

Ανάλυση II - μαθημα 13 - 23/03/2011

Υποθέτιση:  $f: A(\subseteq \mathbb{R}^n) \rightarrow \mathbb{R}$ ,  $\vec{p} \in A$

- Εάν  $\exists \frac{\partial f}{\partial x_i}$  στο  $A$  συνεχείς στο  $\vec{p}$

τότε  $\exists df(\vec{p}): \mathbb{R}^n \rightarrow \mathbb{R}$  γραμμική

$$df(\vec{p})(\vec{h}) = \nabla f(\vec{p}) \cdot \vec{h} = h_1 \frac{\partial f}{\partial x_1}(\vec{p}) + \dots + h_n \frac{\partial f}{\partial x_n}(\vec{p})$$

- $\vec{f} = (f_1, \dots, f_m): A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $f_i: A \rightarrow \mathbb{R}$   $i=1, 2, \dots, m$ .

$$d\vec{f}(\vec{p}) = (df_1(\vec{p}), \dots, df_m(\vec{p}))$$

$$d\vec{f}(\vec{p})(\vec{h}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}_{\vec{p}} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

(συνέχεια) 5)  $\vec{f}(x, y) = (x^2 + y^4, \sin x^3, \log(y^2 + 5))$ ,

$$d\vec{f}(1, 8), J_{\vec{f}}(1, 8)$$

- $f_1(x, y) = x^2 + y^4 \begin{cases} \frac{\partial f_1}{\partial x}(x, y) = 2x \\ \frac{\partial f_1}{\partial y}(x, y) = 4y^3 \end{cases}$  συνεχείς στο  $\mathbb{R}^2 \Rightarrow$

$$\begin{aligned} \Rightarrow \exists df_1(1, 8)(h_1, h_2) &= h_1 \frac{\partial f_1}{\partial x}(1, 8) + h_2 \frac{\partial f_1}{\partial y}(1, 8) = \\ &= h_1 \cdot 2 + h_2 \cdot 4 \cdot 8^3 = 2h_1 + 4 \cdot 8^3 h_2 \end{aligned}$$

$$\bullet f_2(x, y) = \sigma\mu\nu(x^3) \quad \left\{ \begin{array}{l} \frac{\partial f_2}{\partial x}(x, y) = -3x^2\eta\mu x^3 \\ \frac{\partial f_2}{\partial y}(x, y) = 0 \end{array} \right. \quad \text{Συναρτήσεις}$$

$$\Rightarrow \exists df_2(1, 8)(h_1, h_2) = h_1 \frac{\partial f_2}{\partial x}(1, 8) + h_2 \frac{\partial f_2}{\partial y}(1, 8) \\ = h_1(-3\eta\mu 1) + 0h_2$$

$$\bullet f_3(x, y) = \log(x^2 + 5) \quad \left\{ \begin{array}{l} \frac{\partial f_3}{\partial x}(x, y) = 0 \\ \frac{\partial f_3}{\partial y}(x, y) = \frac{2y}{y^2 + 5} \end{array} \right. \quad \text{Συναρτήσεις} \Rightarrow$$

$$\Rightarrow \exists df_3(1, 8)(h_1, h_2) = h_1 \frac{\partial f_3}{\partial x}(1, 8) + h_2 \frac{\partial f_3}{\partial y}(1, 8) = \\ = 0 \cdot h_1 + \frac{16}{69} h_2, \quad (h_1, h_2) \in \mathbb{R}^2$$

Τελικά:

$$d\vec{f}(1, 8)(h_1, h_2) = (2h_1 + 4 \cdot 8^3 h_2, h_1(-3\eta\mu 1) + 0h_2, 0h_1 + \frac{16}{69}h_2) \\ = \begin{pmatrix} 2 & 4 \cdot 8^3 \\ -3\eta\mu 1 & 0 \\ 0 & \frac{16}{69} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad J_f(1, 8) = \begin{pmatrix} 2 & 4 \cdot 8^3 \\ -3\eta\mu 1 & 0 \\ 0 & \frac{16}{69} \end{pmatrix}$$

$(h_1, h_2) \in \mathbb{R}^2$

6)  $\vec{f}(x, y, z) = (x + 20\sqrt{xy}z^2, e^{xy^2})$ ,  
 $d\vec{f}(5, 10, 1), \quad \vec{J}_{\vec{f}}(5, 10, 1)$

