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# Implicit Representation of Rational Parametric Surfaces

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*In many applications we need to compute the implicit representation of rational parametric surfaces. Previously, resultants and Gröbner bases have been applied to this problem. However, these techniques at times result in an extraneous factors along with the implicit equation and fail altogether when a parametrization has base points. In this paper we present algorithms to implicitize rational parametric surfaces with and without base points. One of the strength of the algorithms lies in the fact that we do not use multivariate factorization. The base points blow up to rational curves on the surface and we present techniques to compute the rational parametrization of the blow up curves.*

## 1. Introduction

Many algebraic and geometric algorithms use the parametric form to represent surfaces. For computational reasons, they are restricted to rational functions for parametric representation. A surface represented parametrically by rational functions is known as a *rational surface*. The parametrization of a rational surface represented in terms of homogeneous coordinates is:

$$(x, y, z, w) = (X(s,t), Y(s,t), Z(s,t), W(s,t)), \qquad (1)$$

where $X(s,t)$, $Y(s,t)$, $Z(s,t)$ and $W(s,t)$ are polynomials in the indeterminates $s$ and $t$. The set of rational surfaces is a proper subset of the set of algebraic surfaces. Thus, every rational parametric surface has a corresponding implicit representation and it is desirable to compute it. This process of converting from parametric to implicit is known as *implicitization.* The implicit representation is useful for representing the object as a semi-algebraic set and for surface intersections as shown in Hoffmann (1989) and Prakash & Patriakalakis (1988).

There are two known techniques for implicitization. Both these techniques reduce the problem of implicitizing rational surfaces to eliminating two variables from three parametric equations. The first technique involves the use of Elimination theory. In Hoffmann (1989) the two variables are eliminated in succession by using the Sylvester resultant for two equations. The resulting expression does not correspond to the resultant of three parametric equations and contains an extraneous factor, whose separation can be a time consuming task involving multivariate factorization. The Dixon formulation, given in Dixon (1908), for computing the resultant has been used to implicitize tensor product surfaces in Sederberg et al. (1984). It does not generate an extraneous factor, but is limited to tensor product surfaces and not applicable to total degree bounded parametrizations. Bajaj et al. (1988) use Macaulay's formulation for computing the resultant of three parametric equations for implicitizing. In general, it is believed that techniques based on Elimination theory can result in extraneous factors along with the implicit equation and separating them can be a time consuming task as mentioned in Hoffmann (1989).

The second technique utilizes Gröbner bases. It computes a canonical representation of the ideal generated by the parametric equations, by defining a suitable ordering of the variables as shown in Buchberger(1989) and Hoffmann(1989). However, this method can be extremely slow in practice. In this paper, we formulate the three parametric equations in such a manner, that their resultant corresponds to the implicit representation without generating any extraneous factor.

All the techniques mentioned above fail when a parametrization has base points in the parametric domain. A base point in the domain, say $s = s_0, t = t_0$, corresponds to a common solution of the following four equations

$$X(s,t) = 0, \quad Y(s,t) = 0, \quad Z(s,t) = 0, \quad W(s,t) = 0.$$

The base points also include the common solutions at infinity. In general any faithful parametrization of a rational surface whose algebraic degree is not a perfect square has base points. Furthermore, the base points blow up to rational curves on the surface (known as seam curves).

We present an algorithm to implicitize rational parametrizations with base points and also compute the rational parametrizations of seam curves. In particular, we symbolically perturb the given parametric equations and show that the implicit equation is contained in the lowest degree term of the resultant of the perturbed system (expressed in terms of the perturbing variable). However the lowest degree term contains an extraneous factor along with the implicit equation, as observed in Chionh (1990), and separating it can be a time consuming task involving multivariate factorization. To overcome this problem we consider a particular perturbation, obtained by perturbing one of the three equations and hereby denoted as the *efficient perturbation*, and show that the extraneous factor is independent of one of the variables. This allows us to compute the extraneous factor by two substitutions for that variable followed by a GCD (greatest common divisor) calculation. Moreover, it is shown that in the case of efficient perturbation the extraneous factor corresponds to the projection of the seam curves and is used for computing the rational parametrizations of the seam curves.

The rest of the paper is organized in the following manner. In section 2, we specify our notation and present some background material from algebraic geometry. Section 3 shows how resultants can be used to compute the implicit representation without generating any extraneous factors. In section 4, we analyse parametrizations with base points and show why resultants and Gröbner basis fail on such surfaces. We perturb the given parametric equations in section 5 and show that the implicit equation is contained in the lowest degree term of the resultant of the perturbed system. In section 6 we consider the efficient perturbation and show that the extraneous factor in the lowest degree term is a function of two variables and corresponds to the projection of seam curves. This extraneous factor is used for computing the rational parametrizations of seam curves in section 7.

# 2. Background

A rational parametrization is a vector valued function of the form

$$\mathbf{F}(s,t) = (X(s,t), Y(s,t), Z(s,t), W(s,t)). \tag{2}$$

We use lower case letters like $s$, $t$, $x$ or $y$ to denote scalar variables and upper case letters to represent scalar functions like $W(s,t)$ or $F(x,y,z)$ and homogeneous functions like $\overline{F}(x,y,w)$. Bold face upper case letters, like $\mathbf{F}(s,t)$, are used to represent vector valued functions and lower case bold face letters like $\mathbf{p}$ and $\mathbf{q}$ represent tuples like $(s,t,u)$.

In (2), $X(s,t)$, $Y(s,t)$, $Z(s,t)$ and $W(s,t)$ are bivariate polynomials and assumed to have *power basis* representation. A polynomial $H(x,y,z)$ is independent of $z$, if it is a bivariate polynomial in $x$ and $y$ and all monomials are independent of $z$.

A surface parametrization, (2), represents a mapping of the form

$$\mathbf{F}: \quad R^2 \rightarrow R^3,$$

In fact the domain is often restricted to a finite interval, of the form $[a_1,b_1] \times [a_2,b_2]$ or a triangle. Since the field real numbers is not algebraically closed, we extend this definition to the complexes and also include the points at infinity. As a result, the resulting parametrization corresponds to a mapping of the form

$$\mathbf{F}: \quad P^2 \rightarrow P^3,$$

where $P$ denotes the complex projective space. We use homogeneous coordinates to represent the domain and range of $\mathbf{F}$ and a point in the domain is represented by the tuple $(s, t, u)$. The rational surface $\mathbf{F}(s, t)$ should be interpreted as a representation of the form

$$\overline{\mathbf{F}}(s,t,u) = (\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)) \tag{3}$$

where $\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u)$ and $\overline{W}(s,t,u)$ are homogeneous polynomials in $s$, $t$ and $u$ and each polynomial has the same degree. Moreover,

$$GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)) = 1.$$

## 2.1. ALGEBRAIC SETS

In this section we present some definitions and basic results on the dimension of algebraic sets. We use these results in the rest of the paper.

Let us consider an algebraically closed field, $C$ and define a polynomial ring

$$A = C[x_1, x_2, \ldots, x_m]$$

of $m$ variables over $C$. All the polynomials used in this section are assumed to be defined over this ring.

DEFINITION. The set of common zeros of a system of polynomials $F_1, \ldots, F_n$ in $x_1, \ldots, x_m$ is called an *algebraic set* and is denoted $V(F_1, \ldots, F_n) \subset C^m$. An algebraic set $V(F)$ defined by a single polynomial (which is not identically zero) is called a *hypersurface*.

If all the $F_i$ are homogeneous, it is more convenient to work with the projective space $P^{m-1}$, formed by identifying points in $C^m$ which are scalar multiples of each other. We use the same notation, $V(\overline{F}_1, \ldots, \overline{F}_n) \subset P^{m-1}$ for an algebraic set defined by homogeneous polynomials $\overline{F}_i$.

An algebraic set is said to be *reducible* if it can be expressed as a finite union of proper subsets which are algebraic. Otherwise it is an *irreducible* algebraic set. An irreducible algebraic set is known as a *variety*. An algebraic set can always be expressed as a finite union of irreducible algebraic subsets called *components*. Many results in algebraic geometry apply only to irreducible algebraic sets, and in much of what follows, we work with the individual components of an algebraic set.

DEFINITION. Let $Z$ be the intersection of $n$ hypersurfaces in $m$-dimensional affine or projective space. A component $W$ of $Z$ is said to be *proper* if it has dimension $m - n$. A component of dimension greater than $m - n$ is said to be an *excess* component.

And in fact all components of an intersection must be either proper or excess by the following lemma from Mumford (1976):

LEMMA 1. *If $F_i$ are $n$ non-homogeneous polynomials in $m$ variables, (or homogeneous in $m+1$ variables), then every component of $V(F_1, \ldots, F_n)$ has dimension at least $m-n$.*

# 3. Implicitization

Consider a rational surface

$$\overline{\mathbf{F}}(s,t,u) = (x,y,z,w) = (\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)),$$

where $\overline{X}(s,t,u)$, $\overline{Y}(s,t,u)$, $\overline{Z}(s,t,u)$ and $\overline{W}(s,t,u)$ are homogeneous polynomials of degree $n$. Let $\mathcal{Y}$ denote the image of $\overline{\mathbf{F}}$. It is assumed that $\mathcal{Y}$ is a two dimensional set. In other words, the image of $\overline{\mathbf{F}}$ is not a 1-dimensional curve.

