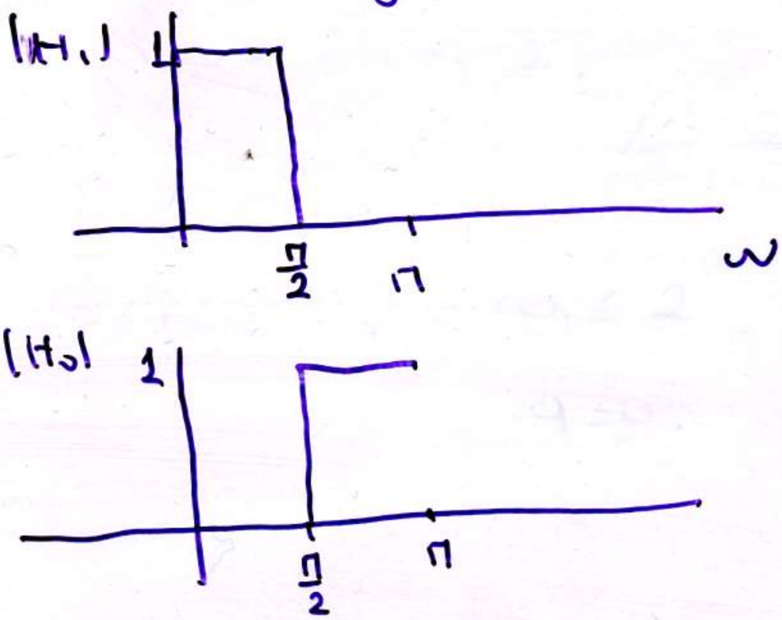
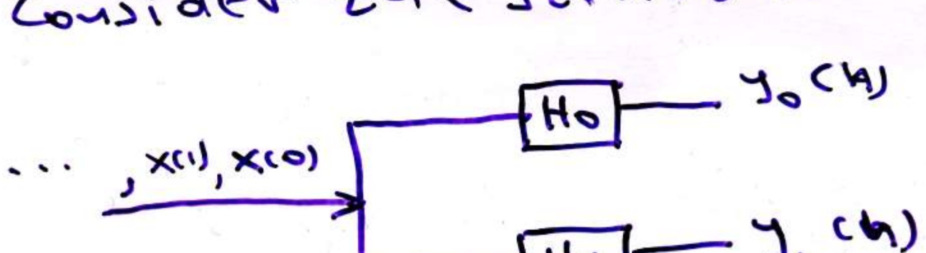
The Discrete Wavelet Transform

- Let $x(0), x(1)$ the set of measurements. Let also two ideal filters, lowpass and highpass



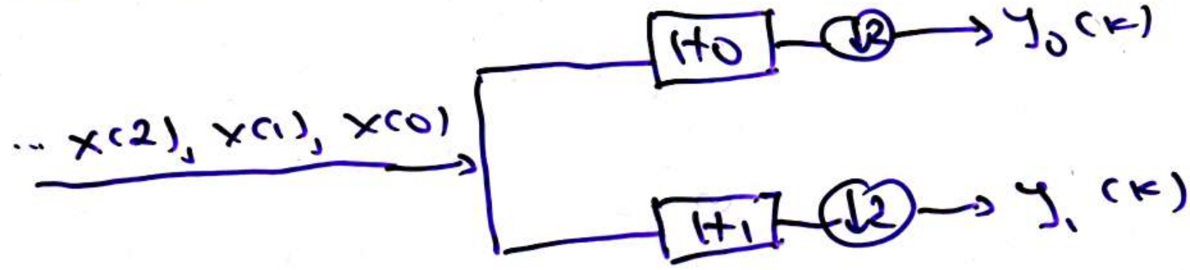
Let $h_0(k), h_1(k)$ the respective impulse responses

- Consider the structure



$y_0(n)$ is the high-pass content of $x(n)$ and $y_1(n)$ the low pass. Both are oversampled since their bandwidth is half of that of $x(n)$