•  $f_1(x, y, z) = x + 20\sqrt{xy}z^2$

$\frac{\partial f_1}{\partial x}(x, y, z) = 1$

$\frac{\partial f_1}{\partial y}(x, y, z) = 0$

$\frac{\partial f_1}{\partial z}(x, y, z) = \frac{20z}{1+z^4}$

$\Rightarrow$  Zuvor in  $\mathbb{R}^3$

$\exists d f_1(5, 10, 1)(h_1, h_2, h_3)$

$= h_1 + 0h_2 + h_3 \cdot 1$

•  $f_2(x, y, z) = e^{xy^2}$

$\frac{\partial f_2}{\partial x}(x, y, z) = y^2 e^{xy^2}$

$\frac{\partial f_2}{\partial y}(x, y, z) = 2xy e^{xy^2}$

$\frac{\partial f_2}{\partial z}(x, y, z) = 0$

$\Rightarrow$  Zuvor in  $\mathbb{R}^3$

$d f_2(5, 10, 1)(h_1, h_2, h_3) =$

$= h_1 \cdot 10^2 e^{5 \cdot 10^5} + h_2 \cdot 5 \cdot 10 e^{5 \cdot 10^5} + 0h_3$

$\vec{J}_{\vec{f}}(5, 10, 1) = \begin{pmatrix} 1 & 0 & 1 \\ 100e^{500} & 50e^{500} & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$

$\vec{J}_{\vec{f}}(5, 10, 1) = \begin{pmatrix} 1 & 0 & 1 \\ 100e^{500} & 50e^{500} & 0 \end{pmatrix}$

7) Να βρεθεί η γραμμικοποίηση των:

i)  $f(x) = x^3 + 1$ ,  $x_0 = 2$

ii)  $g(x, y) = xy + \sqrt{x^2 + 1}$ ,  $(x_0, y_0) = (5, 1)$

$$f(\vec{x}_0 + \vec{h}) = \underbrace{f(\vec{x}_0)}_{L(\vec{x}_0 + \vec{h})} + df(\vec{x}_0)(\vec{h}) + o(\|\vec{h}\|)$$

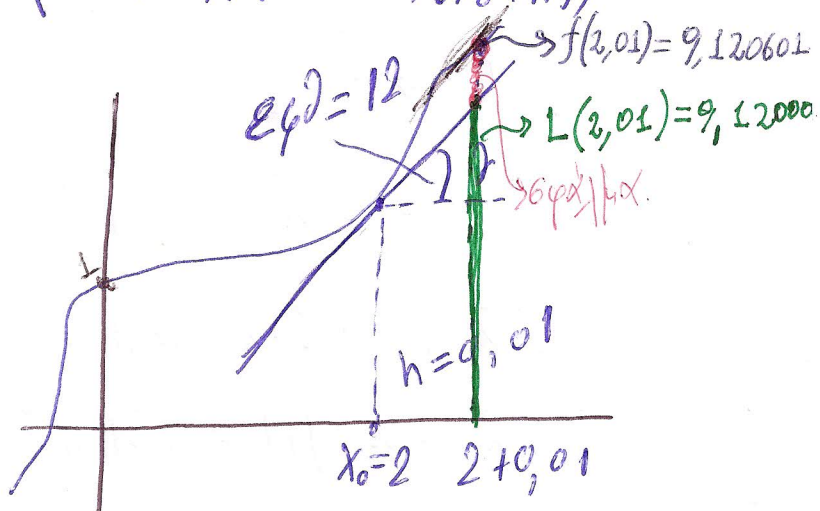
$L(\vec{x}_0 + \vec{h})$  (Το πολυώνυμο 1ου βαθμού που "προσγγίζει" το  $f(\vec{x}_0 + \vec{h})$ )

(i)  $df(2)(h) = 12h$

$L(2+h) = 9 + 12h$

$f(2+h) \approx 9 + 12h$

$(2,01)^3 + 1 \approx 9 + 12(0,01)$



ii)  $\frac{\partial g}{\partial x}(x, y) = y + \frac{x}{\sqrt{x^2 + 1}}$   
 $\frac{\partial g}{\partial y}(x, y) = x$

