Root counts of semi-mixed systems, and an application to counting Nash equilibria *

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ABSTRACT

Semi-mixed algebraic systems are those where the equations can be partitioned into subsets with common Newton polytopes. We are interested in counting roots of semi-mixed multihomogeneous systems, where both variables and equations can be partitioned into blocks, and each block of equations has a given degree in each block of variables. Our motivation is counting the number of totally mixed Nash equilibria in games of several players. We observe that MacMahon's Master theorem can be applied to the multihomogeneous Bézout bound for such systems. Even for more general systems, we obtain a generating function for the maximal number of common roots.

The main contributions of this paper are in relating and unifying the BKK and multivariate Bézout bounds for semi-mixed systems, through mixed volumes and permanents. In particular, we show that certain BKK bounds are directly obtained as a matrix permanent, which offers a faster computation and better approximation. This holds for all multihomogeneous systems, without any requirement of semi-mixed structure, as well as arbitrary systems whose Newton polytopes are products of polytopes in complementary subspaces. The complexities of computing permanents and of computing terms of generating functions are juxtaposed to that of a combinatorial geometric algorithm for semi-mixed volumes, by means of a novel asymptotic analysis of the latter.

Keywords

Multihomogeneous Bézout bound, mixed volume, matrix permanent, generating function, Nash equilibrium

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1. INTRODUCTION

Multihomogeneous polynomial systems are algebraic systems on a product

$$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_S} \tag{1}$$

of projective spaces. The variables are partitioned into S subsets, or blocks, so that each equation is homogeneous of given degree in each block of $n_j + 1$ homogeneous variables parameterizing the projective spaces $\mathbb{P}^{n_j}$, $j \in \{1, \ldots, S\}$. In contrast, *semi-mixed* algebraic systems are those where some equations have the same structure of involved monomials. In the context of sparse systems, the equations can be partitioned into subsets with the same Newton polytope, e.g. [4, 10, 23]. In the case of semi-mixed multihomogeneous systems, the equations in each block are homogeneous of the same degree per block of variables.

An important example of a semi-mixed multihomogeneous system appears in game theory. Consider a game of S players, each with $m_1, \ldots, m_S$ options respectively. The jth player plays a *totally mixed strategy* when he chooses his option randomly with nonzero probabilities $p_1^{(j)}, \ldots, p_{m_j}^{(j)}$. A *totally mixed Nash equilibrium* (TMNE) is a combination of players' totally mixed strategies such that no player can improve his payoff by unilaterally choosing other (pure or mixed) strategy. The equation system for the TMNE is

$$P_1 = \sum_{k_2,\ldots,k_S} a^{(1)}_{i,k_2,\ldots,k_S}\, p^{(2)}_{k_2} \cdots p^{(S)}_{k_S}, \quad 1 \leqslant i \leqslant m_1, \tag{2}$$

$$P_2 = \sum_{k_1,\ldots,k_S} a^{(2)}_{k_1,i,\ldots,k_S}\, p^{(1)}_{k_1} \cdots p^{(S)}_{k_S}, \quad 1 \leqslant i \leqslant m_2, \tag{3}$$

$$\cdots$$

$$P_S = \sum_{k_1,\ldots,k_{S-1}} a^{(S)}_{k_1,\ldots,k_{S-1},i}\, p^{(1)}_{k_1} \cdots p^{(S-1)}_{k_{S-1}}, \; 1 \leqslant i \leqslant m_S \tag{4}$$

Here $a^{(j)}_{k_1,k_2,\ldots,k_S}$ denotes the pre-defined payoff of the jth player, under the scenario when each player chooses the pure option $k_\ell \in \{1, \ldots, m_\ell\}$, resp. The equations imply that the payoff P_j of each player j does not depend on his own strategy i, as long as other players do not change their strategies. Eliminating the P_j's leads to a multihomogeneous (multilinear, actually) system with $n_j = m_j - 1$ in (1). Actual probabilities are determined from the normalizing conditions

$$p_1^{(j)} + p_2^{(j)} + \ldots + p_{m_j}^{(j)} = 1 \qquad \text{for } j \in \{1, 2, \ldots, S\}. \tag{5}$$

We are interested only in the real solutions that yield the numbers $p_i^{(j)} \in (0, 1] \subset \mathbb{R}$ after the normalization.

The generic number of complex solutions of algebraic systems is the Bernstein-Khovanski-Koushnirenko (BKK) bound [1]. If the equation system is dense with respect to some multihomogeneous structure, this number coincides with the *multihomogeneous Bézout bound*, or *m-Bézout bound*, see § 3.1. For sparse systems, the BKK bound may be tighter and, in this sense, it generalizes the classic Bézout bounds. This paper relates characterization and computation of the BKK and m-Bézout bounds.

Furthermore, we relate these root counts to matrix permanents: for the system (2)–(4), the BKK bound was essentially related to the permanent [14, thm 2] More generally, we show that, for any multihomogeneous system, the m-Bézout (and the equal BKK bound) reduces to a matrix permanent, with no requirement on semi-mixed structure. A major result is that the reduction of BKK to permanents can be generalized to systems that are not multihomogeneous, provided their Newton polytopes are products of polytopes in complementary subspaces. The reduction to the permanent offers faster ways of computing root bounds and better accuracy in approximating them.

The BKK bound for the system (2)–(4) was established in [15], in terms of a combinatorial count. It is automatically an upper bound for the number of proper real solutions that give TMNE (when this number is finite). The upper bound is sharp, as [15, § 4] established families of games with as many TMNE. In [19], it is observed that MacMahon's Master theorem [13] can be applied to a standard expression of the m-Bézout bound for multilinear system (2)–(4), where the number of equation blocks equals the number S of variable blocks.