- Subsample $y_0(n)$, $y_1(n)$. The following results



$$y_0(k) = \sum_{\ell} x(\ell)\, h_0(n-\ell)\Big|_{n=2k}$$

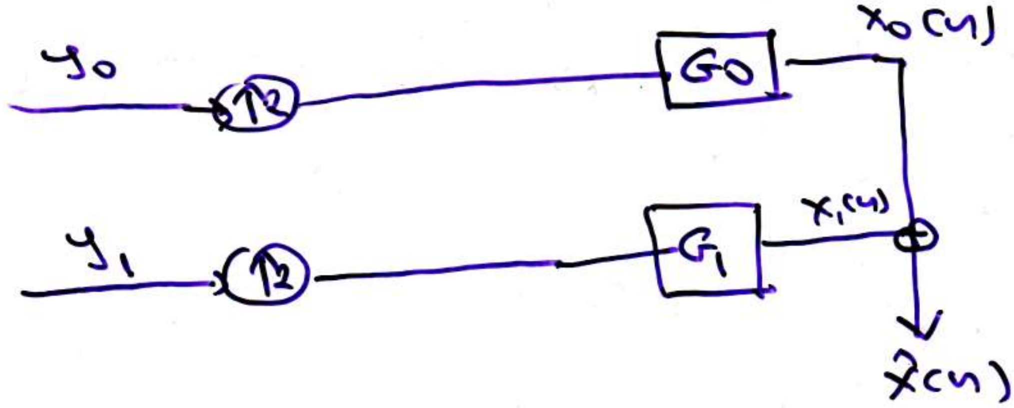
$$y_1(k) = \sum_{\ell} x(\ell)\, h_1(n-\ell)\Big|_{n=2k}$$

- Collect $y_0(k)$, $y_1(k)$ together

$$\begin{bmatrix} \vdots \\ y_0(0) \\ y_1(0) \\ y_0(1) \\ y_1(1) \\ y_0(2) \\ y_1(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} \cdots & h_0(2) & h_0(1), & h_0(0) & h_0(-1) & \cdots & \\ \cdots & \boxed{h_1(2)} & h_1(1), & h_1(0) & h_1(-1) & \cdots & \\ \cdots & \cdots & \cdots & h_0(2) & h_0(1) & \cdots \\ \cdots & \cdots & \cdots & \boxed{h_1(2)} & h_1(1) & \cdots \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ \vdots \end{bmatrix}$$

$$\underline{y} = T_i\, \underline{x}$$

- Can we reobtain $x(n)$ from $y_0(k)$, $y_1(k)$?? The answer is
YES



- The input to G_0 and G_1 are

$$\ldots 0 \; y_0(0) \; 0 \; y_0(1) \; 0 \; y_0(2) \ldots$$

$$\ldots 0 \; y_1(0) \; 0 \; y_1(1) \; 0 \; y_1(2) \ldots$$

Hence

$$x_0(n) = \sum_k y_0(k) \, g_0(n-2k)$$

$$x_1(n) = \sum_k y_1(k) \, g_1(n-2k)$$

$$\begin{bmatrix} \vdots \\ \hat{x}(0) \\ \hat{x}(1) \\ \hat{x}(2) \\ \hat{x}(3) \\ \vdots \end{bmatrix} = \begin{bmatrix} & \vdots & & & \vdots \\ \cdots & g_0(0) & \boxed{g_1(0)} & g_0(-2) & g_1(-2) & \cdots \\ \cdots & g_0(1) & g_1(1) & g_0(-1) & g_1(-1) & \cdots \\ \cdots & g_0(2) & g_1(2) & g_0(0) & \boxed{g_1(0)} & \cdots \\ \cdots & g_0(3) & g_1(3) & g_0(1) & g_1(1) & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \end{bmatrix} \begin{bmatrix} y_0(0) \\ y_1(0) \\ y_0(1) \\ y_1(1) \\ \vdots \end{bmatrix}$$

$$\underline{\hat{x}} = T_0 \, \underline{y} = T_0 T_1 \, \underline{x} \Rightarrow$$

$T_0 T_1 = I$ for perfect reconstruction

$$\boxed{\hat{x} = x}$$

- $T_0 . T_1 = I \Longrightarrow$

$$\sum_n h_i(2k-n)\, g_j(n-2l) = \delta_{kl}\,\delta_{ij}$$

$i, j = 0, 1$

Biorthogonality condition

- Some examples:

$$\sum_n h_1(2k-n)\, g_0(n-2l) = 0$$

$$\sum_n h_1(2k-n)\, g_1(n-2l) = 0$$

$$\sum_n h_1(2k-n)\, g_1(n-2k) = 1$$

- Under the biorthogonality condition

$$x(n) = \sum_k y_0(k)\, g_0(n-2k) +$$

$$\sum_k y_1(k)\, g_1(n-2k)$$

- Remarks:
  - The above is an analysis of $x(n)$ to a set of basis functions
    $$\{ g_0(n-2k), g_1(n-2k) \}$$
  - These are shifted versions of two basic functions
    $$g_0(n), \quad g_1(n)$$
  - The cofficients $y_i(k)$ of the expansion are obtained from $x(n)$ and the biorthogonal functions of $g_0(n)$, $g_1(n)$,
    $$\boxed{y_i(k) = \sum_n x(n)\, h_i(2k-n)}$$
  - The expansion is obtained via the $y_i(k)$,
    $$\boxed{x(n) = \sum_{i=0}^{1} \sum_k y_i(k)\, g_i(n-2k)}$$
  - If $h_i(n)$, $h_j(n)$ are themselves orthogonal,
    $$\sum_n h_i(2k-n)\, h_j(2\ell-n) = \delta_{k\ell}\,\delta_{ij}$$