Let us consider the case when the parametrization, $\overline{\mathbf{F}}$, has *no base points* and the map $\overline{\mathbf{F}}$, is therefore, defined at all points in the domain. Since $P^2$ is a closed, compact and irreducible set of dimension 2 and $\overline{\mathbf{F}}$ is a continuous rational map, the image of $\overline{\mathbf{F}}$ is a closed and irreducible set in $P^3$. This can be proved formally by considering $P^2$ and $P^3$, the domain and range of $\overline{\mathbf{F}}$, as topological spaces with respect to Zariski topology. It is shown in Munkres (1975) that the image of a compact set under a continuous map is compact. As a result $\mathcal{Y}$ is a compact set. Furthermore, every compact subset of a Hausdorff space is closed, as proven in Munkres (1975). Since $P^3$ is a Hausdorff space, $\mathcal{Y}$ is therefore, a closed set. Thus, $\mathcal{Y}$ is a 2 dimensional projective variety in $P^3$. The following lemma from algebraic geometry,(Hartshorne, 1977),

LEMMA 2. *A projective variety $Y \subset \mathrm{P}^m$ has dimension $m-1$, if and only if it is the zero set of a single irreducible and homogeneous polynomial $\overline{G}$ of positive degree.*

implies that the image of $\overline{\mathbf{F}}$ corresponds to the zero set of a single irreducible and homogeneous polynomial, $\overline{G}(x,y,z,w)$. Thus, $\overline{G}(x,y,z,w)$ is the implicit representation of the given surface. It is characterized by the following property:

$$\overline{G}(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)) = 0.$$

Consider the following *parametric equations*

$$\begin{aligned} \overline{F}_1(s,t,u) &= x\overline{W}(s,t,u) - w\overline{X}(s,t,u) = 0, \\ \overline{F}_2(s,t,u) &= y\overline{W}(s,t,u) - w\overline{Y}(s,t,u) = 0, \\ \overline{F}_3(s,t,u) &= z\overline{W}(s,t,u) - w\overline{Z}(s,t,u) = 0. \end{aligned} \tag{4}$$

The solution set of each equation corresponds to a 4-dimensional hypersurface in $P^2 \times P^3$ (spanned by $(s,t,u)$ and $(x,y,z,w)$, respectively). Let's consider the algebraic set, $\overline{Q} =$

$V(\overline{F}_1, \overline{F}_2, \overline{F}_3)$, obtained by the intersection of the three hypersurfaces, obtained as the solution set of the above equations. Let $\overline{\Pi}$ be a projection function

$$\overline{\Pi} : P^2 \times P^3 \rightarrow P^3$$

such that

$$\overline{\Pi}(s,t,u,x,y,z,w) = (x,y,z,w).$$

Lemma 1 implies that each component in $\overline{Q}$ has dimension of at least 2. Since there are no base points, the intersection set consists of the following components:

1.
$$\overline{Q}_1 = \{(s,t,u,x,y,z,w) | x = \overline{X}(s,t,u), y = \overline{Y}(s,t,u), z = \overline{Z}(s,t,u), w = \overline{W}(s,t,u)\}.$$

   $\overline{Q}_1$ is a proper component of $\overline{Q}$ and

$$\overline{\Pi}(\overline{Q}_1) = V(\overline{H}(x,y,z,w)),$$

   where

$$\overline{H}(x,y,z,w) = \overline{G}(x,y,z,w)^k, \quad for \ \ k \geq 1.$$

2.
$$\overline{Q}_2 = \{(s,t,u,x,y,z,w) | \overline{W}(s,t,u) = 0, w = 0\}.$$

   $\overline{Q}_2$ is an excess component of $\overline{Q}$ (of dimension 3). However, $\overline{\Pi}(\overline{Q}_2)$ has dimension 2 and corresponds to the points at infinity in the $(x,y,z,w)$ space.

We see that $\overline{\Pi}(\overline{Q})$ consists of at least two distinct components, whereas we are interested in computing $\overline{G}(x,y,z,w)$ only. We therefore, work with an affine representation of the image space and modify the parametric equations, (4), as

$$\begin{array}{rcl} \overline{F'}_1(s,t,u) & = & x\overline{W}(s,t,u) - \overline{X}(s,t,u) = 0, \\ \overline{F'}_2(s,t,u) & = & y\overline{W}(s,t,u) - \overline{Y}(s,t,u) = 0, \\ \overline{F'}_3(s,t,u) & = & z\overline{W}(s,t,u) - \overline{Z}(s,t,u) = 0. \end{array} \tag{5}$$

This corresponds to substituting $w = 1$ in (4). Let's consider

$$Q = V(\overline{F'}_1, \overline{F'}_2, \overline{F'}_3) \subset P^2 \times C^3,$$

and let $\Pi$ be the projection function

$$\Pi : P^2 \times C^3 \rightarrow C^3$$

such that

$$\Pi(s,t,u,x,y,z) = (x,y,z).$$

THEOREM 1. *If the given parametrization has no base points and the parametrization is faithful then $Q$ consists of a single component. Moreover, that component can be represented as*

$$Q = \{(s,t,u,x,y,z) | x = \frac{\overline{X}(s,t,u)}{\overline{W}(s,t,u)}, y = \frac{\overline{Y}(s,t,u)}{\overline{W}(s,t,u)}, z = \frac{\overline{Z}(s,t,u)}{\overline{W}(s,t,u)}\}.$$

PROOF. The fact that $\overline{Q}_1 \subset \overline{Q}$ implies that $Q_1 \subset Q$. Thus, $Q_1$ is a component of $Q$. Let us assume that $Q$ consists of some other component, say $P$. Since $P \neq Q_1$, $\exists\ \mathbf{p} = (s_1, t_1, u_1, x_1, y_1, z_1) \in P$ and $\mathbf{p} \notin Q_1$. There are two possibilities:

1. $\overline{W}(s_1, t_1, u_1) = 0$.

   We know that $\mathbf{p} \in V(\overline{F'}_1, \overline{F'}_2, \overline{F'}_3)$ and therefore

   $$\overline{F'}_1(s_1, t_1, u_1) = 0,$$

   $$\Rightarrow \overline{X}(s_1, t_1, u_1) = x_1 \overline{W}(s_1, t_1, u_1) = 0.$$

   Similarly, we can show that $\overline{Y}(s_1, t_1, u_1) = 0$ and $\overline{Z}(s_1, t_1, u_1) = 0$. This implies that $(s_1, t_1, u_1)$ is a base point of $\overline{\mathbf{F}}$, which is contrary to our assumption.

2. $\overline{W}(s_1, t_1, u_1) \neq 0$.

   We know that $\mathbf{p} \in Q$ and therefore,

   $$\overline{F'}_1(s_1, t_1, u_1) = 0$$

   $$\Rightarrow x\overline{W}(s_1, t_1, u_1) = \overline{X}(s_1, t_1, u_1)$$

   $$\Rightarrow x_1 = \frac{\overline{X}(s_1, t_1, u_1)}{\overline{W}(s_1, t_1, u_1)}.$$

   Similarly we can show that

   $$y_1 = \frac{\overline{Y}(s_1, t_1, u_1)}{\overline{W}(s_1, t_1, u_1)},$$

   and

   $$z_1 = \frac{\overline{Z}(s_1, t_1, u_1)}{\overline{W}(s_1, t_1, u_1)}.$$

   This implies that $\mathbf{p} \in Q_1$.

Thus, all points in $Q$ also lie in $Q_1$ and therefore,

$$Q = Q_1.$$

Thus, $Q$ consist of one component. Q.E.D.

Since $Q$ is an irreducible algebraic set, each point in $\Pi(Q)$ lies in $\mathcal{Y}$. This follows from the representation of $Q$ in Theorem 1. Since $Q$ and $\Pi(Q)$ are 2 dimensional algebraic sets, $\Pi(Q)$ correspond to the affine portion of the zero set of the implicit representation of $\overline{\mathbf{F}}(s,t,u)$. If the given parametrization is unfaithful, each point in $\Pi(Q)$ has more than one preimage with respect to $\overline{\mathbf{F}}$. In this case, $\Pi(Q)$ corresponds to an algebraic set of multiplicity greater than one. Thus,

$$\Pi(Q) = V(H(x,y,z)), \tag{6}$$

where $H(x,y,z) = G(x,y,z)^k$, $k \geq 1$. $k = 1$ if and only if $\overline{F}$ is a faithful parametrization. Using Bezout's theorem it can be shown that the algebraic degree of $H(x,y,z)$ is $n^2$, where $n$ is the degree of the parametrization. The degree of $G(x,y,z)$ is $n^2/k$. Moreover, $k$ corresponds to the number of points in the $(s,t,u)$ plane, that are the preimages of an arbitrary point in $V(G(x,y,z))$.