συνεχώς στο  $\mathbb{R}^2 \Rightarrow$   
 $\Rightarrow \exists dg(5, 1)(h_1, h_2)$   
 $= h_1 \left(1 + \frac{5}{\sqrt{26}}\right) + 5h_2$

$L(5+h_1, 1+h_2) = (5 + \sqrt{26}) + \left(1 + \frac{5}{\sqrt{26}}\right)h_1 + 5h_2$

$(5+h_1)(1+h_2) + \sqrt{(5+h_1)^2 + 1} \approx (5 + \sqrt{26}) + \left(1 + \frac{5}{\sqrt{26}}\right)h_1 + 5h_2$



Σχέση διαφορισμότητας - συνέχειας -  
μέριμης παραγωγίσιμης.

$$f: A (\subseteq \mathbb{R}^n) \rightarrow \mathbb{R}, \vec{p} \in A$$

\*\*\*  $\rightarrow$  Εάν  $\exists df(\vec{p}) \Rightarrow f$  είναι συνεχής στο  $\vec{p}$   
 $\not\Leftarrow$  π.χ.  $f(x) = |x|$  συνεχής, όχι διαφ. στο  $p=0$

$$f(\vec{p} + \vec{h}) = f(\vec{p}) + df(\vec{p})(\vec{h}) + \|\vec{h}\| g(\vec{h}), \quad \lim_{\vec{h} \rightarrow \vec{0}} g(\vec{h}) = 0 \quad (*)$$

$$= g(\vec{0})$$

$u: \mathbb{R}^n \rightarrow \mathbb{R}$  Γραμμική

$$u(\vec{x}) = \vec{a} \cdot \vec{x}, \quad \vec{x} \in \mathbb{R}^n, \quad \vec{a} = (u(\vec{e}_1), \dots, u(\vec{e}_n))$$

$$|u(\vec{x})| = |\vec{a} \cdot \vec{x}| \leq \|\vec{a}\| \|\vec{x}\|$$

$$|u(\vec{x}) - u(\vec{y})| \leq \|\vec{a}\| \|\vec{x} - \vec{y}\|$$

$$|df(\vec{p})(\vec{h})| = |\nabla f(\vec{p}) \cdot \vec{h}| \leq \|\nabla f(\vec{p})\| \|\vec{h}\|$$

$$\lim_{\vec{h} \rightarrow \vec{0}} df(\vec{p})(\vec{h}) = 0$$

$$(*) \lim_{\vec{h} \rightarrow \vec{0}} f(\vec{p} + \vec{h}) = f(\vec{p}) \text{ δηλ. } f = \text{συνεχής στο } \vec{p}$$

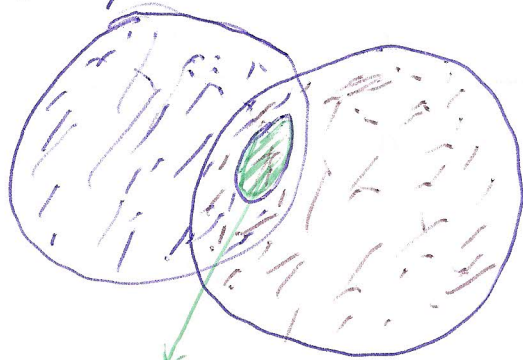
\*\*\*  $\rightarrow$  Εάν  $\exists df(\vec{p}) \Rightarrow \exists \frac{\partial f}{\partial x_i}(\vec{p}), i=1, \dots, n$  (έχει αποδειχθεί)

$$\not\Leftarrow \text{π.χ. } f(x,y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$$

Ασυνεχής  
(δεν είναι διαφορίσιμη)

$$\exists \frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$$

Συνεχείς Συν.