- Remarks:
  - The basis sequences for each level i are power of 2 shifts of a basic mother sequence

$$\boxed{\begin{aligned} \psi_{ik}(n) &= \psi_{i0}(n - 2^r k) \\ \phi_{ik}(n) &= \phi_{i0}(n - 2^r k) \end{aligned}}$$

$r = i+1 \quad i \neq J-1$
$r = J-1 \quad i = J-1$

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  - The power of 2 shifts are the result of the successive splitting by two in the tree structured filter banks

△ Multiresolution Interpretation

- Assume G_0 and G_1 are ideal high and low pass. Observe that this is the case for orthogonal filter banks, where H_0 is low pass and H_1 is high pass and

$$g_i(n) = h_i(-n)$$

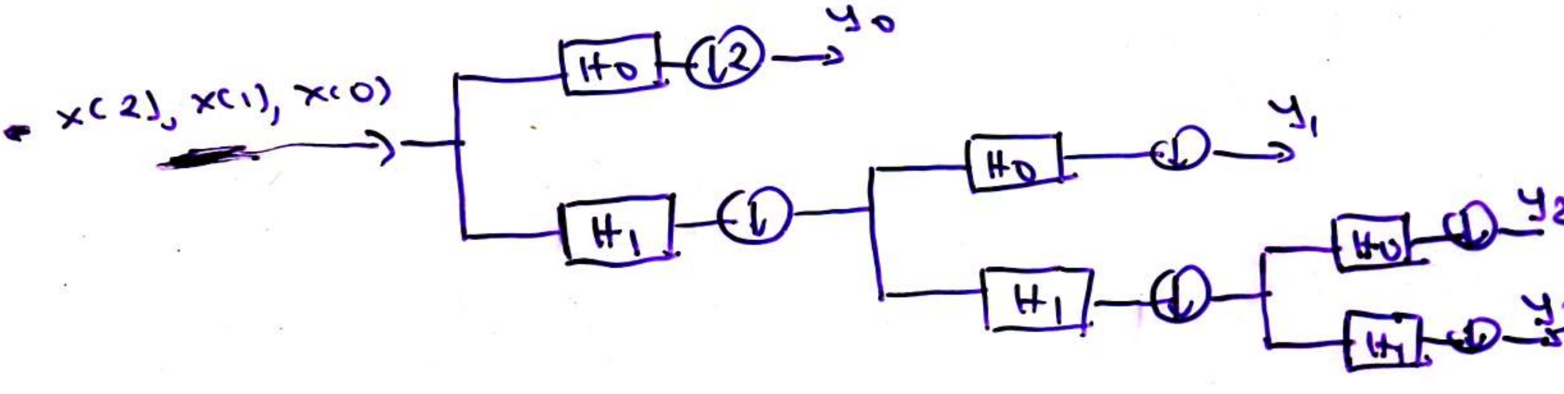
then

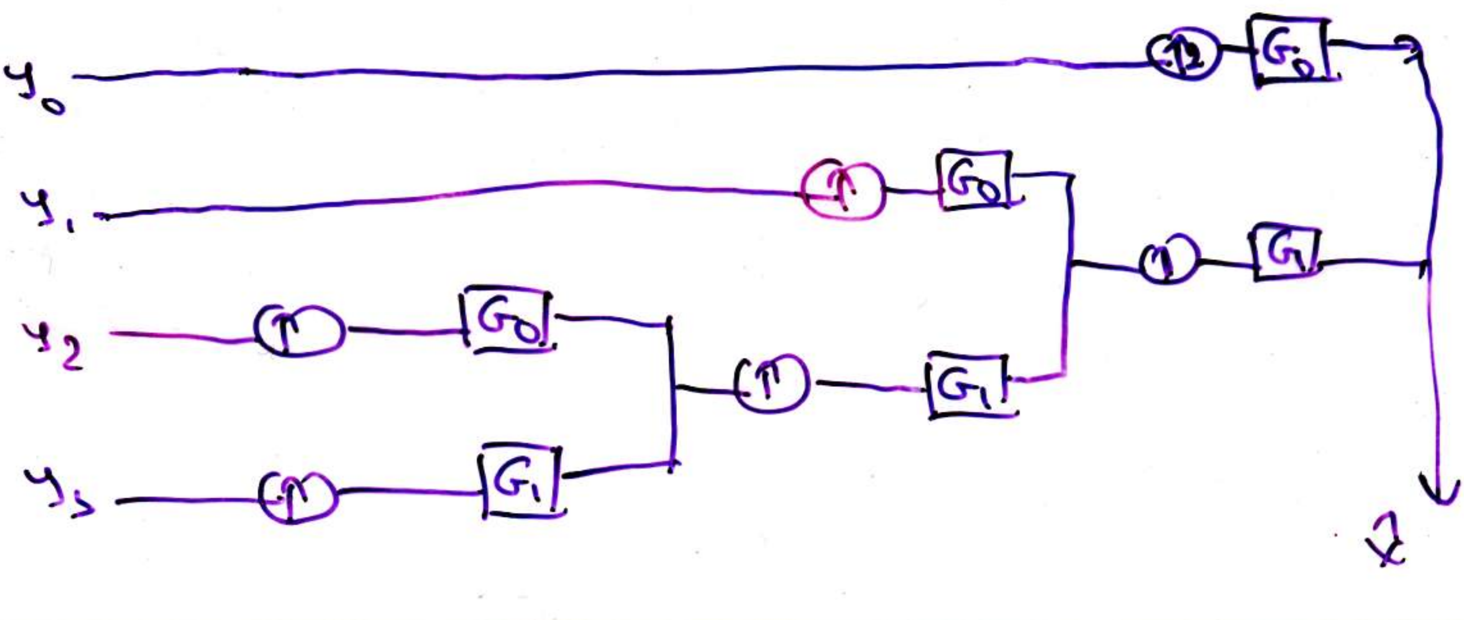
$$g_i(n) = h_i(-n)$$

- The filters $h_i(n)$ are known as **Analysis** and the $g_i(n)$ as **Synthesis** filters.

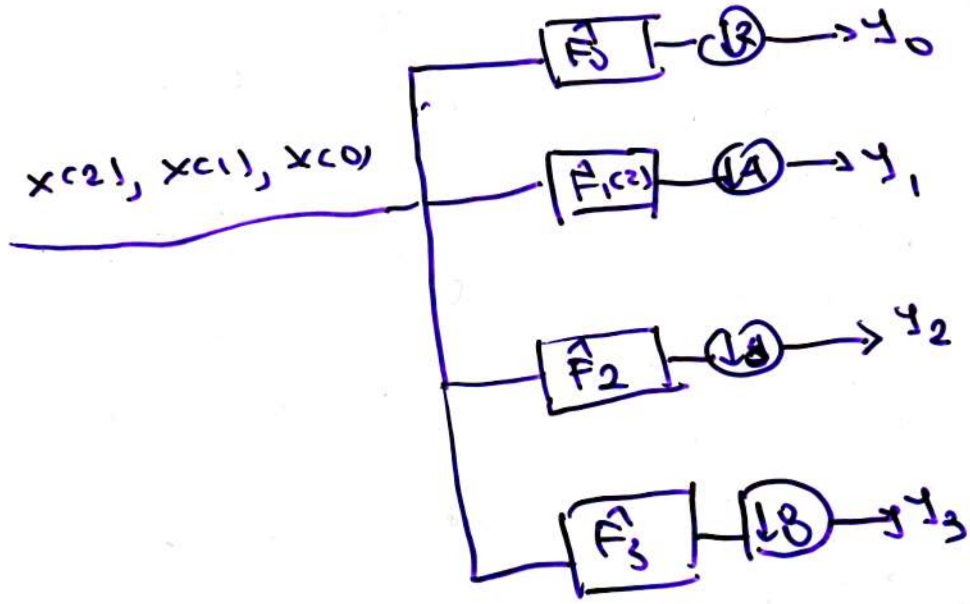
- The coefficients $y_0(k), y_1(k)$ are known as the **Discrete Wavelet Coefficients**.