With computational aspects in mind, this paper generalizes the results in [15, 19] to broader algebraic systems. In particular, we start with semi-mixed multihomogeneous systems where the number of equation blocks equals the number S of variable blocks as in (1), and we have a varying number of equations per block; more importantly, the degree per block is arbitrary. The total number of equations equals the total number of (dehomogenized) variables, hence generically this system has a finite number of solutions. MacMahon's master theorem can be applied yielding a multivariate generating function for the m-Bézout bound of these systems. It is computed via a matrix of dimension S, instead of handling a matrix of dimension equal to that of the system.

Our approach applies to more general systems, thus yielding the generating function of the root count, which is important when we seek root counts for a family of systems. In fact, connection of MacMahon's master theorem to permanents is under active investigation [12], yet apparently it has not been applied to computation of root counts.

The following complexity question is raised: What are "threshold" families of algebraic systems whose BKK bound can be computed in polynomial time? The general problem of computing this bound is #P-hard by reduction of the permanent, e.g. [4, 9]. We juxtapose the various methods examined in this paper in terms of asymptotic complexity, and we also compare them to a general method, relying on mixed subdivisions, which computes semi-mixed volumes. Mixed subdivisions offer the fastest way to compute mixed volumes; for the first time, the method's complexity is analyzed for semi-mixed systems.

Algebraic systems of considered types already occurred in expressing conformation search in robotics and structural bioinformatics. In particular, cyclic mechanisms with 3 degrees of freedom [7] give rise to (sparse) multihomogeneous systems that fit the setting of Cor. 4.4.

Last but not least, a theoretical motivation comes from systems for which optimal determinantal formulae are known for the sparse resultant. These are precisely multihomogeneous systems where each Newton polytope is the product of standard unit simplices and segments, see [3, 6, 18, 20].

This paper is organized as follows. In § 2, we detail the problem of TMNE. We unify root counts in § 3, where we relate the m-Bézout bound, the BKK bound and mixed volume, and the permanent. Generating functions of root bounds are derived in § 4. Complexity issues are studied in § 5, including some open questions.

2. NASH EQUILIBRIA

Game theory offers mathematical modeling of strategic decision making. A key concept is that of *Nash equilibrium*: it is a combination of strategies of participating players, such that no player can improve his payoff by unilaterally changing his strategy. The strategies can be pure (when a player chooses a single option) or mixed (when a player makes a choice randomly, by assigning probabilities to his options). Totally mixed strategies play any available strategy with a nonzero probability.

Nash equilibria always exist, but it is not easy to compute or enumerate them even for games of two players [16]. The equations (2)–(3) for the TMNE are then linear, but one of the two subsystems is overdetermined generically if $m_1 \neq m_2$. If $m_1 = m_2$, there is at most one TMNE generically. But sharp bounds for the total number of Nash equilibria are not known when $\max(m_1, m_2) > 4$. A loose upper bound for the total number of Nash equilibria is

$$\sum_{\ell=1}^{\min(m_1,m_2)} \binom{m_1}{\ell}\binom{m_2}{\ell},$$

which counts pairs of subsets of pure strategies (to be played with nonzero probability) of equal size ℓ. If a Nash equilibrium is not totally mixed, we have inequalities $\geqslant$ instead of $=$ in (2), (3) for those i with $p_i^{(1)} = 0$, $p_i^{(2)} = 0$, respectively.

In games of $S > 2$ players, even the bound for the maximal number of TMNE (in terms of $m_1, \ldots, m_S$) is a non-trivial question. In [15] they answered it in the form of a combinatorial count, by counting certain partitions of

$$N = m_1 + \ldots + m_S - S \tag{6}$$

elements into S sets with $n_j = m_j - 1$ elements each. We have $N = n_1 + \ldots + n_S$.

The combinatorial count can be eloquently formulated as follows: Consider a card recreation of S players, each with $n_1, n_2, \ldots, n_S$ cards originally. All cards are shuffled together, and then each player j receives the same number n_j of cards as originally. Let $E(n_1, n_2, \ldots, n_S)$ denote the number of ways to deal the cards in such a way that no player receives a card that he held originally. The maximal number of TMNE in a generic (i.e., regular [15]) game with $m_1, m_2, \ldots, m_S$ pure options equals

$$E(m_1 - 1, m_2 - 1, \ldots, m_S - 1). \tag{7}$$

As established in [15], this number is the BKK bound for the system (2)–(4), hence the generic number of complex solutions. This is also a sharp upper bound for the number of TMNE (when this number is finite), as [15, § 4] gives a family of real payoffs $a^{(j)}_{k_1,k_2,\ldots,k_S}$ such that all complex (multihomogeneous) solutions normalize by (5) to proper real solutions representing TMNE. In [11], the systems whose roots represent TMNE are parametrized. In [2], system solving is applied to compute TMNE. In [5, Ch. 5] Sylvester-type resultant matrices are computed for such systems.

Note that the combinatorial count gives the correct generic number of TMNE (that is, 1 iff $m_1 = m_2$, and 0 otherwise) for games of 2 players. More generally, $E(n_1, n_2, \ldots, n_S) = 0$ for $n_1 > n_2 + \ldots + n_S$, as then the first player cannot avoid his own cards. For $n_1 = n_2 + \ldots + n_S$ we have

$$E(n_1, n_2, \ldots, n_S) = \frac{n_1!}{n_2!n_3!\cdots n_S!}. \tag{8}$$

$E(1, 1, \ldots, 1)$ equals the number of *derangements* of S elements, i.e., the number of permutations without fixed elements. As well-known, the number of derangements is

$$E(1, 1, \ldots, 1) = S! \sum_{j=0}^{S} \frac{(-1)^j}{j!}.$$

This is the maximal number of TMNE for S players when each has 2 options.

3. ROOT COUNTS

This section explores relations between root counts, mixed volumes, and permanents. To bound the number of common roots, we start with the classic tool of m-Bézout bound. Here is its most general statement.