The problem of implicitizing parametric surfaces without any base points corresponds to computing $\Pi(Q)$ and making sure that the resulting polynomial is square free. This can be done using Grőbner bases or resultants, as shown in Buchberger (1989) and Manocha & Canny (1992), respectively. The resultant of three parametric equations (5) can be expressed as determinant of a matrix. The corresponding formulations are given in Dixon (1908) and Morley & Coble (1927). This holds for tensor product surfaces as well as total degree bounded parametrizations. In practice, this formulation is efficient for computing the implicit representation, as shown in Manocha & Canny (1992).

# 4. Base Points

A base point is a common solution of

$$\overline{X}(s,t,u) = 0, \quad \overline{Y}(s,t,u) = 0, \quad \overline{Z}(s,t,u) = 0, \quad \overline{W}(s,t,u) = 0.$$

The solution set of any of the polynomials, say $\overline{X}(s,t,u) = 0$, corresponds to an algebraic plane curve in the $P^2$ plane (denoted by homogeneous coordinates $s$, $t$ and $u$). Each curve may have more than one component and the base point corresponds to the intersection of these curves. The *multiplicity* of each base point is equal to the multiplicity of the curves at that point. In other words, a base point has multiplicity $k$, if it is a $k$-fold point of $\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u)$ and $\overline{W}(s,t,u)$. The multiplicity of a curve is defined in Semple & Roth (1985). Let

$$\mathcal{S} = V(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u))$$

be the set of base points. Since

$$GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)) = 1,$$

$\mathcal{S}$ is therefore, a finite set. Let $\mathbf{p} = (s_0, t_0, u_0) \in \mathcal{S}$. Moreover,

$$\overline{\mathbf{F}}(\mathbf{p}) = \overline{\mathbf{F}}(s_0, t_0, u_0) = (0, 0, 0, 0),$$

which does not correspond to any point in the image space. It has been known that base points blow up to rational curves on the surface (known as *seam curves*), given in detail in Clebsch (1868), Semple & Roth (1985) and Snyder et al. (1970). Furthermore, the degree of the seam curve is bounded by the multiplicity of the corresponding base point.

Since $\overline{\mathbf{F}}$ is not defined at the base points, we modify its domain and define it as a mapping of the form

$$\overline{\mathbf{F}'} : P^2 \setminus \mathcal{S} \rightarrow P^3$$

$$\overline{\mathbf{F}'}(s, t, u) = \overline{\mathbf{F}}(s, t, u),$$

where $P^2 \setminus \mathcal{S}$ represents the difference of two sets. $P^2 \setminus \mathcal{S}$ is an open and irreducible set of dimension 2. Let $\mathcal{K}$ be the image of $\overline{\mathbf{F}'}$. We know that $\mathcal{K}$ is a 2-dimensional set and $\mathcal{K} \subset P^3$. In general, $\mathcal{K}$ is a proper subset of an algebraic set $V(\overline{H}(x, y, z, w))$. The problem of implicitization corresponds to computing $\overline{H}(x, y, z, w)$. The base points decrease the degree of the implicit equation as explained in Manocha & Canny (1992). A base point of multiplicity $k$ decreases the degree of the implicit equation by at least $k^2$. The total number of base points (counted properly) correspond to $n^2 - d$, where $n$ is the degree of the parametrization and $d$ is the degree of its implicit representation. Thus, a base point of multiplicity $k$ is counted at least $k^2$ times.

## 4.1. IMPLICITIZING SURFACES WITH BASE POINTS

Given $\overline{\mathbf{F}}$, a parametrization with base points, we use resultants to compute the implicit equation. The resultant of the parametric equations (5), by considering them as polynomials in $s$, $t$ and $u$, is zero. This can be explained in the following manner.

Given $\mathbf{p} = (s_0, t_0, u_0)$, a base point in the parametrization. From the definition of a base point it follows that

$$\overline{F}_1(s_0, t_0, u_0) = 0 \quad \overline{F}_2(s_0, t_0, u_0) = 0, \quad \overline{F}_3(s_0, t_0, u_0) = 0.$$

Thus, the given system of equations, (5), has a non trivial solution $(s_0, t_0, u_0)$. Moreover, this solution is independent of the coefficients, $x$, $y$ and $z$. The resultant is therefore, identically zero.

The Gről̋bner bases approach to implicitizing parametric surfaces considers the ideal generated by the parametric equations. More details of this approach are given in Buchberger (1989) and Hoffmann (1989). It uses a particular ordering of the variables and compute the Gről̋bner base of the ideal. One of the polynomials in the Gről̋bner base is independent of $s$ and $t$ and therefore, corresponds to the implicit representation. However, the technique fails if a parametrization has base points in the affine domain as shown in Manocha & Canny (1992).

Grőbner bases offer us the flexibility of working in the affine space. As a result, it is possible to implicitize parametrizations with base points only at infinity. All polynomial parametrizations (with or without base points) can therefore, be implicitized using Grőbner bases.

# 5. Perturbation

In the previous sections, we have shown the use of resultants and Grőbner bases for implicitizing parametric surface. However these techniques fail when a parametrization has base points. For example, the resultant of the parametric equations is identically zero due to the presence of an excess component in the image space. Thus, the problem of implicitizing corresponds to: *computing the proper component in the presence of excess component.* Some similar problems have been encountered while solving system of polynomial equations and techniques for dealing with such problems have been highlighted in Canny (1990) and Ierardi (1989). The technique corresponds to perturbing the given equations, such that the resulting algebraic set (in the higher dimensional space defined by adding the perturbing variable) has no excess component. The projections of the proper components of the algebraic set corresponding to the unperturbed system can be obtained from the projections of the algebraic set corresponding to the perturbed system by applying limiting arguments.

We will carry out the rest of perturbation analysis with resultants. The technique is also applicable with Grőbner bases. However we recommend resultants for their efficiency, as shown in Manocha & Canny (1992).

Lets consider the parametrization

$$\overline{\mathbf{F}}(s,t,u) = (x,y,z,w) = (\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)),$$

of degree $n$, which has base points in the domain, represented by set $\mathcal{S}$. The resultant of the parametric equations, (5), is identically zero. Lets perturb the given system of equations and the resulting parametric equations are

$$\begin{aligned} \overline{G}_1(s,t,u) &= x\overline{W}(s,t,u) - \overline{X}(s,t,u) + \lambda\overline{X}_1(s,t,u) = 0, \\ \overline{G}_2(s,t,u) &= y\overline{W}(s,t,u) - \overline{Y}(s,t,u) + \lambda\overline{Y}_1(s,t,u) = 0, \\ \overline{G}_3(s,t,u) &= z\overline{W}(s,t,u) - \overline{Z}(s,t,u) + \lambda\overline{Z}_1(s,t,u) = 0, \end{aligned} \tag{7}$$

where $\lambda$ is the perturbing variable and $\overline{X}_1(s,t,u), \overline{Y}_1(s,t,u)$ and $\overline{Z}_1(s,t,u)$ are homogeneous polynomials of degree $n$ such that

$$V(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u), \overline{X}_1(s,t,u), \overline{Y}_1(s,t,u), \overline{Z}_1(s,t,u)) = \phi.$$

In other words, the perturbed system of parametric equations, (7), has no trivial solutions and therefore, their resultant does not vanish. A simple procedure is to choose random polynomials, $\overline{X}_1(s,t,u), \overline{Y}_1(s,t,u)$ and $\overline{Z}_1(s,t,u)$. The resulting system of perturbed equations has a base point if and only if their resultant of $\overline{G}_1, \overline{G}_2$ and $\overline{G}_3$ is zero. This process of choosing random polynomials can be repeated until the resultant is non-zero. The probability of success is very close to 1.

Let

$$Q = V(\overline{G}_1, \overline{G}_2, \overline{G}_3) \subset P^2 \times C^3 \times C^1,$$

and $\Pi$ be the projection function

$$\Pi : P^2 \times C^3 \times C^1 \rightarrow C^3 \times C^1, \tag{8}$$

such that

$$\Pi(s,t,u,x,y,z,\lambda) = (x,y,z,\lambda).$$

According to Lemma 1 every component of $Q$ has dimension greater than or equal to 3. Let $R(x,y,z,\lambda)$ be the resultant of the perturbed system, (7), i.e.

$$R(x,y,z,\lambda) = \Pi(Q).$$

Let us express the resultant as a polynomial in $\lambda$, while the coefficients are polynomials in $x$, $y$ and $z$:

$$R(x,y,z,\lambda) = P_i(x,y,z)\lambda^i + \ldots + P_d(x,y,z)\lambda^d. \tag{9}$$

The fact that specializing $\lambda = 0$ makes the resultant of (7) equal to zero implies that $i > 0$ in (9).