▲ Extension to many bands





- Equivalently

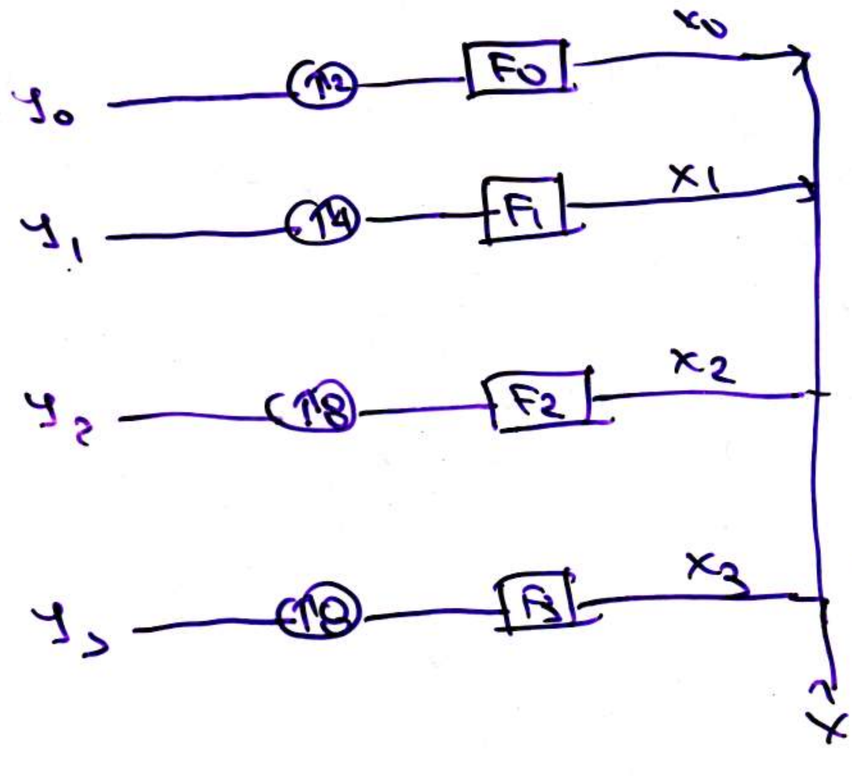


$\hat F_0(z) = H_0(z)$

$\hat F_1(z) = H_0(z^2)H_1(z)$

$\hat F_2(z) = H_0(z^4)H_1(z^2)H_1(z)$

$\hat F_3(z) = H_1(z^4)H_1(z^2)H_1(z)$



$F_0(z) = G_0(z)$

$F_1(z) = G_0(z^2)G_1(z)$

$F_2(z) = G_0(z^4)G_1(z^2)G_1(z)$

$F_3(z) = G_1(z^4)G_1(z^2)G_1(z)$

- If H_0, H_1, G_0, G_1 satisfy the biorthogonality condition

$\hat x(n) = x(n)$

- For J bands:

$$x(n) = \sum_{i=0}^{J-2} \sum_k y_i(k) f_i(n - 2^{i+1} k) + \sum_k y_{J-1}(k) f_{J-1}(n - 2^{J-1} k)$$

$$y_i(k) = \sum_n x(n) \hat{f}_i(2^{i+1} k - n) \quad i = 0, 1, \dots J-2$$

$$y_{J-1}(k) = \sum_n x(n) \hat{f}_{J-1}(2^{J-1} k - n)$$

- Set:

$$\psi_{ik}(n) = f_i(n - 2^{i+1} k) \quad i = 0, 1, \dots, J-2$$

$$\psi_{(J-1)k}(n) = f_{J-1}(n - 2^{J-1} k)$$

$$\phi_{ik}(n) = \hat{f}_i(2^{i+1} k - n) \quad i = 0, 1, \dots, J-2$$

$$\phi_{(J-1)k}(n) = \hat{f}_{J-1}(2^{J-1} k - n)$$

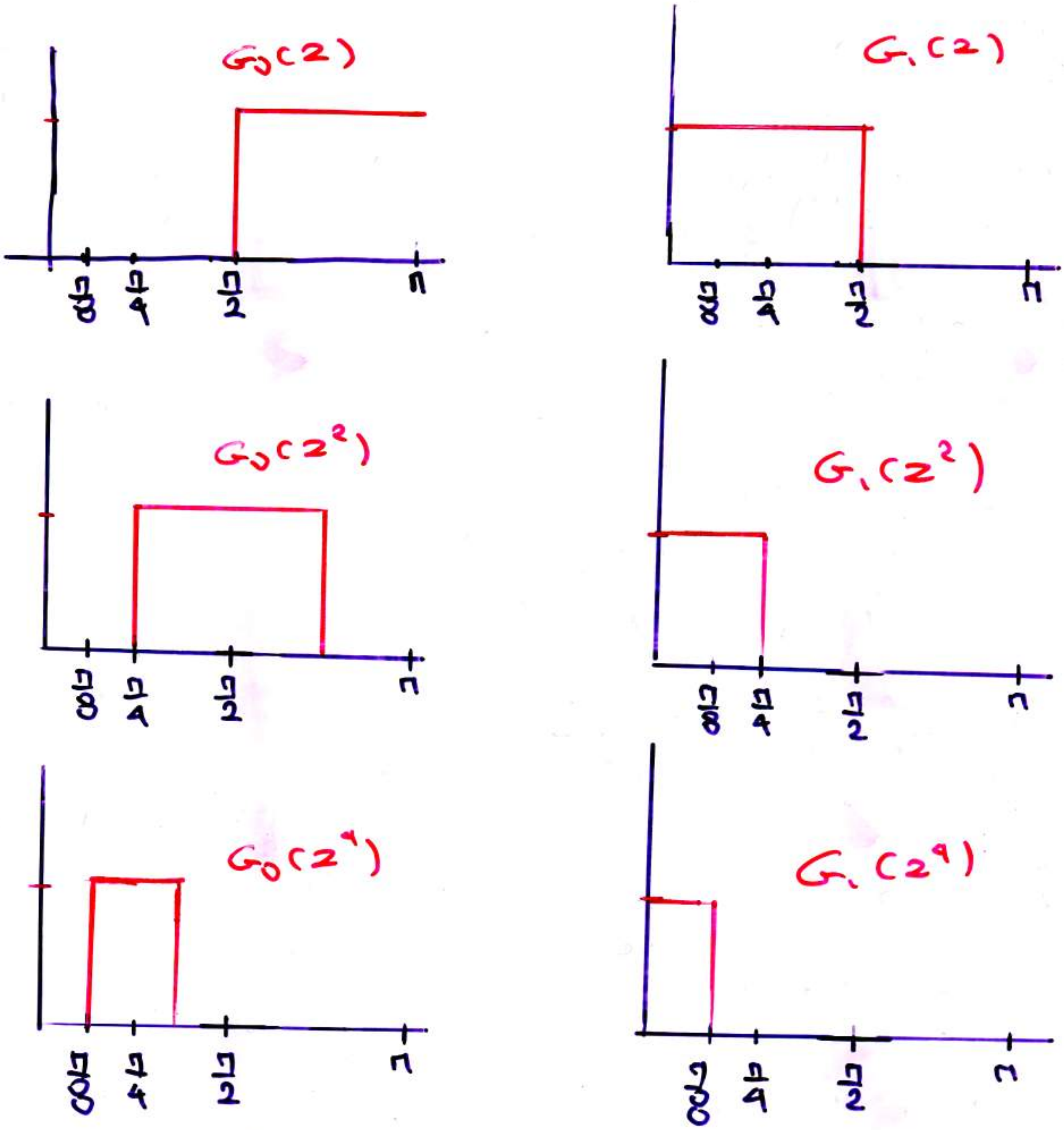
- Then

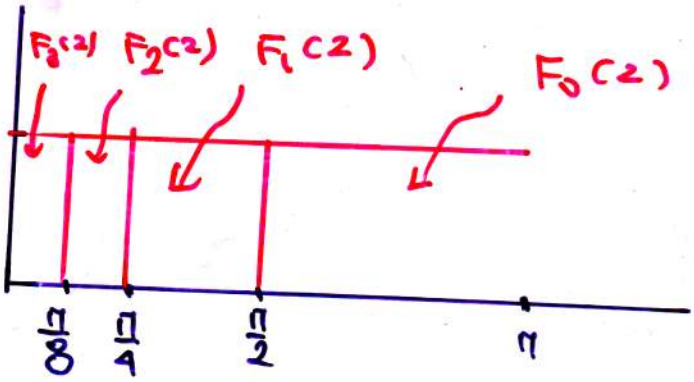
$$y_i(k) = \sum_n x(n) \phi_{ik}(n)$$

$$x(n) = \sum_i \sum_k y_i(k) \psi_{ik}(n)$$

$$\sum_n \phi_{ik}(n) \psi_{j\ell}(n) = \delta_{k\ell} \delta_{ij} \quad \text{Biorthogonality}$$

$$\psi_{ik}(n) \; \phi_{ik}(n) \quad \text{orthogonality}$$





- Remarks
  - $y_0(k) \leftrightarrow F_0(z)$ contains the detail (high pass) content of $x(n)$ $\longrightarrow$ Detail resolution
  - $y_3(k) \leftrightarrow F_3(z)$ contains coarser (low pass) content of $x(n)$