THEOREM 3.1 (M-BÉZOUT BOUND). *Consider a system of N equations in N affine variables, partitioned into S subsets so that the j-th subset includes n_j affine variables, and $N = n_1 + \cdots + n_S$. Let d_{ij} be the degree of the i-th equation in the j-th variable subset, for $i = 1, \ldots, N$ and $j = 1, \ldots, S$. Then, the coefficient of $x_1^{n_1} \cdots x_S^{n_S}$ in*

$$\prod_{i=1}^{N} (d_{i1}x_1 + \cdots + d_{iS}x_S) \tag{9}$$

bounds the number of the system's complex roots in (1). *For generic coefficients this bound is tight.*

In the TMNE system (2)–(4), let $P_j(i)$ denote the expression of P_j with the specified i. A full sequence of multihomogeneous equations is

$P_1(1) = P_1(2),\ P_1(1) = P_1(3),\ \ldots,\ P_1(1) = P_1(m_1),$
$P_2(1) = P_2(2),\ P_2(1) = P_2(3),\ \ldots,\ P_2(1) = P_2(m_2),$
etc.

With $m_j = n_j + 1$, we have

$$d_{ij} = \begin{cases} 0, & \text{if } \sum_{\ell=1}^{j-1} n_\ell < i \leqslant \sum_{\ell=1}^{j} n_\ell, \\ 1, & \text{otherwise.} \end{cases}$$

If we set $X = x_1 + x_2 + \ldots + x_S$, the product in (9) can be written as

$$\prod_{j=1}^{S} (X - x_j)^{n_j}. \tag{10}$$

We are looking for the coefficient to $x_1^{n_1} \cdots x_S^{n_S}$ in the expansion of this product. As noticed in [19], MacMahon's Master theorem [13] can be applied here immediately. We discuss the application and its generalization in §4 here.

We shall consider semi-mixed systems whose Newton polytopes have standard shapes, such as parallelotopes and products of simplices. We show that their mixed volume is reduced to computing a permanent. Reducing mixed volume to computing a matrix permanent is useful for approximating the mixed volume, since the permanent admits fast and accurate approximations.

3.1 Mixed volume

The BKK bound is defined via mixed volumes by establishing a powerful connection between convex and algebraic geometry. We start with convex geometric notions and algorithms, then state the BKK bound.

Let vol$(\cdot)$ denote Euclidean N-dimensional volume that assigns the unit volume to the hypercube of unit edge length.

DEFINITION 3.2. *Let $MV(Q_1, \ldots, Q_N)$ denote the* mixed volume *of convex polytopes $Q_1, \ldots, Q_N \subset \mathbb{R}^N$, defined as the unique real-valued function which is invariant under permutation of the Q_i, and multilinear with respect to scalar multiplication and Minkowski addition on the Q_i, such that*

$$MV(Q_1, \ldots, Q_1) = N!\, vol(Q_1).$$

Multilinearity is formally written as follows: For $\mu, \rho \in \mathbb{R}_{\geqslant 0}$ and a convex polytope $Q'_k \subset \mathbb{R}^N$,

$$\mathrm{MV}(Q_1, \ldots, \mu Q_k + \rho Q'_k, \ldots, Q_N) =$$
$$\mu \mathrm{MV}(Q_1, \ldots, Q_k, \ldots, Q_N) + \rho \mathrm{MV}(Q_1, \ldots, Q'_k, \ldots, Q_N).$$

Let us generalize our discussion to the case of repeated polytopes. Given a system of N polytopes in $\mathbb{R}^N$, if there are only S distinct polytopes, then the mixed volume is called *semi-mixed volume* and is denoted by

$$\mathrm{MV}(Q_1, k_1; \ldots; Q_S, k_S), \quad k_1 + \cdots + k_S = N,$$

where polytope Q_i is repeated $k_i \geqslant 1$ times, $1 \leqslant i \leqslant S$. To stress the different cases, we refer to the case $S = N$ as fully mixed.

Equivalently, these notions can be defined via mixed subdivisions, which have offered the most efficient means of computing mixed volumes. A different, randomized algorithm appeared in [4], but requires one polytope to be repeated a number of times close to N.

Suppose we are given polytopes $Q_1, \ldots, Q_S \subset \mathbb{R}^N$. Let A_i be their respective vertex set. We first lift the A_i to $\mathbb{R}^{N+1}$, thus defining pointsets $\widehat{A}_i$. by taking S sufficiently generic linear forms $l_i : \mathbb{R}^N \to \mathbb{R}$, $1 \leqslant i \leqslant S$. This is implemented by randomly picking the l_i with bitsize which depends on the error probability ϵ (see § 5). For reducing the bitsize of the l_i, see [23]. Then, apply each l_i to the respective Q_i thus defining lifted pointsets

$$\widehat{A}_i = \{(a_i, l_i(a_i)) : a_i \in A_i\} \subset \mathbb{R}^{N+1}, \quad i = 1, \ldots, S.$$

Their Minkowski sum is an $(N+1)$-dimensional polyhedral complex, and its *lower hull* is an N-dimensional polyhedral complex defined as the union of all N-dimensional faces, or facets whose inner normal vector has positive last component. The genericity of the l_i ensures that the lower hull projects bijectively onto the Minkowski sum $\sum_{i=1}^{S} Q_i$ of the original polytopes. A subdivision of this Minkowski is induced by the subdivision of the lower envelope by projecting each k-face onto a k-dimensional cell.

The induced subdivision is called *regular*. *Maximal* cells are those with dimension equal to the dimension of the subdivision. The subdivision is *mixed* since its cells are expressed (or, can be written) as Minkowski sums of convex hulls of point subsets from the A_i's; equivalently, this expression is read off the lower hull. The genericity of the l_i ensures that every lower hull facet is a unique sum of faces $\widehat{F}_i$ from the $\widehat{Q}_i$ for $i = 1, \ldots, S$ such that $\sum_{i=1}^{S} \dim \widehat{F}_i = N$. *Fine (or tight)* cells are those whose dimension equals the sum of its summands' dimensions. The constructed subdivision is *fine (or tight)* since all its cells are fine. More precisely, due to the linearity of the lifting functions l_i, $\dim F_i = \dim \widehat{F}_i$ therefore $\sum_{i=1}^{S} \dim F_i = j$, if $\sum_{i=1}^{S} F_i$ define a k-dimensional cell.

