Test, 8 Ιουνίου 2012 (1 ώρα)

Γ+Δ+επί πτυχίω

Θ1 Να υπολογιστεί το επικαμπύλιο ολοκλήρωμα $I=\int_\Gamma \vec{F}\cdot d\vec{r}$, όπου $\Gamma : \{(x,y) : 9x^2+4y^2=1$ με $x,y \geq 0\}$ και $\vec{F}(x,y)=(x,y)$, $(x,y)\in\mathbb{R}^2$.

Θ2 Να υπολογιστεί το επιφανειακό ολοκλήρωμα $J=\left|\iint_S \vec{F}\cdot d\vec{S}\right|$, όπου $S : \{(x,y,z) : z=x^2+y^2$ με $z\leq 100\}$ και $\vec{F}(x,y,z)=(y,x,z^2)$, $(x,y,z)\in\mathbb{R}^3$.

Θ3 i) Εάν $f(x,y)=$ τοξεφ$(xy^2)+10^y+\ln\left(\sqrt[3]{\text{συν}y+10x}\right)$ να υπολογιστεί η κλίση $\nabla f\left(1,\frac{\pi}{2}\right)$

ii) Εάν $\vec{F}(\vec{r})=\frac{\vec{r}}{r^3}$, $\vec{r}=(x,y,z)\in\mathbb{R}^3\setminus\{(0,0,0)\}$ και $r=\|\vec{r}\|$ να υπολογιστεί η απόκλιση $\operatorname{div}\vec{F}(\vec{r})$.

(0,7+0,7+0,6)

Test, 8 Ιουνίου 2012 (1 ώρα)

Α+Β

Θ1 Να υπολογιστεί το επικαμπύλιο ολοκλήρωμα $I=\int_\Gamma f\,ds$, όπου $\Gamma : \{(x,0,z) : x^2+z^2=1$ με $z\geq 0\}$ και $f(x,y,z)=2-z$, $(x,y,z)\in\mathbb{R}^3$.

Θ2 Να υπολογιστεί το εμβαδόν της σφαίρας κέντρου (x_0,y_0,z_0) και ακτίνος α $(\alpha>0)$

Θ3 i) Εάν $\vec{F}(x,y)=\left(\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2}\right)$, $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$ να υπολογιστεί ο στροβιλισμός $\nabla\times\vec{F}(x,y)$, $(x,y)\neq(0,0)$

ii) Εάν $\vec{F}(x,y,z)=\left(\int_0^x \text{τοξεφ}(t^2+y^2)\,dt,\ \int_0^1\left(\int_0^y e^{-x^2}dx\right)dy,\ \ln\left(\sqrt{(x^2+z^2+5)^5}\right)\right)$ να υπολογιστεί η απόκλιση $\operatorname{div}\vec{F}(1,0,2)$

(0,7+0,7+0,6)

Ενδεικτικές λύσεις   Γ+Δ τεύχη πτυχίων

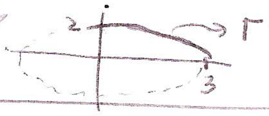
Θ1 $\vec{r}(t)=\left(\frac{1}{3}\text{συν}t,\ \frac{1}{2}\text{ημ}t\right),\ t\in\left[0,\frac{\pi}{2}\right]$ $\left(\begin{array}{l} x(t)=\frac{1}{3}\text{συν}t \\ y(t)=\frac{1}{2}\text{ημ}t \end{array}\right\}\ 9x^2+4y^2=1,\ x,y\geq 0\Bigg)$

$\vec{r}\,'(t)=\left(-\frac{1}{3}\text{ημ}t,\ \frac{1}{2}\text{συν}t\right)$

$$\underline{\underline{I}}=\int_0^{\pi/2}\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt=\int_0^{\pi/2}\left(\frac{1}{3}\text{συν}t,\ \frac{1}{2}\text{ημ}t\right)\cdot\left(-\frac{1}{3}\text{ημ}t,\ \frac{1}{2}\text{συν}t\right)dt=$$

$$=\int_0^{\pi/2}\left(-\frac{1}{9}\text{ημ}t\,\text{συν}t+\frac{1}{4}\text{ημ}t\,\text{συν}t\right)dt=\frac{5}{36}\int_0^{\pi/2}\text{ημ}t\,\text{συν}t\,dt=\frac{5}{36}\left[\frac{1}{2}\text{ημ}^2t\Big|_0^{\pi/2}\right]$$

$$=\frac{5}{72}$$



Θ2 $\vec{r}(\theta,r)=(r\,\text{συν}\theta,\ r\,\text{ημ}\theta,\ r^2),\ (\theta,r)\in[0,2\pi]\times[0,10]$ $\left(\begin{array}{l} x(r,\theta)=r\,\text{συν}\theta \\ y(r,\theta)=r\,\text{ημ}\theta \\ z=r^2,\ z\in[0,100] \end{array}\right)$