Coarse resolution

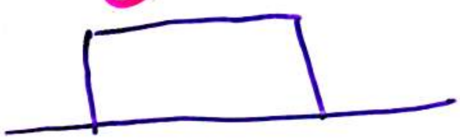
- Observe that

Low Pass ⟵•⟶ High Pass

Small bandwidth ⟷ Large Bandwidth

- Remember that

Large Bandwidth ⟺ Short Impulse response



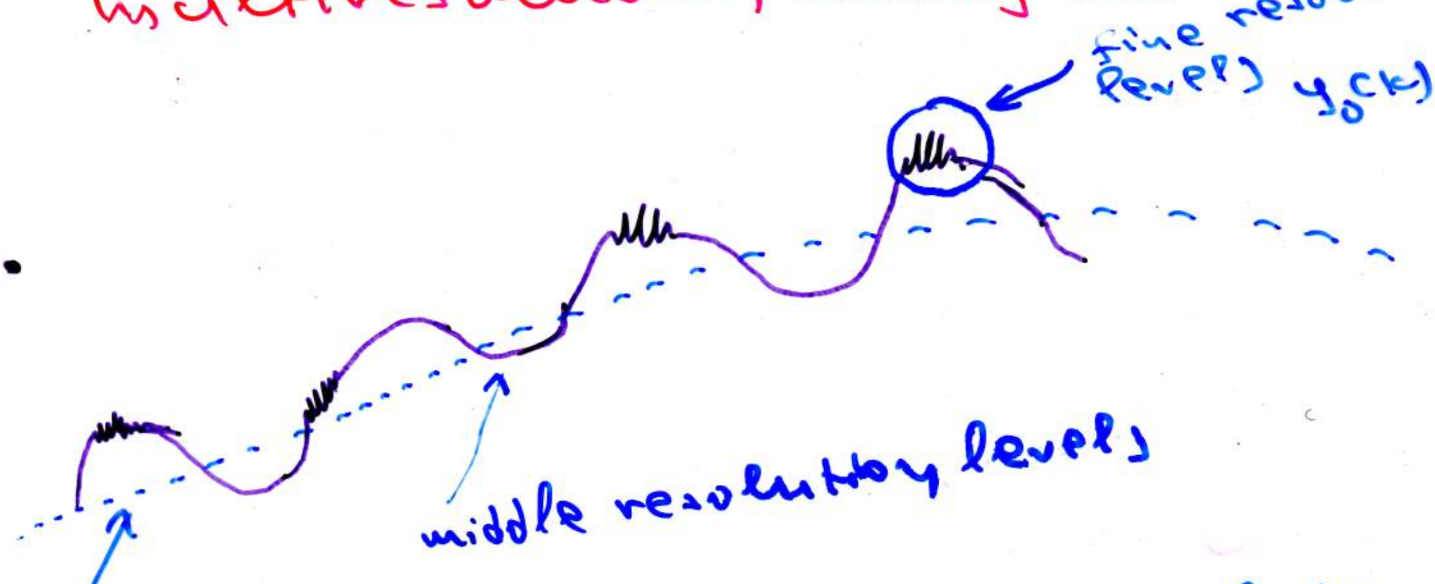
Small Bandwidth ⟺ Large IR

- Sudden changes ⟶ High frequency ⟶ analyzed with filters of short I.R. Otherwise they would spread out, if they had to be convolved with large IR

- Slow changes ⟶ Low frequency ⟶ analysed with filters of large I.R.

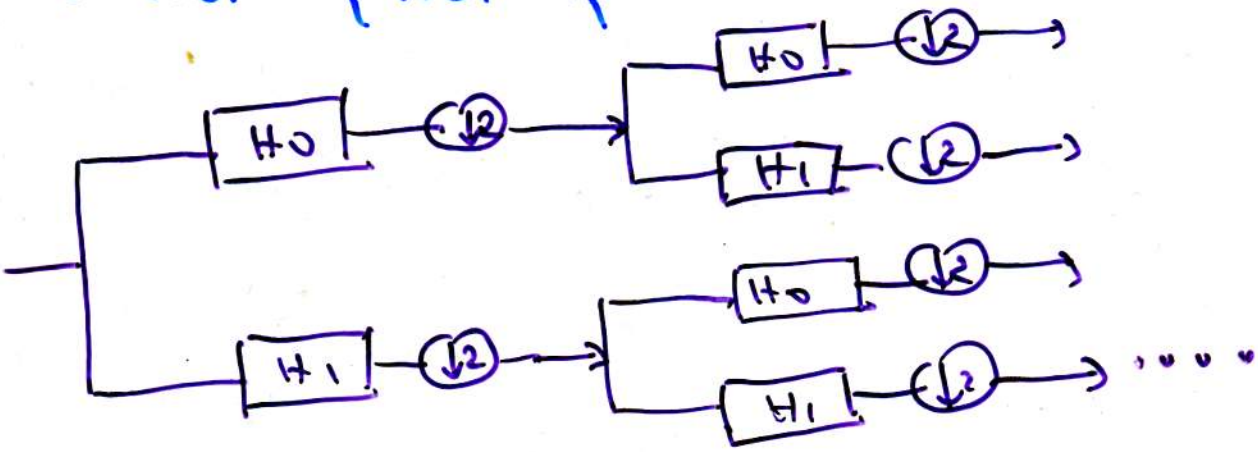
- The resolution matches the scale of the activity under investigation