THEOREM 2. *$H(x,y,z)$, the implicit representation of $\overline{\mathbf{F}}(s,t,u)$ is contained in $P_i(x,y,z)$, i.e.*

$$H(x,y,z) \ \mid \ P_i(x,y,z),$$

*where $P_i(x,y,z)$ is the coefficient of the lowest degree term of $R(x,y,z,\lambda)$, expressed as a polynomial in $\lambda$.*

PROOF. Let

$$P = V(\overline{F}_1(s,t,u), \overline{F}_2(s,t,u), \overline{F}_3(s,t,u))$$

where $\overline{F}_i(s,t,u)$ is an unperturbed parametric equation and

$$P \subset P^2 \times C^3.$$

Let $B$ be the component of $P$ defined as

$$\begin{aligned} B \ = \ & \{(s,t,u,x,y,z) \mid x = \frac{\overline{X}(s,t,u)}{\overline{W}(s,t,u)}, y = \frac{\overline{Y}(s,t,u)}{\overline{W}(s,t,u)}, z = \frac{\overline{Z}(s,t,u)}{\overline{W}(s,t,u)}, \ (s,t,u) \in P^2 \setminus \mathcal{S}\} \\ & \cup \{(s,t,u,x,y,z) \mid (s,t,u) \in \mathcal{S} \ and \ (x,y,z) \in C_{(s,t,u)}(x,y,z)\}, \end{aligned}$$

where $C_{(s,t,u)}(x,y,z)$ is the set of all points lying on the seam curves corresponding to $(s,t,u)$. $B$ is a proper component of $P$.

With the addition of a complex variable $\lambda$, the zero set of $Q$ lies in $P^2 \times C^3 \times C^1$. Since $\overline{F}_i(s,t,u)$ and $\overline{G}_i(s,t,u)$ are identical when $\lambda = 0$, it follows that

$$P \times \{0\} = Q \cap (\lambda = 0).$$

Thus, $B \times \{0\} \subset Q$. Since every component of $Q$ has dimension greater than or equal to 3, $B \times \{0\}$ must be contained in some 3 (or higher) dimensional component $B'$ of $Q$. Every point of $B'$ has a 3 dimensional neighbourhood whose intersection with the hypersurface $\lambda = 0$ is a 2 dimensional set. Thus, for every point $\mathbf{q} = (s_k, t_k, u_k, x_k, y_k, z_k, 0) \in B \times \{0\}$, there is a sequence of points $\mathbf{q}_j = (s_j, t_j, u_j, x_j, y_j, z_j, \lambda_j)$ in $B' - B \times \{0\}$ which converges to $\mathbf{q}$. Moreover $R(\Pi(\mathbf{q}_j)) = 0$ for all j's. Thus, $R(x_j, y_j, z_j, \lambda_j) = 0$. Divide the polynomial throughout by $(\lambda_j)^i$ (which is non-zero) and we obtain

$$P_i(x_j, y_j, z_j) + P_{i+1}(x_j, y_j, z_j)(\lambda_j) + \ldots + P_d(x_j, y_j, z_j)(\lambda_j)^{d-i} = 0$$

for all $\mathbf{q}_j$. This is a polynomial in the coordinates of $\mathbf{q}_j$ and is, therefore, a continuous function of the coordinates. Since it is zero for $\mathbf{q}_j \rightarrow \mathbf{q}$, it must be zero at $\mathbf{q}$. But $\mathbf{q}$ is a point lying on the hypersurface $\lambda = 0$, so $P_i(x_k, y_k, z_k) = 0$. Since

$$V(H(x,y,z)) = \{(x_k, y_k, z_k) \mid \mathbf{q} = (s_k, t_k, u_k, x_k, y_k, z_k, 0) \in B \times \{0\}\},$$

$V(H(x,y,z)) \subset V(P_i(x,y,z))$. If $\overline{\mathbf{F}}$ is a faithful parametrization, $H(x,y,z)$ is an irreducible polynomial and therefore, $H(x,y,z) \mid P_i(x,y,z)$. Else let any generic point in $\mathcal{Y}$ have $m$ preimages $(m > 1)$. Thus, $H(x,y,z) = G(x,y,z)^m$. Let $(x_1, y_1, z_1) \in \mathcal{Y}$ and $(s_i, t_i, u_i)$, $1 \leq i \leq m$ be its preimages. In other words $\mathbf{q}_i = (s_i, t_i, u_i, x_1, y_1, z_1) \in B$ for all $i$. As a result $\mathbf{q}_i \times 0 \in B'$ and it has a 3-dimensional neighborhood in $B' - B \times 0$ which converges to $\mathbf{q}_i \times 0$. Since $R(x,y,z,\lambda) = \Pi(Q)$, we can use the limiting argument to show that $(x_1, y_1, z_1)$ is a point of multiplicity $m$ in $V(P_i(x,y,z))$. Thus,

$$H(x,y,z) \mid P_i(x,y,z). \qquad \text{Q.E.D.}$$

The same result hold when we use the Dixon eliminant on tensor product parametrizations or Grőbner bases on any parametrization as shown in Manocha & Canny (1992). We illustrate the technique on the following examples.

EXAMPLE 1. Let

$$\mathbf{F}(s,t) = (x,y,z) = (\frac{s^2 - 1 - t^2}{s^2 + 1 + t^2}, \frac{2s}{s^2 + 1 + t^2}, \frac{2st}{s^2 + 1 + t^2})$$

be the parametrization of a rational surface (a sphere in this case), which has a base point at $(s,t) = (0,i)$, where $i = \sqrt{-1}$. The ideal generated by the parametric equations is

$$\mathcal{I} = \{x(s^2 + 1 + t^2) - s^2 + 1 + t^2, y(s^2 + 1 + t^2) - 2s, z(s^2 + 1 + t^2) - 2st\}.$$

None of the polynomials in $\mathcal{I}$ is independent of $s$ and $t$. Lets perturb the parametric equations and the ideal corresponding to the resulting parametric equations is

$$\mathcal{J} = \{x(s^2+1+t^2) - s^2 + 1 + t^2 - \lambda t, y(s^2+1+t^2) - 2s - \lambda, z(s^2+1+t^2) - 2st + \lambda s\}.$$

Compute the Grőbner bases of $\mathcal{J}$ with a variable ordering

$$z < y < x < \lambda < s < t.$$

The first polynomial in the Grőbner bases is independent of $s$ and $t$. It can be expressed as a polynomial in $\lambda$ as

$$\lambda(2+\lambda^2)(\lambda^2 y^2(-3-x^2-y^2-4z-z^2) - 2\lambda y(-x+x^3-2y^2+xy^2-z-4xz-x^2z-3y^2z-2z^2)$$
$$-2\lambda y(-3xz^2-z^3) - 2(x^2+y^2+z^2-1)(1+2x+x^2+y^2+2z+2xz+z^2)),$$

whose lowest degree term is

$$-2(x^2+y^2+z^2-1)(1+2x+x^2+y^2+2z+2xz+z^2).$$

Thus, the implicit representation of the sphere, $x^2+y^2+z^2-1$, is contained in the lowest degree term. Q.E.D.

EXAMPLE 2. Lets consider a tensor product parametrization

$$\mathbf{F}(s,t) = (x,y,z,w) = (st^2 - t, st + s, 2s - 2t, st^2),$$

which has a base point at $(s,t) = (0,0)$. The resulting parametric equations are

$$\begin{aligned} xst^2 - st^2 + t &= 0, \\ yst^2 - st - s &= 0, \\ zst^2 - 2s + 2t &= 0, \end{aligned}$$

whose Dixon eliminant is zero. Lets perturb these equations and the resulting system is

$$\begin{aligned} xst^2 - st^2 + t + \lambda(s+2) &= 0, \\ yst^2 - st - s + \lambda t^2 &= 0, \\ zst^2 - 2s + 2t + \lambda(s+4) &= 0. \end{aligned}$$

The Dixon eliminant of these equations is polynomial in $x, y, z$ and $\lambda$ and after expressing it as a polynomial in $\lambda$, the lowest degree term is

$$8(-2+2x-z)(-4x+4x^2-8y+8xy+4y^2+2z-4xz-4yz+z^2).$$

In this case, $(-2+2x-z)$ is an extraneous factor and $(-4x+4x^2-8y+8xy+4y^2+2z-4xz-4yz+z^2)$ is the implicit representation. Q.E.D.

Thus, we can perturb the given parametric equations such that the lowest degree term of the resultant of the perturbed system contains the implicit equation. However, there is always an extraneous factor present in the lowest degree term and extracting the implicit representation involves multivariate factorization. Furthermore, we need to test each irreducible polynomial, obtained after factorization, whether it corresponds to the implicit equation. In many cases this process can be a time consuming task.