We have thus defined a *regular fine mixed subdivision* of $\sum_i Q_i$, to which we refer as mixed subdivision. We define the *cells of type* $(j_1, \ldots, j_S)$ to be precisely those where the i-th summand is j_i-dimensional.

LEMMA 3.3. *For fully mixed systems where $S = N$, the mixed volume is*

$$MV(Q_1, \ldots, Q_N) = \sum_{\text{mixed } \sigma} vol(\sigma),$$

where the sum ranges over all mixed cells σ, i.e. all cells of type $(1, \ldots, 1)$, in a (fine regular) mixed subdivision of $Q_1 + \cdots + Q_N$.

This could be used as a definition of mixed volume. More generally, the semi-mixed volume is obtained from a mixed subdivision of the Minkowski sum $Q_1 + \cdots + Q_S$. The following lemma subsumes Lem. 3.3.

LEMMA 3.4. [10, Thm.2.4] *The semi-mixed volume, for $k_1 + \cdots + k_S = N$, $k_i \geqslant 1$, is*

$$MV(Q_1, k_1; \ldots; Q_S, k_S) = \prod_{i=1}^{S} k_i! \sum_{\text{type}(\sigma)=(k_1,\ldots,k_S)} vol(\sigma),$$

where the sum ranges over all cells σ of type $(k_1, \ldots, k_S)$ in a mixed subdivision of $Q_1 + \cdots + Q_S$.

To apply these notions to root counting, let us consider a system of N Laurent polynomials in N variables:

$$f_i = \sum_{j=1}^{k_i} c_{ij} x^{a_{ij}}, \quad c_{ij} \neq 0, \quad 1 \leqslant i \leqslant N, \tag{11}$$

where $x = (x_1, \ldots, x_N)$, $x^e = \prod_i x_i^{e_i}$, and $\{a_{i1}, \ldots, a_{ik_i}\} \subset \mathbb{Z}^n$ is the *support* of f_i. Let Q_i be the *Newton polytope* of f_i defined as the convex hull of the support.

Let us examine the Newton polytopes corresponding to the TMNE system (2)–(4). It is an important example because no Newton polytope is fully dimensional and they are defined by simplices, namely

$$Q_i = \Delta_{n_1} \times \cdots \Delta_{n_{i-1}} \times \Delta_{n_{i+1}} \times \cdots \times \Delta_{n_S}, \ 1 \leqslant i \leqslant S, \tag{12}$$

where Δ_{n_j} is the unit simplex in $\mathbb{R}^{n_j}$, and all such simplices lie in complementary subspaces. Equivalently, the Q_i the following Minkowski sum:

$$Q_i = \Delta_{n_1} + \cdots + \Delta_{n_{i-1}} + \Delta_{n_{i+1}} + \cdots + \Delta_{n_S}, \ 1 \leqslant i \leqslant S. \tag{13}$$

We can now state the most general root count.

THEOREM 3.5 (BKK). *For $f_1, \ldots f_N \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ with Newton polytopes $Q_1, \ldots, Q_N$, the number of common isolated solutions, multiplicities counted, in the corresponding toric variety, which contains $(\mathbb{C}^*)^N$ as a dense subset, does not exceed $MV(Q_1, \ldots, Q_N)$ (independently of the corresponding variety's dimension).*

In the case of homogeneous systems, the corresponding toric variety equals $\mathbb{P}^N$. For multihomogeneous systems, the toric variety equals $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_S}$, see Thm 3.1.

When the coefficients corresponding to vertices of the Q_i are generic, then the BKK bound is tight.

3.2 Permanent

Let $\mathcal{S}_N$ denote the group of $N!$ permutations of N integers. Recall that the *permanent* of a square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \tag{14}$$

is defined similarly as the determinant but without multiplying by the sign $\text{sgn}(\varrho)$ of permutation ϱ:

$$\text{perm } A = \sum_{\varrho \in \mathcal{S}_N} \prod_{i=1}^{N} a_{i\varrho(i)}, \quad \det A = \sum_{\varrho \in \mathcal{S}_N} \text{sgn}(\varrho) \prod_{i=1}^{N} a_{i\varrho(i)}.$$

A standard and very useful property states that the permanent equals the coefficient of $x_1 \cdots x_N$ in

$$\prod_{i=1}^{N} (a_{i1}x_1 + \ldots + a_{iN}x_N). \tag{15}$$

Ryser [17] derived the formula

$$\text{perm } A = (-1)^N \sum_{Z \subseteq \{1,\ldots,N\}} (-1)^{|Z|} \prod_{i=1}^{N} \sum_{j \in Z} a_{ij}. \tag{16}$$

The degree structure of the TMNE system (2)–(4) is represented by the $N \times N$ matrix

$$L = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & \mathbf{0} \end{bmatrix}, \tag{17}$$

shown here in the form of $S \times S$ blocks. The diagonal $\mathbf{0}$'s are square blocks of size $n_j \times n_j$ with all entries equal to 0, while the $\mathbf{1}$'s are blocks of size $n_i \times n_j$ $(i \neq j)$ with all entries equal to 1. The columns represent N equations of the multihomogeneous system, and the rows represent dehomogenized variables on $\mathbb{P}^{n_1}, \ldots, \mathbb{P}^{n_S}$. The equations for P_j in (2)–(4) contain no probabilities of the jth player (hence the $\mathbf{0}$'s) and are multilinear in the other variables (hence the $\mathbf{1}$'s). The BKK bound and the permanent are related as follows.