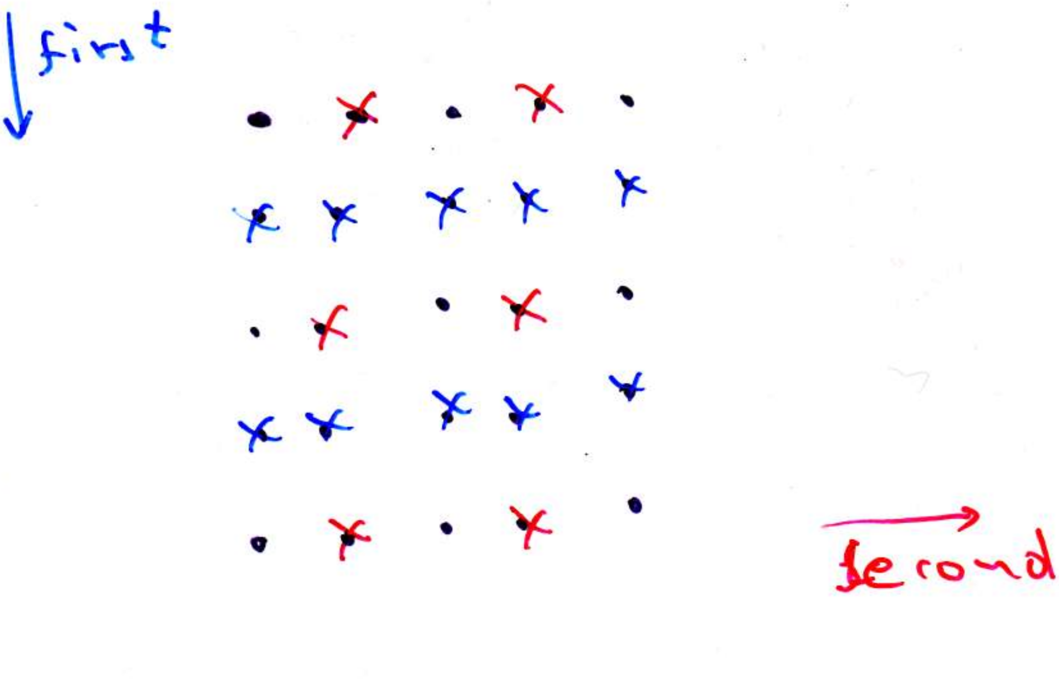
- The wavelet transform provides the means of analysing the signal into a number of different resolution levels in a hierarchical fashion. This is known as multiresolution analysis

- 

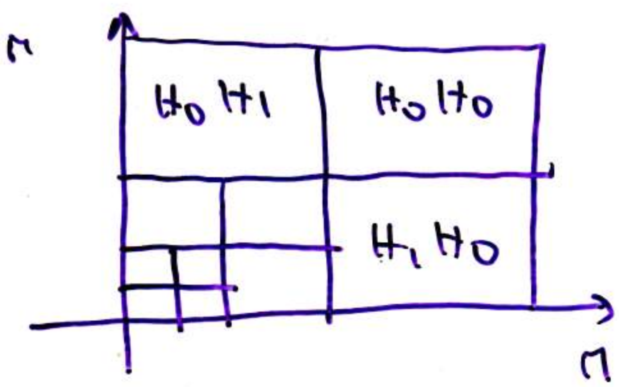
Two Dimensional Generalization

- Filter first columnwise and subsample then filter row-wise and sampsample

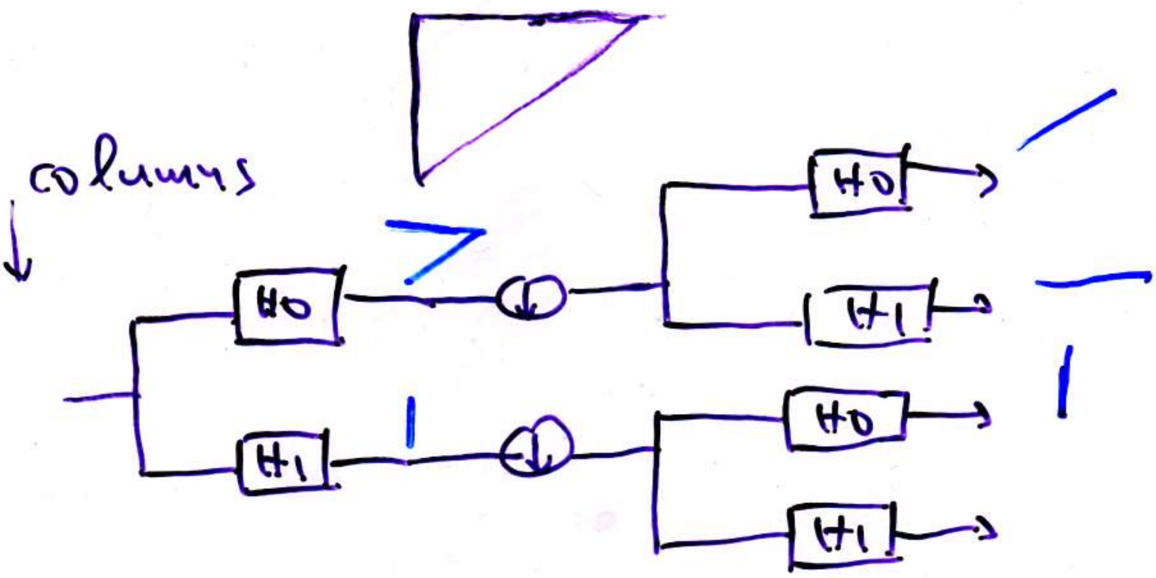




- The frequency domain filter banks



- An example



Thus, the original image can be composed from its multiresolution components, and each component 'encodes' different characteristic of the original image.