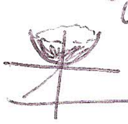
$\vec{r}_\theta=(-r\,\text{ημ}\theta,\ r\,\text{συν}\theta,\ 0)$

$\vec{r}_r=(\text{συν}\theta,\ \text{ημ}\theta,\ 2r)$

$\vec{r}_\theta\times\vec{r}_r=\left(2r^2\text{συν}\theta,\ 2r^2\text{ημ}\theta,\ \underbrace{-r\,\text{ημ}^2\theta-r\,\text{συν}^2\theta}_{=-r}\right)$

$$\underline{\underline{J}}=\left|\int_0^{10}\int_0^{2\pi}\vec{F}(\vec{r}(\theta,r))\cdot(\vec{r}_\theta\times\vec{r}_r)\,d\theta\,dr\right|=\left|\int_0^{10}\int_0^{2\pi}(r\,\text{ημ}\theta,\ r\,\text{συν}\theta,\ r^4)\cdot(2r^2\text{συν}\theta,\ 2r^2\text{ημ}\theta,\ -r)\,d\theta\,dr\right|$$

$$=\left|\int_0^{10}\int_0^{2\pi}(2r^3\text{ημ}\theta\,\text{συν}\theta+2r^3\text{συν}\theta\,\text{ημ}\theta-r^5)\,d\theta\,dr\right| \quad \int_0^{2\pi}\text{ημ}\theta\,\text{συν}\theta\,d\theta=0 \quad \left|2\pi\int_0^{10}(-r^5)\,dr\right|=\frac{\pi}{3}10^6$$



Θ3 i) $\frac{\partial f}{\partial x}=\frac{y^2}{1+x^2y^4}+\frac{1}{3}\frac{10}{\text{συν}y+10x},\quad \frac{\partial f}{\partial x}\left(1,\frac{\pi}{2}\right)=\frac{(\pi/2)^2}{1+(\pi/2)^4}+\frac{1}{3}$

$\frac{\partial f}{\partial y}=\frac{2xy}{1+x^2y^4}+10^y\ln(10)-\frac{1}{3}\frac{\text{ημ}y}{\text{συν}y+10x},\quad \frac{\partial f}{\partial y}\left(1,\frac{\pi}{2}\right)=\frac{\pi}{1+(\pi/2)^4}+10^{\pi/2}\ln(10)-\frac{1}{30}$

$$\nabla f\left(1,\frac{\pi}{2}\right)=\left(\frac{(\pi/2)^2}{1+(\pi/2)^4}+\frac{1}{3},\ \frac{\pi}{1+(\pi/2)^4}+10^{\pi/2}\ln(10)-\frac{1}{30}\right)$$

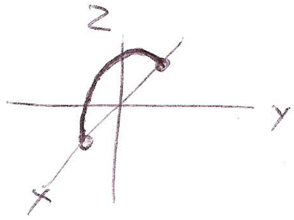
ii) $\vec{F}=(P,Q,R),\quad P=\frac{x}{(x^2+y^2+z^2)^{3/2}},\quad \frac{\partial P}{\partial x}=\frac{(x^2+y^2+z^2)^{3/2}-3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$

$Q=\frac{y}{(x^2+y^2+z^2)^{3/2}},\quad \frac{\partial Q}{\partial y}=\frac{(x^2+y^2+z^2)^{3/2}-3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$

$R=\frac{z}{(x^2+y^2+z^2)^{3/2}},\quad \frac{\partial R}{\partial z}=\frac{(x^2+y^2+z^2)^{3/2}-3z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$ $\Bigg\}\ ⊛$

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=\underline{\text{div}\vec{F}}\overset{⊛}{=}\frac{3(x^2+y^2+z^2)^{3/2}-3(x^2+y^2+z^2)^{1/2}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^3}=0$$

Ενδεικτικές λύσεις Α+Β



Θ1 $\vec{r}(t)=(\cos t,\, 0,\, \sin t)$, $t\in[0,\pi]$ $(z\geq 0)$

$\vec{r}\,'(t)=(-\sin t,\, 0,\, \cos t)$, $\|\vec{r}\,'(t)\|=\sqrt{(-\sin t)^2+\cos^2 t}=1$

$$\underline{I}=\int_0^\pi f(\vec{r}(t))\cdot\|\vec{r}\,'(t)\|\,dt=\int_0^\pi(2-\sin t)\,dt=2\pi-\int_0^\pi \sin t\,dt=2\pi+\cos t\Big|_0^\pi=2\pi-2$$