Μερ. παραίσιμες

Διαφορίσιμες

Ⓑ Εφαρμογή του :

$f: A (\subseteq \mathbb{R}^n) \rightarrow \mathbb{R}, \vec{p} \in A$  και

$$\exists \nabla f(\vec{p}) = \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) (\vec{p}).$$

$$\text{Τότε } \exists df(\vec{p}) \iff \lim_{\vec{h} \rightarrow \vec{0}} \frac{f(\vec{p} + \vec{h}) - f(\vec{p}) - \nabla f(\vec{p}) \cdot \vec{h}}{\|\vec{h}\|} = 0$$

(Σε διητάδες συναρτήσεις)

$$1) f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

⇒ υπάρχει το διαφ της  $f$  στο  $(0, 0)$ ;

$$\begin{aligned} \frac{\partial f}{\partial x}(0, 0) &= \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \\ &= \lim_{h \rightarrow 0} \frac{\frac{h^3}{h^2}}{h} = \lim_{h \rightarrow 0} \frac{h^3}{h^3} = \lim_{h \rightarrow 0} 1 = 1 \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h \rightarrow 0} \frac{f(0,0+h) - f(0,0)}{h} = \\ &= \lim_{h \rightarrow 0} \frac{0}{h} = \lim_{h \rightarrow 0} 0 = 0 \end{aligned}$$

$$\lim_{\vec{h}=(h_1, h_2) \rightarrow (0,0)} \frac{f(0+h_1, 0+h_2) - f(0,0) - \nabla f(0,0) \cdot (h_1, h_2)}{\| (h_1, h_2) \|}$$

$$= \lim_{(h_1, h_2) \rightarrow (0,0)} \frac{\frac{h^3}{h_1^2 + h_2^2} - (h_1 \cdot 1 + h_2 \cdot 0)}{\sqrt{h_1^2 + h_2^2}} =$$

$$= \lim_{(h_1, h_2) \rightarrow (0,0)} \frac{-h_1 h_2^2}{(h_1^2 + h_2^2) \sqrt{h_1^2 + h_2^2}} (= 0)$$

$$h_1 = h_2 = h > 0, \quad \lim_{h \rightarrow 0} \frac{-h^3}{h^2 h \sqrt{2}} = -\frac{1}{\sqrt{2}}$$

Το ίδιο (δεν μας απασχολεί αν υπάρχει) δεν είναι 0

Άρα η f είναι διαφορ. στο  $(0,0)$

$$2) f(x, y) = \begin{cases} \frac{x^2 y}{\sqrt{x^2 + y^2}} + 2x, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Eivan diapop. o ro  $(0, 0)$ ;

$$\frac{\partial f}{\partial x}(0, 0) = \dots = 2$$

$$\frac{\partial f}{\partial y}(0, 0) = \dots = 0$$

$$\lim_{(h_1, h_2) \rightarrow (0, 0)} \frac{f(0+h_1, 0+h_2) - f(0, 0) - \nabla f(0, 0)(h_1, h_2)}{\|(h_1, h_2)\|} =$$

$$= \lim_{(h_1, h_2) \rightarrow (0, 0)} \frac{\left( \frac{h_1^2 h_2}{\sqrt{h_1^2 + h_2^2}} + 2h_1 \right) - (2h_1 + 0h_2)}{\sqrt{h_1^2 + h_2^2}} =$$

$$= \lim_{(h_1, h_2) \rightarrow (0, 0)} \frac{h_1^2 h_2}{h_1^2 + h_2^2} \quad (*)$$

$$\left| \frac{h_1^2 h_2}{h_1^2 + h_2^2} \right| = \frac{|h_1|^2 |h_2|}{h_1^2 + h_2^2} = \frac{|h_1|^2 |h_2|}{\|(h_1, h_2)\|^2} \leq \frac{\|(h_1, h_2)\|^2 \|(h_1, h_2)\|}{\|(h_1, h_2)\|^2} =$$

$$= \|(h_1, h_2)\| \xrightarrow{(h_1, h_2) \rightarrow (0, 0)} 0 \quad \text{Apa } (*) = 0$$

$$\text{Apa } \exists df(0, 0)(h_1, h_2) = 2h_1 + 0 \cdot h_2$$



\* 3) Έστω  $f(x, y) = \begin{cases} (x^2 + y^2) \eta \mu \frac{1}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

i)  $\exists \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \forall (x_0, y_0) \in \mathbb{R}^2$

ii) Να αποδείξετε ότι αυτές είναι ασυνεχείς στο  $(0, 0)$

iii)  $\exists df(0, 0)(h_1, h_2) = 0 \cdot h_1 + 0 \cdot h_2$ .