THEOREM 3.6. *Given the Newton polytopes Q_i of the system expressing TMNE as in (12), and matrix L in (17),*

$$MV(Q_1, n_1; \ldots; Q_S, n_S) = \frac{1}{n_1! \cdots n_S!} \text{ perm } L.$$

PROOF. Let us write the Q_i as Minkowski sums following equations (13). By multilinearity, $\mathrm{MV}(Q_1, n_1; \ldots; Q_S, n_S)$ equals the sum of mixed volumes of the form

$$\mathrm{MV}(\Delta_{n_1}, k_1; \ldots; \Delta_{n_S}, k_S), \quad k_i \geqslant 0, \sum_{i=1}^{S} k_i = N,$$

since mixed volume is invariant under reordering of its arguments. If some $k_i = 0$, the mixed volume vanishes, following Lem. 3.3, because the Minkowski sum of all arguments is not full-dimensional. If some $k_i < n_i$, then there exists some j such that $k_j > n_j$. Hence the cells contributing to the mixed volume must contain a summand of dimension k_j from Δ_{n_j}, which is infeasible. Therefore all $k_i = n_i$. Each such mixed volume is 1, namely

$$\mathrm{MV}(\Delta_{n_1}, n_1; \ldots; \Delta_{n_S}, n_S) = 1. \tag{18}$$

It equals the volume of a single mixed cell of type $(n_1, \ldots, n_S)$.

It remains to show that the number of terms of the form (18) equals $\mathrm{perm}L/\prod_i n_i!$. Recall L is a $S \times S$ block matrix, whose block at position (i, j) if of size $n_i \times n_j$. Let us associate the rows of L to the Q_i, each repeated n_i times, and the columns to the Δ_{n_j}, each repeated n_j times.

Every mixed volume in expression (18) is specified by picking a cover of row and column blocks, i.e. a permutation of $(1, \ldots, S)$. For every such choice, there are $n_1! \cdots n_S!$ permanent terms that pick the same blocks, hence give rise to the same mixed volume in expression (18). All terms in the expansion of $\mathrm{perm}L$ equal to 1, hence the claim follows. □

The relation between mixed volumes and permanents generalizes to equations of arbitrary degrees a_{ij} in the variable blocks, thus trivializing the semi-mixed structure. The following generalized formulation applies to any multihomogeneous system, including the fully mixed. It reduces any m-Bézout bound to a permanent.

THEOREM 3.7. *Consider an algebraic system on* (1) *of* $N = n_1 + \ldots + n_S$ *equations. Assume that the* i*-th equation has degree* a_{ij} *in the* j*-th variable block. Let* A *be an* $N \times N$ *matrix with the entries* a_{ij}*, with the columns repeated* n_j *times. Then:*

- *The Newton polytopes are* $Q_i = \sum_{j=1}^{S} a_{ij} \Delta_{n_j}$*, for* $i \in \{1, 2, \ldots, N\}$.
- *The corresponding m-Bézout bound equals*

$$MV(Q_1, n_1; \ldots; Q_S, n_S) = \frac{1}{n_1! \cdots n_S!} \text{ perm } A.$$

PROOF. The first claim specifying the Q_i is clear. The second claim follows mutatis mutandis from the proof of Thm 3.6. The main difference is that the mixed volume is the sum of quantities of the form

$$\mathrm{MV}(\ldots, a_{i_1 j}\Delta_{n_j}, \ldots, a_{i_{n_j} j}\Delta_{n_j}, \ldots) \tag{19}$$

where we showed precisely the n_j arguments that equal to scalar multiples of Δ_{n_j}, for $j = 1, \ldots, S$. The union of all indices $\cup_{j=1}^{S} \{i_1, \ldots, i_{n_j}\}$ equals set $\{1, \ldots, N\}$. The various mixed volumes correspond to choices of indices which range over all permutations of $(1, \ldots, N)$, corresponding to the terms in the expansion of $\mathrm{perm}A$. For each mixed volume,

$$\mathrm{MV}(\ldots, a_{i_1 j}\Delta_{n_j}, \ldots, a_{i_{n_j} j}\Delta_{n_j}, \ldots) = \prod_{j=1}^{S} \prod_{k=1}^{n_j} a_{i_k j},$$

by multilinearity and the value of (18), which yields the value of a term in $\mathrm{perm}A$. □

An important special case is $n_j = 1$ for $j \in \{1, \ldots, S\}$. Then we have a multihomogeneous system on $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. The Newton polytopes are then *cuboids,* or axis-aligned parallelotopes: the i-th cuboid is the direct product of axis-aligned segments with one endpoint at the origin: $\prod_{j=1}^{N} (0, a_{ij})$, for $1 \leqslant i \leqslant N$. It is known [9] that their mixed volume equals the permanent of matrix $A = [a_{ij}]$. A direct extension includes parallelotopes that are not necessarily axis aligned, namely the mixed volume of $Q_i = \sum_{j=1}^{N} a_{ij} e_j$, $1 \leqslant i \leqslant N$, equals permanent of $A = [a_{ij}]$, where the vectors $e_1, \ldots, e_N \in \mathbb{R}^N$ span $\mathbb{R}^N$.

The broadest generalization to arbitrary (nonhomogeneous) systems follows. For each block of variables there is a n_j-dimensional polytope Γ_j, each in a separate complementary space, $1 \leqslant j \leqslant S$. Assume the Newton polytopes are direct products of scalar multiples (by a_{ij}) of the Γ_j's. We obtain an algebraic system on the product of toric varieties corresponding to Γ_j, of dimension n_j. The proof follows from that of Thm 3.7.

THEOREM 3.8. *Let* A *be the matrix of* a_{ij}*'s, with columns repeated* n_j *times. If the number of equations is* $N = \sum n_j$*, then*

$$MV(Q_1, \ldots, Q_N) = \prod_{j=1}^{S} vol(\Gamma_j) \, \mathrm{perm}A, \quad Q_i = \prod_{j=1}^{S} a_{ij}\Gamma_j.$$

3.3 TMNE counting

Now let us unify the discussion, using our running example, namely the TMNE system (2)–(4).