# 6. Efficient Perturbation

In this section we present an efficient perturbation such that the implicit equation can be extracted from the lowest degree term of the resultant by computing the GCD of bivariate polynomials. Furthermore, the extraneous factor in the lowest degree term of the perturbed system contains interesting information about the seams or blow-ups of the base points. In particular, we choose our perturbation so that we get the $X$-$Y$ projection of the blow-up curves. This is useful because the polynomial we obtain is the product of the implicit equation and a polynomial that depends on $x$ and $y$ only. This makes it easy to separate the implicit equation, assuming that it depends on $z$ (which it will after a generic change of coordinates). As a result we do not need to factorize, and the GCD we compute involves only bivariate polynomials.

Before we present the efficient perturbation and carry out the analysis, we make certain assumptions on the given parametrization, $\overline{\mathbf{F}}$. They are:

1. The implicit representation is not independent of $z$. In other words, it is not of the form $H(x, y) = 0$.

2. $\overline{W}(s, t, u))$ does not divide $\overline{Z}(s, t, u)$. Otherwise the implicit representation is of the form $z - k = 0$, where $k = \frac{\overline{Z}(s,t,u)}{\overline{W}(s,t,u)}$ and we can compute it directly.

The base points blow up to rational curves of the form $(X(t), Y(t), Z(t), W(t))$ on the surface. Since these curves lie on the surface, they are characterized by the property that

$$H(\frac{X(t)}{W(t)}, \frac{Y(t)}{W(t)}, \frac{Z(t)}{W(t)}) = 0,$$

where $H(x, y, z)$ is the implicit representation of the surface. Lets consider the projection of one of these curves on the $X$-$Y$ plane. The projected curve has a rational parametrization of the form $(X(t), Y(t), W(t))$ and it can be implicitly represented as the zero set of an irreducible polynomial, say $F(x, y)$. Later we show that the lowest degree term of this efficient perturbation can be expressed as a product of $H(x, y, z)$ and $F(x, y)$ (corresponding to each seam curves).

Given a parametrization with base points, let us perturb one of the three parametric equations, (5), say $\overline{F}_3(s,t,u)$ and the resulting perturbed system is

$$\begin{aligned} \overline{G}_1(s,t,u) &= x\overline{W}(s,t,u) - \overline{X}(s,t,u) = 0, \\ \overline{G}_2(s,t,u) &= y\overline{W}(s,t,u) - \overline{Y}(s,t,u) = 0, \\ \overline{G}_3(s,t,u) &= z\overline{W}(s,t,u) - \overline{Z}(s,t,u) + \lambda\overline{Z}_1(s,t,u) = 0, \end{aligned} \tag{10}$$

where $\overline{Z}_1(s,t,u)$ is a homogeneous polynomial of degree $n$ such that

$$V(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u), \overline{Z}_1(s,t,u)) = \phi.$$

We will denote this perturbed parametrization as $\overline{\mathbf{G}}$. It is still possible that for all choices of $\overline{Z}_1(s,t,u)$ the resultant of $\overline{G}_1(s,t,u), \overline{G}_2(s,t,u)$ and $\overline{G}_3(s,t,u)$ is zero. Let

$$Q = V(\overline{G}_1, \overline{G}_2, \overline{G}_3) \subset P^2 \times C^3 \times C^1,$$

and $\Pi$ be the projection function from $P^2 \times C^3 \times C^1$ to $C^3 \times C^1$, as defined in (8).

THEOREM 3. *Given a set of three equations of the form,* $\overline{G}_1(s,t,u), \overline{G}_2(s,t,u)$ *and* $\overline{G}_3(s,t,u)$*, where* $\overline{Z}_1(s,t,u)$ *is chosen such that*

$$V(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u), \overline{Z}(s,t,u), \overline{Z}_1(s,t,u)) = \phi.$$

*The necessary and sufficient condition that the resultant of* $\overline{G}_1$*,* $\overline{G}_2$ *and* $\overline{G}_3$ *does not vanish is that*

$$\overline{P}(s,t,u) = GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u))$$

*is a constant.*

PROOF. *Necessity*

Let us assume that $\overline{P}(s,t,u)$ is a polynomial of positive degree. Let us consider the set

$$M = \{(s,t,u,x,y,z,\lambda) \mid \overline{P}(s,t,u) = 0, -\overline{Z}(s,t,u) + \lambda\overline{Z}_1(s,t,u) = 0\}$$

and

$$M \subset P^2 \times C^3 \times C^1.$$

Let $\mathbf{p} = (s_1, t_1, u_1, x_1, y_1, z_1, \lambda_1) \in M$. Thus,

$$\overline{P}(s_1,t_1,u_1) = 0$$

and therefore

$$\begin{aligned} \overline{G}_1(s_1,t_1,u_1) &= x_1\overline{W}(s_1,t_1,u_1) - \overline{X}(s_1,t_1,u_1) = 0 - 0 = 0, \\ \overline{G}_2(s_1,t_1,u_1) &= y_1\overline{W}(s_1,t_1,u_1) - \overline{Y}(s_1,t_1,u_1) = 0 - 0 = 0, \\ \overline{G}_3(s_1,t_1,u_1) &= z_1\overline{W}(s_1,t_1,u_1) - \overline{Z}(s_1,t_1,u_1) + \lambda_1\overline{Z}_1(s_1,t_1,u_1) \\ &= -\overline{Z}(s_1,t_1,u_1) + \lambda_1\overline{Z}_1(s_1,t_1,u_1) = 0. \end{aligned}$$

Thus, $\mathbf{p} \in V(\overline{G}_1, \overline{G}_2, \overline{G}_3) \Rightarrow \mathbf{p} \in Q$. In other words, $M \subset Q$. $M$ is a 4–dimensional set. Given any 4-tuple, $(x, y, z, \lambda) = (x_1, y_1, z_1, \lambda_1)$, one can find $(s_1, t_1, u_1)$ such that $(s_1, t_1, u_1, x_1, y_1, z_1, \lambda_1) \in M$. Thus, $M$ is an excess component of $Q$ and $\Pi(M)$ is a 4–dimensional set, too. Therefore the resultant of $\overline{G}_1, \overline{G}_2$ and $\overline{G}_3$ is zero.

*Sufficiency*

Let $\overline{P}(s,t,u)$ be a constant polynomial. To prove the non-vanishing of the resultant it is sufficient to show that there is some value of $x$, $y$, $z$ and $\lambda$ such that for those values $\overline{G}_1$, $\overline{G}_2$ and $\overline{G}_3$ have no common solution.

First pick $x = 0$. Now choose a value of $y$ so that $\overline{G}_2(s,t,u)$ has a finite number of intersections with $\overline{G}_1(s,t,u)_{x=0}$, i.e. $\overline{X}(s,t,u)$. Since $GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u))$ is a constant, for almost all values of $y$, $\overline{X}(s,t,u)$ and $\overline{G}_2(s,t,u)$ intersect in $n^2$ points, according to Bezout's theorem. Let these points be $(s_i, t_i, u_i)$, $\;1 \leq i \leq n^2$.

Once there are a finite number of solutions for the $\overline{G}_1(s,t,u)$ and $\overline{G}_2(s,t,u)$, it is easy to choose $z$ and $\lambda$ such that $\overline{G}_3(s_i,t_i,u_i) \neq 0$. At any of the $n^2$ solution points, say $(s_i,t_i,u_i)$, $\overline{X}(s_i,t_i,u_i) = 0$. Pick $z$ and $\lambda$ such that for each solution they satisfy the following constraint. The constraint depends on the value of $\overline{W}(s_i,t_i,u_i)$:

1. Case $\overline{W}(s_i,t_i,u_i) = 0$.

   The fact that $\overline{X}(s_i,t_i,u_i) = 0$ implies that $\overline{Y}(s_i,t_i,u_i) = 0$. The polynomial $\overline{Z}_1(s,t,u)$ is chosen to be non-zero at the common roots of $\overline{X}(s,t,u)$, $\overline{Y}(s,t,u)$ and $\overline{W}(s,t,u)$ and therefore, $\overline{Z}_1(s_i,t_i,u_i) \neq 0$. In this case

   $$\lambda \neq \frac{\overline{Z}(s_i,t_i,u_i)}{\overline{Z}_1(s_i,t_i,u_i)}.$$

2. Case $\overline{W}(s_i,t_i,u_i) \neq 0$.

   Let $\lambda$ take any value choose $z$ such that

   $$z \neq \frac{\overline{Z}(s_i,t_i,u_i) - \lambda \overline{Z}_1(s_i,t_i,u_i)}{\overline{W}(s_i,t_i,u_i)}.$$

Thus, for almost all choices of $z$ and $\lambda$, the given equations have no common solution and therefore, the resultant does not vanish. Q.E.D.