• Σημείωση: Από αυτή την άσκ. και από τις προηγ. έχουμε:



Λύση i)  $\frac{\partial f}{\partial x}(x, y) = \begin{cases} 2x \eta \mu \left( \frac{1}{\sqrt{x^2 + y^2}} \right) - \frac{x}{\sqrt{x^2 + y^2}} \epsilon \omega \frac{1}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} h \eta \mu \frac{1}{|h|} = 0, & (x, y) = (0, 0) \end{cases}$

ii) Ασυνεχώς στο  $(0, 0)$ : Διαφορίσιμη  $(x, y) \neq (0, 0) \Rightarrow x > 0, \lim_{x \rightarrow 0^+} \frac{\partial f}{\partial x}(x, 0) = 0 - \lim_{x \rightarrow 0^+} \epsilon \omega \frac{1}{x}$   
 Ανάγωγα  $\approx \frac{\partial f}{\partial y}(x, y)$  δω ν διαρχε

iii)  $\lim_{(h_1, h_2) \rightarrow (0, 0)} \frac{f(0+h_1, 0+h_2) - f(0, 0) - \nabla f(0, 0) \cdot (h_1, h_2)}{\|(h_1, h_2)\|} =$   
 $= \lim_{(h_1, h_2) \rightarrow (0, 0)} \sqrt{h_1^2 + h_2^2} \eta \mu \frac{1}{\sqrt{h_1^2 + h_2^2}} = 0$ . Άρα  $\exists df(0, 0)(h_1, h_2) = 0 \cdot h_1 + 0 \cdot h_2$   
 (Ανδραφοσφίση)

Παράγωγος Ανωτέρας Τάξης

$$f: A (\subseteq \mathbb{R}^n) \rightarrow \mathbb{R}$$

$$\text{Έστω } \exists g_i = \frac{\partial f}{\partial x_i} : A (\subseteq \mathbb{R}^n) \rightarrow \mathbb{R}$$

$$\text{Εάν } \exists \frac{\partial g_i}{\partial x_j} (\vec{p}) \stackrel{\text{συμ}}{=} \frac{\partial^2 f}{\partial x_j \partial x_i} (\vec{p})$$

$$\text{Αν } i=j, \frac{\partial^2 f}{\partial x_i \partial x_i} \stackrel{\text{συμ}}{=} \frac{\partial^2 f}{\partial x_i^2}$$

$$\text{Ανάλογα } \frac{\partial^3 f}{\partial x_k \partial x_j \partial x_i} \quad \text{κ.ο.κ.}$$

Άσκηση

$$f(x, y) = x + y^2 + e^x \sin y \quad 0, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2},$$

Τι συμπαίνει;

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= 1 + e^x \sin y, \quad \frac{\partial^2 f}{\partial x^2}(x, y) = \\ &= e^x \sin y, \quad \frac{\partial^2 f(x, y)}{\partial y \partial x} = -e^x \eta \mu y \end{aligned}$$

$$\frac{\partial f}{\partial y}(x, y) = 2y - e^x \eta \mu x, \quad \frac{\partial^2 f}{\partial y^2}(x, y) = 2 - e^x \cos y,$$

*μεταφορικά*

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = -e^x \eta \mu y,$$

Παρατηρούμε ότι:  $\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y)$

Δεν συμβαίνει πάντα!!)

Διήγημα Clairaut ή Μερικών Τοποθετήσεων

Εάν  $\exists \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$  να είναι συνεχείς

τότε είναι ίσες!

Παραδειγμα: όπου  $\frac{\partial^2 f}{\partial x \partial y}(x, y) \neq \frac{\partial^2 f}{\partial y \partial x}(x, y)$

$$f(x, y) = \begin{cases} xy & \frac{x^2 - y^2}{x^2 + y^2} \\ 0, & (x, y) = (0, 0) \end{cases}, (x, y) \neq (0, 0)$$

Ανάλυση

$$\frac{\partial f}{\partial x}(0, y) = \begin{cases} -\frac{y^5}{y^4} = -y & y \neq 0 \\ 0 & y = 0 \end{cases} = -y, \quad \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) (0, 0) = -1$$

$$\frac{\partial f}{\partial y}(x, 0) = x, \quad \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) (0, 0) = 1$$