(*i*) The BKK bound is equal to the mixed volume of the Newton polytopes Q_i of the N equations. The Newton polytopes are products of simplexes Δ_k, with $k \in \{n_1, \ldots, n_S\}$, as in (12). This polytope is repeated n_i times, so the root bound is given by the following semi-mixed volume:

$$\mathrm{MV}(Q_1, n_1; \ldots; Q_S, n_S), \quad n_1 + \cdots + n_S = N.$$

(*ii*) The mixed volume is expressed by the permanent, by Thm 3.6. Expansion of the permanent gives the combinatorial count in [15]. The factors $1/n_j!$ reflect grouping the permanent terms into the partitions of N cards (of the formulated card recreation).

(*iii*) Let $Y = (y_{11} + \ldots + y_{1n_1}) + \cdots + (y_{S1} + \ldots + y_{Sn_S})$, the sum of N distinct variables. With reference to (15), the bound equals the coefficient of $(y_{11} \cdots y_{1n_1}) \cdots (y_{S1} \cdots y_{Sn_S})$ in

$$\frac{1}{n_1! \cdots n_S!} \prod_{j=1}^{S} (Y - y_{j1} - \ldots - y_{jn_j})^{n_j}. \tag{20}$$

This follows from the characterization of the permanent in terms of the product in (15). This bound is recognizable in [15, Cor. 3.2].

(*iv*) Let $X = x_1 + \ldots + x_S$. The m-Bézout bound equals the coefficient of $x_1^{n_1} \cdots x_S^{n_S}$ in (10). This directly follows by grouping $y_{j1} + \ldots + y_{jn_j} = x_j$ in (20).

The relation between these four items generalizes to general multihomogeneous systems (preferably semi-mixed), and to the systems described at the end of the previous subsection.

When applying the BKK bound to count TMNE, we must note that the bound is not the exact number of relevant solutions under the following conditions:

- $a^{(1)}_{i,k_2,\ldots,k_S} = a^{(1)}_{j,k_2,\ldots,k_S}$ for $i \neq j$, or a similar equality of payoffs for other player. Then a Newton polygon "loses" a corner vertex, and the mixed volume might be smaller.

- The system is not generic, so that it either has an infinite family of solutions (including some at toric infinity), or some solutions coalesce into a solution with multiplicity.

- Not all solutions are real.

- Not all real solutions give proper probabilities $p_i^{(j)} \in (0,1]$. Real representatives $\left(p_1^{(j)} : p_2^{(j)} : \ldots : p_{m_j}^{(j)}\right)$ of a homogeneous solution must be either all positive or all negative in each block $j \in \{1,2,\ldots,S\}$.

4. GENERATING FUNCTIONS

MacMahon's master theorem [13] has powerful applications to counting restricted partitions and proving binomial identities. Interestingly, the m-Bézout bound (9) for properly semi-mixed systems begs for application of this theorem.

THEOREM 4.1 (MACMAHON'S MASTER THEOREM). *Let A be a complex $S \times S$ matrix as in (14). Let $x_1, \ldots, x_S$ be formal variables, and let V denote the diagonal matrix with the nonzero entries $x_1, \ldots, x_S$. The coefficient of $x_1^{n_1} \ldots x_S^{n_S}$ in*

$$\prod_{j=1}^{S} (a_{j1}x_1 + \ldots + a_{jS}x_S)^{n_j} \tag{21}$$

equals the coefficient of $x_1^{n_1} \ldots x_S^{n_S}$ in the multivariate Taylor expansion of

$$f(x_1, \ldots, x_S) = \frac{1}{\det(I - VA)} \tag{22}$$

around $(x_1, \ldots, x_S) = (0, 0, \ldots, 0)$.

THEOREM 4.2. *Consider a multihomogeneous system on $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_S}$ of $N = n_1 + \ldots + n_S$ equations, where the equations are partitioned into S subsets of exactly $n_1, \ldots, n_S$ equations. We assume that for any $i, j \in \{1, 2, \ldots, S\}$, the polynomial equations in the i-th subset have degree a_{ij} in the variables of the j-th variable subset. Let A be the $S \times S$ matrix defined by the a_{ij}'s. Then the m-Bézout bound for the for the multihomogeneous system equals the coefficient of $x_1^{n_1} \cdots x_S^{n_S}$ in the multivariate Taylor expansion of*

$$1/\det(I - VA)$$

around $(x_1, \ldots, x_S) = (0, 0, \ldots, 0)$.

PROOF. The expression (9) becomes (21) for the semi-mixed system under consideration. MacMahon's master theorem immediately applies. □

For the TMNE system (2)–(4), we have $a_{ii} = 0$, and $a_{ij} = 1$ for $i \neq j$. Thus A has the same shape as the matrix L in expression (17), with blocks of minimal 1×1 size:

$$A = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}, \tag{23}$$

The function in (22) is then the generating function for the numbers $E(n_1, \ldots, n_S)$. Let $M := I - VA$, then

$$M = \begin{bmatrix} 1 & -x_1 & -x_1 & \cdots & -x_1 \\ -x_2 & 1 & -x_2 & \cdots & -x_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -x_S & -x_S & \cdots & -x_S & 1 \end{bmatrix}.$$

In order to compute $\det M$, we start with some definitions. Let σ_j denote the j-th elementary symmetric polynomial in the variables $x_1, \ldots, x_S$:

$$\sigma_1 = \sum_{i=1}^{S} x_i, \ \sigma_2 = \sum_{i=2}^{S}\sum_{h=1}^{i-1} x_h x_i, \ \ldots, \ \sigma_S = \prod_{i=1}^{S} x_i.$$