To circumvent this problem of vanishing resultant in certain cases we perform a change of coordinates and let the new parametrization be

$$\overline{\mathbf{F}}'(s,t,u) = (x', y', z', w') = (x, y + kz, z, w)$$

$$= (\overline{X}(s,t,u), \overline{Y}(s,t,u) + k\overline{Z}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)),$$

where $k$ is a scalar. The corresponding parametric equations are

$$\begin{aligned}\overline{G}_1'(s,t,u) &= x\overline{W}(s,t,u) - \overline{X}(s,t,u) = 0,\\ \overline{G}_2'(s,t,u) &= y\overline{W}(s,t,u) - \overline{Y}(s,t,u) - k\overline{Z}(s,t,u) = 0,\\ \overline{G}_3(s,t,u) &= z\overline{W}(s,t,u) - \overline{Z}(s,t,u) + \lambda\overline{Z}_1(s,t,u) = 0.\end{aligned}$$

Since $GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)) = 1$, for any generic $k$,

$$GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u) + k\overline{Z}(s,t,u), \overline{W}(s,t,u)) = 1,$$

too. We compute the implicit representation in terms of $x', y', z'$ and $w'$ and substitute them to obtain an implicit equation in terms of $x, y, z$ and $w$. From now onwards we assume that it is possible to choose $\overline{Z}_1(s,t,u)$ such that the resultant of $\overline{G}_1(s,t,u), \overline{G}_2(s,t,u)$ and $\overline{G}_3(s,t,u)$, $R(x,y,z,\lambda)$, is non–zero. Moreover the resultant can be expressed as a polynomial of the form

$$R(x,y,z,\lambda) \quad = \quad \lambda^i S(x,y,z,\lambda), \tag{11}$$

where $S(x,y,z,0) \neq 0$.

LEMMA 3. *The total number of base points (counted properly) of* $\overline{\mathbf{F}}$ *correspond to* $i$ *in (11).*

PROOF. Let $\overline{\mathbf{F}}$ has $m$ base points (counted properly). Base points of multiplicity $k$ are counted at least $k^2$ times. Thus, its implicit representation has degree $n^2 - m$. $R(x,y,z,\lambda)$ is the resultant of $\overline{G}_1(s,t,u)$, $\overline{G}_2(s,t,u)$ and $\overline{G}_3(s,t,u)$. $\overline{G}_1(s,t,u)$ and $\overline{G}_2(s,t,u)$ correspond to plane curves of degree $n$ each and according to Bezout's theorem intersect in $n^2$ points (counted properly). Let the points be $(s_i, t_i, u_i)$, $1 \leq i \leq n^2$. If $(s_0, t_0, u_0)$ is a base point of $\overline{\mathbf{F}}$, $\overline{G}_1(s_0,t_0,u_0) = \overline{G}_2(s_0,t_0,u_0) = 0$. Thus, the intersection set consist of these $m$ base points and $n^2 - m$ other intersections (which are functions of $x$ and $y$). Let $(s_j, t_j, u_j)$, $1 \leq j \leq m$ correspond to the base points. Using properties of resultants, highlighted in Salmon (1885), it follows

$$\begin{aligned} R(x,y,z,\lambda) &= \prod_{i=1}^{n^2} \overline{G}_3(s_i,t_i,u_i)\\ &= \prod_{i=1}^{n^2} (z\overline{W}(s_i,t_i,u_i) - \overline{Z}(s_i,t_i,u_i) + \lambda\overline{Z}_1(s_i,t_i,u_i))\\ &= \alpha\lambda^m \prod_{i=m+1}^{n^2} \overline{G}_3(s_i,t_i,u_i), \end{aligned}$$

where $\alpha = \prod_{i=1}^{m} \overline{Z}_1(s_i,t_i,u_i) \neq 0$. Thus, the lowest degree term in $\lambda$ in $R(x,y,z,\lambda)$ has degree at least $m$. Since the points, $(s_i,t_i,u_i)$, $m < i \leq n^2$ do not correspond to the base points, at least $\overline{W}(s_i, t_i, u_i)$ or $\overline{Z}(s_i, t_i, u_i)$ does not vanish. Thus, the lowest degree term of the resultant has degree exactly equal to $m$. Q.E.D.

For a generic choice of $\overline{Z}_1(s, t, u)$ it is possible to show that $S(x, y, z, \lambda)$ is an irreducible polynomial. This follows from the fact, that for any generic choice of $\lambda = \lambda_i$, the resulting parametrization $\overline{\mathbf{G}}$ has no base points and $R(x, y, z, \lambda_i)$ corresponds to its implicit representation. Therefore, $R(x, y, z, \lambda_i)$ is equal to some power of an irreducible polynomial and for a generic choice of $\overline{Z}_1(s, t, u)$, $R(x, y, z, \lambda_i)$ is an irreducible polynomial. Thus, $V(R(x, y, z, \lambda))$ consist of the following components:

1. $V(\lambda)$ of multiplicity $i$.
2. $V(S(x, y, z, \lambda))$.

As a result, $Q$ consists of $i + 1$ components. $i$ of these components are of the form

$$\{(s_0, t_0, u_0, x, y, z, 0), \}$$

where $(s_0, t_0, u_0)$ is a base point and the $(i + 1)st$ component can be represented as

$$Q^{'} = \{\mathbf{q} = (s_j, t_j, u_j, x_j, y_j, z_j, \lambda_j) \mid \mathbf{q} \in Q,\ S(x_j, y_j, z_j, \lambda_j) = 0\}.$$

Let us express the resultant as a polynomial in $\lambda$, and let $P_i(x, y, z)$ be the constant term of $S(x, y, z, \lambda)$. We know from Theorem 2 that

$$P_i(x, y, z) = H(x, y, z)F(x, y, z),$$

where $H(x, y, z)$ corresponds to some power of the implicit equation and $F(x, y, z)$ is the extraneous factor. Our aim is to extract $H(x, y, z)$ without resorting to multivariate factorization.

THEOREM 4. *$F(x, y, z)$ is independent of $z$. In other words $F(x, y, z)$ is a bivariate polynomial in $x$ and $y$. Moreover, $F(x, y, z)$ corresponds exactly to the projections of the seam curves on the $X$-$Y$ plane.*

PROOF. Every component of $Q$ has dimension 3. Let $P$ and $B$ be algebraic sets as defined in the proof of Theorem 3. For every point $\mathbf{q} \in B \times \{0\}$, there is a sequence of points $(\mathbf{q}_j) \in Q^{'} - B \times \{0\}$ in its neighbourhood, which converges to $\mathbf{q}$. Furthermore, $\mathbf{q}$ has a 3-dimensional neighbourhood for defining such sequence of points. As a result we are able to show that $H(x, y, z) \mid P_i(x, y, z)$. Let $(s_0, t_0, u_0)$ be a base point of $\overline{\mathbf{F}}(s, t, u)$ and let $\mathbf{q} = (s_0, t_0, u_0, x_0, y_0, z_0, 0)$, where $(x_0, y_0, z_0)$ is a point on the seam curve corresponding to $(s_0, t_0, u_0)$. Let $(x_j, y_j, z_j, \lambda_j)$ be a point in the neighbourhood of $(x_0, y_0, z_0, 0)$ such that $S(x_j, y_j, z_j, \lambda_j) = 0$. For each such $(x_j, y_j, z_j, \lambda_j)$ there exists $(s_j, t_j, u_j)$ such that $\mathbf{q}_j = (s_j, t_j, u_j, x_j, y_j, z_j, \lambda_j) \in Q^{'} - B \times \{0\}$. As a result we are able to define a sequence of points $\mathbf{q}_j$ converging to $\mathbf{q}$. Corresponding to every point in this sequence let us consider another sequence of points $\mathbf{q}_j^{'} = (s_j, t_j, u_j, x_j, y_j, z_j^{'}, \lambda_j^{'})$ such that

$$\begin{aligned} z_j^{'} &= kz_j, \\ \lambda_j^{'} &= \frac{-kz_j\overline{W}(s_j,t_j,u_j) + \overline{Z}(s_j,t_j,u_j)}{\overline{Z}_1(s_j,t_j,u_j)}, \end{aligned}$$

where $k$ is any arbitrary constant. The fact $\mathbf{q}_j \in Q^{'}$ implies that $\mathbf{q}_j^{'} \in Q^{'}$. As a result $R(x_j, y_j, kz_j, \lambda_j^{'}) = 0$.

Consider the sequence of points approaching $\mathbf{q}$, and from the limiting argument it follows that $(x_0, y_0, z_0) \in V(P_i(x, y, z))$. Moreover,

$$\lim_{(s_j,t_j,u_j)\rightarrow(s_0,t_0,u_0)} \lambda_j^{'} = \lim_{(s_j,t_j,u_j)\rightarrow(s_0,t_0,u_0)} \frac{-kz_j\overline{W}(s_j,t_j,u_j) - \overline{Z}(s_j,t_j,u_j)}{\overline{Z}_1(s_j,t_j,u_j)} = 0.$$

This is because $(s_0, t_0, u_0)$ is a base point and therefore, $\overline{W}(s_0, t_0, u_0) = 0$, $\overline{Z}(s_0, t_0, u_0) = 0$ and $\overline{Z}_1(s_0, t_0, u_0) \neq 0$. Thus, $\mathbf{q}_j^{'} \rightarrow (s_0, t_0, u_0, x_0, y_0, kz_0, 0)$ and from the limiting arguments it follows that $P_i(x_0, y_0, kz_0) = 0$. Furthermore, $(x_0, y_0, z_0)$ can correspond to any point on the seam curve and the choice of $k$ is arbitrary.