Θ2 Αν υπολογιστεί το εμβαδόν επιφάνειας της $S=\{(x,y,z): x^2+y^2+z^2=\alpha^2\}$ τότε αυτό είναι ίσο με το ζητούμενο:

- $\vec{r}(\theta,\varphi)=(\alpha\cos\theta\sin\varphi,\, \alpha\sin\theta\sin\varphi,\, \alpha\cos\varphi)$, $(\theta,\varphi)\in[0,2\pi]\times[0,\pi]$

$\vec{r}_\theta(\theta,\varphi)=(-\alpha\sin\theta\sin\varphi,\, \alpha\cos\theta\sin\varphi,\, 0)$

$\vec{r}_\varphi(\theta,\varphi)=(\alpha\cos\theta\cos\varphi,\, \alpha\sin\theta\cos\varphi,\, -\alpha\sin\varphi)$

$\vec{r}_\theta\times\vec{r}_\varphi=(-\alpha^2\cos\theta\sin^2\varphi,\, -\alpha^2\sin\theta\sin^2\varphi,\, -\alpha^2\sin^2\theta\sin\varphi\cos\varphi-\alpha^2\cos^2\theta\sin\varphi\cos\varphi)$

$\|\vec{r}_\theta\times\vec{r}_\varphi\|=\alpha^2\sqrt{\cos^2\theta\sin^4\varphi+\sin^2\theta\sin^4\varphi+\sin^2\varphi\cos^2\varphi}=\alpha^2\sqrt{\sin^4\varphi+\sin^2\varphi\cos^2\varphi}=$

$=\alpha^2\sqrt{\sin^2\varphi}=\alpha^2\sin\varphi \quad (\varphi\in[0,\pi])$

$$\underline{A}=\int_0^{2\pi}\int_0^\pi\|\vec{r}_\theta\times\vec{r}_\varphi\|\,d\varphi\,d\theta=2\pi\alpha^2\left(\int_0^\pi\sin\varphi\,d\varphi\right)=2\pi\alpha^2\left[-\cos\varphi\Big|_0^\pi\right]=4\pi\alpha^2$$

Θ3 i) $P(x,y)=\dfrac{-y}{x^2+y^2}$, $Q(x,y)=\dfrac{x}{x^2+y^2}$, $(x,y)\neq(0,0)$

$\dfrac{\partial P}{\partial y}(x,y)=\dfrac{-(x^2+y^2)+y\cdot 2y}{(x^2+y^2)^2}=\dfrac{y^2-x^2}{(x^2+y^2)^2}$

$\dfrac{\partial Q}{\partial x}=\dfrac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}=\dfrac{y^2-x^2}{(x^2+y^2)^2}$

$\operatorname{curl}\vec{F}(x,y)=\left(\dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}\right)\vec{k}=0\cdot\vec{k}=(0,0,0)$

ii) $P(x,y,z)=\int_0^x \text{τοξεφ}(t^2+y^2)\,dt \Rightarrow \dfrac{\partial P}{\partial x}(x,y,z)=\text{τοξεφ}(x^2+y^2)$, $\dfrac{\partial P}{\partial x}(1,0,2)=\text{τοξεφ}\,1=\dfrac{\pi}{4}$

$Q(x,y,z)=\int_0^1\left(\int_0^y e^{-x^2}dx\right)dy=$ σταθερός αριθμός $\Rightarrow \dfrac{\partial Q}{\partial y}(x,y,z)=0$, $\forall (x,y,z)\in\mathbb{R}^3$

$R(x,y,z)=\dfrac{5}{2}\ln(x^2+z^2+5) \Rightarrow \dfrac{\partial R}{\partial z}(x,y,z)=\dfrac{5}{2}\cdot\dfrac{2z}{x^2+z^2+5}$, $\dfrac{\partial R}{\partial z}(1,0,2)=\dfrac{5}{2}\cdot\dfrac{2\cdot 2}{1+2^2+5}=1$

Άρα $\operatorname{div}\vec{F}(1,0,2)=\left(\dfrac{\partial P}{\partial x}+\dfrac{\partial Q}{\partial y}+\dfrac{\partial R}{\partial z}\right)_{(1,0,2)}=\dfrac{\pi}{4}+1$