LEMMA 4.3. *Let $\sigma_1, \sigma_2, \ldots, \sigma_S$ be the elementary symmetric polynomials. Then,*

$$\det M = 1 - \sigma_2 - 2\sigma_3 - \ldots - (S-1)\sigma_S.$$

PROOF. The determinant of M is a symmetric function of $x_1, \ldots, x_S$, at most linear in each variable. Hence it is a linear combination of $\sigma_0 = 1$ and $\sigma_1, \ldots, \sigma_S$. The linear combination can be recovered from the diagonal specialization $x_1 = \ldots = x_S$. If we set all variables equal to $1/\lambda$,

$$\det M = \frac{1}{\lambda^S} \det(\lambda - A). \tag{24}$$

Here $\det(\lambda I - A)$ is the characteristic polynomial of A. The rank of $(I + A)$ equals 1, hence $\lambda = -1$ is an eigenvalue of A with multiplicity $S - 1$. The other eigenvalue is $\lambda = S - 1$, with an eigenvector consisting of all 1's. Hence $\det(\lambda I - A) = (\lambda + 1)^{S-1}(\lambda - S + 1)$ and

$$\det M = (1 + x_1)^{S-1}(1 - (S-1)x_1) = \sum_{j=0}^{S} (1 - j)\binom{S}{j} x_1^j$$

when $x_1 = \ldots = x_S$. For each $j \in \{1, 2, \ldots, S\}$, the term with x_1^j represents $\binom{S}{j}$ summands of σ_j. Without the diagonal specialization, $\det M = \sum_j (1 - j)\sigma_j$ as claimed. □

As obtained in [19], the generating function for the m-Bézout bound for the family of TMNE systems with variable $n_1, \ldots, n_S$ is

$$F(x_1, \ldots, x_S) = \frac{1}{1 - \sigma_2 - 2\sigma_3 - \ldots - (S-1)\sigma_S}. \tag{25}$$

In other words, the coefficient of $x_1^{n_1} \cdots x_S^{n_S}$ in the multivariate Taylor expansion at $(x_1, x_2, \ldots, x_S) = (0, 0, \ldots, 0)$ is equal the number $E(n_1, n_2, \ldots, n_S)$. The same m-Bézout bound holds for any semi-mixed multilinear system on (1), where the equations are partitioned into the blocks of $n_1, n_2, \ldots, n_S$ equations, and i-th subset of equations contains none of the variables in the i-th variable subset for $i \in \{1, 2, \ldots, S\}$.

The total number $N = n_1+\cdots+n_S$ of equations and of variables is not constant.

The generating function for the maximal number of TMNE differs from (25) by the factor σ_S, due to the adjustment (7):

$$F_{\rm N}(x_1, x_2, \ldots, x_S) = \frac{\sigma_S}{1 - \sigma_2 - 2\sigma_3 - \ldots - (S-1)\sigma_S}. \quad (26)$$

The coefficient of $x_1^{m_1} x_2^{m_2} \cdots x_S^{m_S}$ in the multivariate Taylor expansion at $(x_1, x_2, \ldots, x_S) = (0, 0, \ldots, 0)$ of this function equals the tight bound in [15] for the number of TMNE in games of S players, each with (respectively) $m_1, m_2, \ldots, m_S$ pure options. The absence of linear terms (particularly, σ_1) in the denominator implies the discussed TMNE count when $n_1 \geqslant n_2 + \ldots + n_S$; see (8). In particular, $F_{\rm card}(x_1, x_2) = 1/(1-x_1x_2)$, consistent with the established count of TMNE for two players.

Further generalization and applications.

The generating function in (25) is symmetric in the variables $x_1, x_2, \ldots, x_S$, reflecting a symmetry of blocks of equations and blocks of variables. A straightforward generalization is the following.

LEMMA 4.4. *Consider a multihomogeneous system on* (1) *of $N = n_1+\ldots+n_S$ equations, with the equations partitioned into S subsets of exactly $n_1, n_2, \ldots, n_S$ equations. We assume that the degree of every equation in the i-th subset is b in the variables of the ith block, and the degree is c in any other block of variables. Then the m-Bézout bound equals the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_S^{n_S}$ in the multivariate Taylor expansion of*

$$\frac{1}{1 + (d-c)\sigma_1 + (d-2c)d\sigma_2 + \ldots + (d-Sc)d^{S-1}\sigma_S},$$

at $(x_1, x_2, \ldots, x_S) = (0, 0, \ldots, 0)$. Here $d = c - b$.

PROOF. The corresponding matrix A of Thm 4.2 has the entries b on the main diagonal, and the entries c elsewhere. The determinant of $M = I - VA$ is a multilinear symmetric function of $x_1, x_2, \ldots, x_S$, hence it is enough to consider the specialization $x_1 = x_2 = \ldots = x_S$, leading to considering (24). The rank of $(b-c)I - A$ equals $S-1$, and the other eigenvalue is $b + (S-1)c$. Hence

$$\begin{aligned}\det(\lambda I - A) &= (\lambda + c - b)^{S-1}(\lambda - b - (S-1)c) \\ &= (\lambda + d)^{S-1}(\lambda + d - Sc).\end{aligned}$$

Hence $\det M$ with specialized $x_1 = x_2 = \ldots = x_S$ equals

$$(1 + dx_1)^{S-1}(1 + (d-Sc)x_1) = \sum_{j=0}^{S}(d - jc)\binom{S}{j}x_1^j,$$

and the claim follows as in Lem. 4.3. □

If $c = 0$, the multihomogeneous system has the same structure of utilized variables as the TMNE system, but the equations are of degree $d = b$ in each block of utilized variables (rather than multilinear). The generating function

$$\frac{1}{1 - d^2\sigma_2 - 2d^3\sigma_3 - \cdots - (S-1)d^S\sigma_S}$$

is obtained from (25) by the substitutions $x_j \mapsto d\,x_j$ for $j \in \{1, 2, \ldots, S\}$. An algebraic system of this type was obtained [7] by solving all configurations of a cyclic mechanism with 3 degrees of freedom or, equivalently, the conformations of cyclohexane. This is a 3×3 system on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with degree pattern

$$\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

We can also make independent substitutions $x_j \to d_j x_j$ in the generating function, which is equivalent to multiplying corresponding rows of A by d_j. Applied to (23), this modifies the TMNE system by setting the degrees of all equations in the j-th variables to d_j (if the degree is not zero), or (by the transposition symmetry) all jth equations to have degree d_j (or zero) in each block of variables.