The fact $P_i(x_0, y_0, kz_0) = 0$ implies either $H(x_0, y_0, kz_0) = 0$ or $F(x_0, y_0, kz_0) = 0$. We have assumed that $H(x, y, z)$ is not independent of $z$ and therefore, it is not possible that for all points $(x_0, y_0, z_0)$ on a seam curve $(x_0, y_0, kz_0) \in V(H(x, y, z))$, for any choice of $k$. Therefore, $F(x_0, y_0, kz_0) = 0$ for all $k$. Since $V(F(x, y, z))$ is a polynomial in $x$, $y$ and $z$, this is possible if and only if $F(x_0, y_0, z) = 0$ for all such $(x_0, y_0)$, where $x_0$ and $y_0$ represent the $x$ and $y$ coordinates of a point on a seam curve. Let $\beta_j(x, y)$, $1 \leq j \leq m$ correspond to the implicit representation of the projection of seam curves (where $m$ correspond to the number of seam curves and $m \leq i$) on the $X$-$Y$ plane and therefore

$$V(\beta_j(x, y)) \subset V(F(x, y, z)), \quad for \;\; 1 \leq j \leq m.$$

It is still possible that $V(P_i(x, y, z))$ may consist of some other component, besides the implicit representation and the projection of seam curves. Let that component be the zero set of $\alpha(x, y, z)$. Since $\alpha(x, y, z)$ is distinct from $H(x, y, z)$ and $\beta_j(x, y)$s, there exist $(x_1, y_1, z_1) \in V(\alpha(x, y, z))$ such that $H(x_1, y_1, z_1) \neq 0$ and $\beta_j(x_1, y_1) \neq 0$.

Let us consider the point $\mathbf{p} = (s_0, t_0, z_0, x_1, y_1, z_1, 0)$. Since $\mathbf{p} \in Q^{'}$, we can choose a sequence $\mathbf{p}_j = (s_j, t_j, z_j, x_j, y_j, z_j, \lambda_j)$ in the neighbourhood of $\mathbf{p}$ such that $\mathbf{p}_j \in Q^{'}$. We can similarly choose a sequence $\mathbf{p}_j^{'} = (s_j, t_j, u_j, x_j, y_j, kz_j, \lambda_j^{'})$, such that $\mathbf{p}_j^{'} \in Q^{'}$, and from the argument used above it follows that $(x_1, y_1, kz_1) \in V(\alpha(x, y, z))$ for all $k$. Thus, $\alpha(x, y, z)$ is independent of $z$ and we may represent it as $\alpha(x, y)$. Moreover $\alpha(x_1, y_1) = 0$.

A seam curve corresponding to $(s_0, t_0, u_0)$ is the set of limit points $(\overline{x}, \overline{y}, \overline{z})$ such that $\mathbf{q} = (s_0, t_0, u_0, \overline{x}, \overline{y}, \overline{z}, 0) \in Q^{'}$ and $\mathbf{q}$ has a 3-dimensional neighbourhood in $Q^{'}$. Since

$S(x_1, y_1, z, 0) = 0$ and $\mathbf{p} = (s_0, t_0, u_0, x_1, y_1, z, 0) \in Q'$ for all $z$, there exists a sequence of points $\mathbf{p}_j \in Q'$ in the neighbourhood of $\mathbf{p}$. The fact that there exists such a sequence implies that $(x_1, y_1)$ must correspond to the $(x, y)$ coordinates of a point on a seam curve, which is contrary to our assumption.

Thus, $F(x, y, z)$ exactly corresponds to the projections of all the seam curves on the $X$-$Y$ plane. Q.E.D.

From now onwards we will represent the lowest degree term of the resultant of the perturbed system as

$$P_i(x, y, z) = H(x, y, z)F(x, y),$$

where $F(x, y)$ is the extraneous factor. Our aim is to extract $F(x, y)$ out of $P_i(x, y, z)$ without resorting to multivariate factorization. Let $P_i(x, y, z)$ and $H(x, y, z)$ be polynomials of degree $d$ $(d > 0)$ and they can be expressed as

$$P_i(x, y, z) = p_0(x, y) + p_1(x, y)z + p_2(x, y)z^2 + \ldots + p_d(x, y)z^d,$$

$$H(x, y, z) = h_0(x, y) + h_1(x, y)z + \ldots, +h_d(x, y)z^d,$$

Since $H(x, y, z)$ corresponds to some power of an irreducible polynomial

$$GCD(h_d(x, y), h_{d-1}(x, y), \ldots, h_0(x, y)) = 1.$$

As a result,

$$GCD(p_0(x, y), p_1(x, y), p_2(x, y), \ldots, p_d(x, y)) = F(x, y).$$

Hence, we can extract the extraneous factor by taking the GCD of $d + 1$ bivariate polynomials.

In general, for almost all two distinct values of $z$, say $z_1$ and $z_2$,

$$F(x, y) = GCD(P_i(x, y, z_1), P_i(x, y, z_2))$$

Thus, the implicit equation can be represented as

$$H(x, y, z) = \frac{P_i(x, y, z)}{GCD(P_i(x, y, z_1), P_i(x, y, z_2))}.$$

Let the parametric equations be polynomials belonging to a ring $\mathcal{F}[s, t, u]$.

COROLLARY 1. *If $\mathcal{F}$ is an infinite field, there exists an implicit equation belonging to the ring $\mathcal{F}[x, y, z]$.*

PROOF. If the parametrization has no base points, then the implicit equation corresponds to the resultant expressed as determinant of a matrix. Each entry of the matrix is of the form $ax + by + cz + d$, where $a, b, c, d \in \mathcal{F}$, and therefore the coefficients of the implicit equation belong to the same field.

If the parametrization has base points, we can always choose a perturbing polynomial $\overline{Z}_1(s,t,u) \in \mathcal{F}(s,t,u)$ and let $R(x,y,z,\lambda)$ be the resultant of the perturbed system. Each coefficient of $R(x,y,z,\lambda)$ and therefore, of $P_i(x,y,z,\lambda)$ lies in $\mathcal{F}$. The implicit equation can be expressed as ratio of two polynomials, whose coefficients belong to $\mathcal{F}$. Thus, the implicit equation has the same coefficient field as the parametric equations. Q.E.D.

# 7. Rational Parametrization of Seam Curves

In the previous section we presented the technique for computing the implicit representation from the parametrization by making use of the GCD operation. The extraneous factor corresponds to the projection of seam curves on the $X$-$Y$ plane. Given a parametrization, $\overline{\mathbf{F}}$, we can use efficient perturbation and perturb the equations containing the $x$ and $y$ variable so that we are able to compute the projections of seam curves on the $Y-Z$ and $X-Z$ planes, respectively. Given these projections, we present an algorithm to compute the rational parametrizations of seam curves.

Perform a transformation on the coordinates of a parametrization and let the projections of the seam curves of the resulting parametrization be $P(x,y)$, $Q(y,z)$ and $R(x,z)$ on the $X$-$Y$, $Y$-$Z$ and $X$-$Z$ planes, respectively. For a generic transformation, each of these polynomials would consist of projections of all the seam curves.

Every rational space curve is birationally equivalent to an algebraic plane curve, as explained in Walker (1950). For a generic choice of coordinates such a birational equivalence can be established between a space curve $\mathbf{B}(t) = (x(t), y(t), z(t), w(t)$ and its projection on $X$-$Y$ plane, $\mathbf{C}(t) = (x(t), y(t), w(t)$. In our case, $P(x,y)$ is the product of the implicit representations of $\mathbf{C}(t)$ corresponding to each seam curve. Thus, given $P(x,y)$ we use a factorization algorithm to decompose it into irreducible polynomials of the form

$$P(x,y) = P_1(x,y)P_2(x,y)\ldots P_m(x,y),$$

where $P_i(x,y)$ is an irreducible polynomial. The factorization algorithms are given in Kaltofen (1983).

Each plane curve, $P_i(x,y) = 0$, is a curve of *genus 0* and therefore, has a rational parametrization. Given any algebraic plane curve of *genus 0* techniques of computing its rational parametrization are well known in algebraic geometry, as explained in Walker (1950). The computational details are worked out in Abhyankar & Bajaj (1988). Thus, we are able to compute the rational parametrization, $\mathbf{C}_i(t) = (x(t), y(t), w(t))$ of the projection of each seam curve.

For the choice of coordinates it is assumed that each seam curve

$$\mathbf{B}_i(t) = (x_i(t), y_i(t), z_i(t), w_i(t))$$

is birationally equivalent to $\mathbf{C}_i(t)$. Thus, our problem is reduced to computing the rational function

$$z = \frac{\phi(x,y)}{\psi(x,y)}$$

expressing the relation between the $x$, $y$ and $z$ coordinates of almost all the points on any seam curve.

