}(a, b, c, d)_{n-2} = \deg(I) - n$.
BP5: $\operatorname{Syz}(a, b, c)_{n-2} = \{0\}$.

In this case, the base point conditions imply that

$$F^h = \det(M),$$

where M is a $(n^2-n)/2 \times (n^2-n)/2$ matrix M with

$$\begin{aligned} \deg(I) - n &\text{ rows coming from moving planes of degree } n-2\\ (n^2+n)/2 - \deg(I) &\text{ rows coming from moving quadrics of degree } n-2. \end{aligned}$$

Here is an example of this result.

Example 5.4. Consider the following parametrization (taken from [**12**]) of a cubic surface with 6 base points:

$$\begin{aligned} a &= s^2t + 2t^3 + s^2u + 4stu + 4t^2u + 3su^2 + 2tu^2 + 2u^3\\ b &= -s^3 - 2st^2 - 2s^2u - stu + su^2 - 2tu^2 + 2u^3\\ c &= -s^3 - 2s^2t - 3st^2 - 3s^2u - 3stu + 2t^2u - 2su^2 - 2tu^2\\ d &= s^3 + s^2t + t^3 + s^2u + t^2u - su^2 - tu^2 - u^3. \end{aligned}$$

One can check with Macaulay 2 that I is saturated and LCI of degree 6, and its regularity is 3. As shown in [**12**], we have the following basis of syzygies of degree $n-2=1$:

$$\begin{aligned} &s(z-y) + t(-x+2w) + u(x-y)\\ &s(x+w) + t(2y-z) + u(y+2w)\\ &s\,x + t\,y + u\,z. \end{aligned} \tag{5.1}$$

The third syzygy shows that $\mathrm{Syz}(a,b,c)_1 \neq 0$, so that BP5 is not verified. But if we consider a,b,d instead, then it is straightforward to check that all base point conditions are satisfied.

In this case, M has $\deg(I) - n = 3$ linear rows and $(n^2+n)/2 - \deg(I) = 0$ quadratic rows. Thus the above syzygies give the matrix

$$M = \begin{pmatrix} z-y & -x+2w & x-y\\ -x-w & z-2y & -y-2w\\ x & y & z \end{pmatrix}.$$

The determinant of M is computed in [**12**] and is the implicit equation of the surface. □

Special μ-Bases for Surfaces. Given a rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$, it can happen that $I = \langle a,b,c,d\rangle$ is saturated. We saw an example of this in Example 4.5. While this is a relatively rare phenomenon, it does have some nice consequences. For example, according to [**4**], we know that

$$I = \langle a,b,c,d\rangle \text{ is saturated} \iff \mathrm{Syz}(a,b,c,d) \text{ is a free module.}$$

In this situation, one can prove that $\mathrm{Syz}(a,b,c,d)$ has free generators p,q,r of degrees μ_1,μ_2,μ_3 such that

$$\mu_1 + \mu_2 + \mu_3 = n,$$

where, as usual, n is the degree of a,b,c,d. In analogy with the curve case discussed in Section 1, we call p,q,r a *special μ-basis* (the adjective "special" refers to the fact that most of the time I isn't saturated, in which case p,q,r don't exist).

In the curve case, the implicit equation is (up to the generic degree) given by the resultant of the μ-basis. Is the same true when a special μ-basis exists? The answer is "not always" (see [**2**] for examples). However, when the base points are LCI, then things work out nicely. Here is the precise result from [**2**].

THEOREM 5.5. *Assume that $I = \langle a, b, c, d\rangle$ satisfies BP1 and BP2 (so the base points are LCI). If I is saturated and p, q, r is a special μ-basis, then*

$$F^h = \mathrm{Res}(p, q, r),$$

where h is the generic degree of ϕ.

We can explain the necessity for LCI base points as follows. Since p, q, r have degrees μ_1, μ_2, μ_3, the theory of multivariable resultants (see [**7**]) implies that

$$\deg(\mathrm{Res}(p, q, r)) = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3.$$

One the other hand, it is shown in [**4**] that I has degree

$$\deg(I) = n^2 - (\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3).$$

It follows that

$$\deg(\mathrm{Res}(p, q, r)) = n^2 - \deg(I) = n^2 - \sum_{p\in\mathbf{V}(I)} \deg(I, p).$$

Since

$$\deg(F^h) = n^2 - \sum_{p\in\mathbf{V}(I)} e(I, p),$$

we see that Theorem 5.5 holds only if the degree equals the sum of the multiplicities, which by Section 3 happens only when the base points are LCI. Hence we get another example where LCI base points have an interesting role to play.

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