5. COMPLEXITY

This section analyzes mixed subdivisions to compute semi-mixed volumes, then juxtaposes it to using permanents or generating functions. Besides asymptotic complexity, we consider briefly practical complexity.

To compute $\mathrm{MV}(Q_1, k_1; \ldots; Q_S, k_S)$, by Lem. 3.4, we sum the volumes of lower hull facets on the Minkowski sum $\sum_i \widehat{Q}_i$ which project to cells of type $(k_1, \ldots, k_S)$. If Q_i has v_i vertices, then the Minkowski sum has $O(v_1 \cdots v_S)$ vertices. Fukuda [8] computes all Minkowski sum faces of dimension $\leqslant j$, $0 \leqslant j < N$, with bit complexity $O(f\delta\lambda(N, m))$, where δ is the sum of the maximum vertex degrees in each summand, f is the number of faces of dimension $\leqslant j$, and $\lambda(N, m)$ bounds the complexity of linear programming in N variables and m constraints, where m is the number of nonparallel edges in the Q_i.

If $v = \max_i\{v_i\}$, then $\delta = O(Sv)$, $m = O(Sv^2)$. Applying [22] and using l to bound the bitsize of the constraint coefficients,

$$\lambda(N, m) = O(((N+m)N^2 + (m+N)^{1.5}N)l) = O(S^{1.5}Nv^3l),$$

assuming $v \geqslant N$, which holds if $\dim Q_i = N$ and for the Q_i of the TMNE system. The overall bit complexity for computing all lower hull facets is

$$O(f\,S^{2.5}N\,v^4 l).$$

In our setting, the coefficients are sums of vertex coordinates, hence l depends on two quantities: (i) $\lg d$, where d is the maximum degree of any polynomial in any variable and (ii) the bitsize of the randomized lifting functionals, which is a function of the error probability ϵ; assuming the latter as fixed, we can ignore this dependence [5]. Since cells have integer volume, f is bounded by the semi-mixed volume V. Applying $v > N$, we get

$$O(V\,S^{2.5}v^5 \lg d).$$

Computing the permanent is a famous #P-complete problem [21]. In particular, the mixed volume (of cuboids) is shown to be #P-hard by reduction of the permanent [9], Direct application of Ryser's formula (16) requires $O(2^N N^2)$ arithmetic operations. This number reduces to $O(2^N N)$ if the subsets are handled in a proper order [21]. Ryser's formula for the permanent in Thm 3.6 becomes

$$\mathrm{perm}\ L = \sum_{k_1=0}^{n_1} \cdots \sum_{k_S=0}^{n_S} (-1)^{N-K} \prod_{j=1}^{S} \binom{n_j}{k_j} (K - k_j)^{n_j}, \quad (27)$$

where $K = k_1 + \ldots + k_S$. Instead of $2^N = \prod_j 2^{n_j}$ sum terms, we have $\prod_j n_j$ terms here. Let all $n_j \leqslant n$, then the total complexity is $O^*(n^S)$, where $O^*(\cdot)$ ignores polylog factors in the arguments. Indeed, computing each product term has amortized complexity in $O(S \log n)$. In view of the generalized Thm 3.7, the formula is modified with general a_{ij}'s.

Concerning implementation, our Maple code for Ryser's formula yields the permanent for $S = 4$ and n at about 60 in ≈ 4 minutes on a Dell Laptop with four 2.4GHz processors (Intel Core i3-2370M), whereas permanents are practically infeasible for matrix dimension $N > 20$.

Generating functions.

Instead of computing permanents, MacMahon's theorem allows computation of the m-Bézout bound of properly semi-mixed systems by computing the determinant of a $S \times S$ matrix (in S variables). The determinant can be computed in polynomial time, but (computing a coefficient of) the multivariate Taylor expansion of the (multiplicative) inverse of the multivariate determinant should require exponential time in general. Nevertheless, complexity is reduced due to replacing the size N of the system by the number of blocks S in the exponents.

The generating function (25) promises faster ways to compute the numbers $E(n_1, \ldots, n_S)$ than by the root bounds or the permanent. In particular, [19] derived recurrences for these E-numbers, and explicit formulas for the $S = 3$ case in terms of terminating hypergeometric ${}_3\mathrm{F}_2(\pm 1)$ sums.

The generating function (25) can be expanded first in the elementary symmetric polynomials. The number of terms $\sigma_2^{\ell_2} \ldots \sigma_S^{\ell_S}$ of the weighted degree $N = \sum_{j=2}^{S} j\ell_j$ equals the coefficient to q^N in $1/\prod_{j=2}^{S}(1-q^j)$, which is asymptotically [21, Partition]

$$\sim \frac{N^{S-2}}{S!(S-2)!} \qquad \text{as } N \to \infty. \tag{28}$$

The coefficient of $\sigma_2^{\ell_2} \ldots \sigma_S^{\ell_S}$ is straightforwardly

$$\frac{(\ell_2 + \ldots + \ell_S)!}{\ell_2!\,\ell_3! \cdots \ell_S!} \prod_{j=2}^{S} (j-1)^{\ell_j}.$$

Computing the coefficient of $x_1^{n_1} \cdots x_S^{n_S}$ in the expanded $\sigma_2^{\ell_2} \cdots \sigma_S^{\ell_S}$ (or just σ_k^{ℓ}) is a combinatorial problem, leading to counting interesting paths in multi-dimensional spaces. If the combinatorial problem can be solved in polynomial time, we would have a speed-up of (27) computation by a factor such as $O(N^2(S-2)!)$.

An interesting open question is whether $E(n, \ldots, n)$ can be computed in polynomial time; here the input consists of just n, S. If the answer is positive, can we compute the numbers $E(n_1, \ldots, n_S)$ in polynomial time?

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