## 7.1. Remainder Sequences

Let us treat $Q(y,z)$ and $R(x,z)$ as polynomials in $z$ and its coefficients are in the ring $\mathcal{F}[x,y]$. Without loss of generality we assume that the degree of $R(x,z)$ is less than or equal to that of $Q(y,z)$. Let

$$S_1(z) = Q(y,z),$$

$$S_2(z) = R(x,z),$$

$$\alpha_i S_i(z) = \beta_i S_{i+1}(z) - S_{i+2}(z),$$

where $S_i(z) \in \mathcal{F}[x,y][z]$, $degree(S_{i+2}(z)) < degree(S_{i+1}(z))$ for $1 \leq i \leq d$ and $\alpha_i$, $\beta_i \in \mathcal{F}[x,y][z]$ such that

$$GCD(\alpha_i, \beta_i) = 1.$$

The sequence $S_1(z), S_2(z), \ldots, S_k(z)$ is a *remainder sequence*, as defined in Loos (1983). $S_k(z)$ is independent of $z$ and corresponds to the resultant of $Q(y,z)$ and $R(y,z)$ with respect to $z$. Let $(x_1, y_1, z_1)$ be any point lying on any seam curve. Thus,

$$osP(x_1, y_1) = 0; \quad Q(y_1, z_1) = 0; \quad R(x_1, z_1) = 0.$$

The fact $S_i(z_1)_{x=x_1, y=y_1} = 0$ and $S_{i+1}(z_1)_{x=x_1, y=y_1} = 0$ implies that

$$S_{i+2}(z_1)_{x=x_1, y=y_1} = 0.i$$

As a result all the polynomials in the remainder sequence vanish when $(x,y,z)$ corresponds to any point on any seam curve. Let's consider the polynomial $S_{k-1}(z)$, which is a linear function in $z$ and can be expressed as

$$S_{k-1}(z) = \psi(x,y)z - \phi(x,y),$$

where $\phi(x,y)$ and $\psi(x,y)$ are polynomials in $x$ and $y$. Since this polynomial vanishes for all points on any seam curve, the points on a seam satisfy the equation

$$z = \frac{\phi(x,y)}{\psi(x,y)}. \tag{12}$$

Thus, the rational parametrizations of the seam curves are

$$\mathbf{B}_i(t) = (\frac{x_i(t)}{w_i(t)}, \frac{y_i(t)}{w_i(t)}, \frac{\phi(\frac{x_i(t)}{w_i(t)}, \frac{y_i(t)}{w_i(t)})}{\psi(\frac{x_i(t)}{w_i(t)}, \frac{y_i(t)}{w_i(t)})})$$

corresponding to each $\mathbf{C}_i(t)$.

EXAMPLE 3. Let's consider the parametrization of a sphere (same as Example 1)

$$\mathbf{F}(s,t) = (x,y,z) = (\frac{s^2-1-t^2}{s^2+1+t^2}, \frac{2s}{s^2+1+t^2}, \frac{2st}{s^2+1+t^2}).$$

Since the parametrization has base points, let's perturb the given system and the corresponding parametric equations are

$$\begin{aligned}
\overline{G}_1(s,t,u) &= x(s^2+t^2+1)-(s^2-1-t^2)=0,\\
\overline{G}_2(s,t,u) &= y(s^2+t^2+1)-2s=0,\\
\overline{G}_3(s,t,u) &= y(s^2+t^2+1)-2st+\lambda(2s^2+3t^2+4)=0,
\end{aligned}$$

The resultant, $R(x,y,z,\lambda)$ is a polynomial in the four variables and the lowest degree of $\lambda$ is 2 (equal to the number of base points in $\overline{\mathbf{F}}$). The coefficient of $\lambda^2$ is

$$\begin{aligned}
P_2(x,y,z) &= -64-128x+128x^3+64x^4+64y^2+128xy^2\\
&\quad +64x^2y^2+64z^2+128xz^2+64x^2z^2.
\end{aligned}$$

Choose 2 generic values of $z$, say $z=1$ and $z=2$ and the extraneous factor is

$$F(x,y) = GCD(P_i(x,y,1), P_i(x,y,2)) = 64+128x+64x^2.$$

Thus, the implicit equation is

$$H(x,y,z) = \frac{P_i(x,y,z)}{F(x,y)} = x^2+y^2+z^2-1.$$

Apply a linear transformation on the coordinates and obtain

$$\begin{aligned}
x &= \overline{x}-2\overline{y}-\overline{z}\\
y &= \overline{x}-\overline{y}-\overline{z}\\
z &= -\overline{y}-\overline{z}
\end{aligned}$$

and the inverse transformation is

$$\begin{aligned}
\overline{x} &= y-z\\
\overline{y} &= y-x\\
\overline{z} &= x-y-z
\end{aligned} \tag{13}$$

The resulting parametrization is

$$\mathbf{F}^{'} = (2s - 2st, 2s - s^2 + 1 + t^2, s^2 - 1 - t^2 - 2s - 2st, s^2 + t^2 + 1).$$

This parametrization has the same base points as $\overline{\mathbf{F}}$ and we perturb each of the parametric equations to obtain the following extraneous factors, which correspond to the projections of seam curves on $X^{'} - Y^{'}$, $Y^{'} - Z^{'}$ and $X^{'} - Z^{'}$ planes.

$$\begin{aligned} P(\overline{x}, \overline{y}) &= 2 + 2\overline{x} + \overline{x}^2 - 4\overline{y} - 2\overline{xy} + 2\overline{y}^2 \\ Q(\overline{y}, \overline{z}) &= 1 - 2\overline{y} + 2\overline{y}^2 + 2\overline{yz} + \overline{z}^2 \\ R(\overline{x}, \overline{z}) &= 1 + \overline{x}^2 + 2\overline{z} + \overline{z}^2 \end{aligned}$$

$P(\overline{x}, \overline{y})$ can be factorized as

$$P(\overline{x}, \overline{y}) = ((y-1)(i-1) - ix)((y-1)(i+1) - ix),$$

where $i = \sqrt{-1}$. The resulting parametrizations are

$$\mathbf{C}_1(t^{'}) = (\overline{x}, \overline{y}, \overline{w}) = (it^{'} - t^{'}, it^{'} + 1, 1)$$

and

$$\mathbf{C}_2(t^{'}) = (\overline{x}, \overline{y}, \overline{w}) = (-it^{'} - t^{'}, -it^{'} + 1, 1).$$

Let's consider the polynomial remainder sequence defined as

$$S_1(\overline{z}) = Q(\overline{y}, \overline{z}),$$

$$S_2(\overline{z}) = R(\overline{x}, \overline{z}).$$

As a result

$$S_3(\overline{z}) = S_1(\overline{z}) - S_2(\overline{z}) = -2\overline{y} + 2\overline{y}^2 + 2\overline{yz} - \overline{x}^2 - 2\overline{z}.$$

Since $S_3(\overline{z})$ is a linear polynomial in $\overline{z}$, we are able to express the rational function from the plane curves to the space curves as

$$z = \frac{\phi(x,y)}{\psi(x,y)} = \frac{\overline{x}^2 + 2\overline{y} - 2\overline{y}^2}{2\overline{y} - 2}.$$

Thus,

$$\mathbf{B}_1(t^{'}) = (\overline{x}, \overline{y}, \overline{z}, \overline{w}) = (it^{'} - t^{'}, it^{'} + 1, -1 - t^{'} - it^{'}, 1)$$

and

$$\mathbf{B}_2(t^{'}) = (\overline{x}, \overline{y}, \overline{z}, \overline{w}) = (-it^{'} - t^{'}, -it^{'} + 1, -1 - t^{'} + it^{'}, 1)$$

Now we can apply the inverse transform according to (13) and obtain the parametrization of the original seam curves as

$$\mathbf{B}_1(t) = (x, y, z, w) = (-1, it, t, 1)$$

and

$$\mathbf{B}_2(t) = (x, y, z, w) = (-1, -it, t, 1)$$

These seam curves lie on the surface and we can verify that by substituting their parametrizations into the surface equation, $H(x, y, z) = 0$.

# 8. Conclusion

In this paper we presented algorithms to compute the implicit representation of rational parametric surfaces. If a parametrization has no base points the implicit representation corresponds to a matrix determinant (using results from Elimination theory), otherwise we use perturbation techniques. In particular, we presented an efficient perturbation such that the computation of implicit representation involved GCD of polynomials as opposed to multivariate factorization. Moreover, the extraneous factors obtained can be used to compute the rational parametrizations of seam curves. The techniques presented in this paper can be used to implicitize rational hypersurfaces in higher dimensional space. The implicit equation can always be extracted from the resultant of the perturbed system by GCD operation and the extraneous factors can be used to compute the images of base points (and its higher dimensional equivalents). This follows from the proofs of Theorems 2 and 4, which utilize the properties of the algebraic sets defined by the parametric equations for a given parametrization. More details on the implementation of the algorithm and its performance are given in Manocha & Canny (1992).

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