Ασκήσεις 6

1) Έστω $f : \mathbb{R}^2 \to \mathbb{R}$ διαφορίσιμη

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f_{xx} + f_{yy}$$

Εάν $F(r,\varphi) = f(x(r,\varphi), y(r,\varphi)) = F(r\,\text{συν}\varphi, r\,\text{ημ}\varphi)$

Ν.Δ.Ο:

i) $$\|\nabla f\|^2 = \left(\frac{\partial F}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial F}{\partial \varphi}\right)^2$$

ii) $$\nabla^2 f = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2}$$

<u>Λύση</u>

i)
$$\begin{tikzcd}
(r,\varphi) \arrow[r, "\vec{g}"] \arrow[dr, "F = f\circ\vec{g}"'] & (x,y) \arrow[d, "f"] \\
& \mathbb{R}
\end{tikzcd}$$

$\vec{g}(r,\varphi) = (r\,\text{συν}\varphi, r\,\text{ημ}\varphi)$, $\quad r\,\text{συν}\varphi = x$, $\quad r\,\text{ημ}\varphi = y$

$$\frac{\partial f}{\partial r} = \frac{\partial}{\partial r} f(x(r,\varphi), y(r,\varphi)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} =$$

$$= f_x\,\text{συν}\varphi + f_y\,\text{ημ}\varphi$$

$$\frac{\partial F}{\partial \varphi} = \frac{\partial}{\partial \varphi} f(x(r,\varphi), y(r,\varphi)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varphi} =$$

$$= -f_x \cdot r\,\text{ημ}\varphi + r f_y\,\text{συν}\varphi$$

$$\left(\frac{\partial F}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial F}{\partial \varphi}\right)^2 = f_x^2\,\sigma\upsilon\nu^2\varphi + f_y^2\,\eta\mu^2\varphi + \cancel{2f_x f_y\,\eta\mu\varphi\,\sigma\upsilon\nu\varphi}$$

$$+ f_y^2\,\sigma\upsilon\nu^2\varphi + f_x^2\,\eta\mu^2\varphi - \cancel{2f_x f_y\,\eta\mu\varphi\,\sigma\upsilon\nu\varphi} =$$

$$= f_x^2\left(\sigma\upsilon\nu^2\varphi + \eta\mu^2\varphi\right) + f_y^2\left(\eta\mu^2\varphi + \sigma\upsilon\nu^2\varphi\right) = f_x^2 + f_y^2 =$$

$$= \|\nabla f\|^2.$$

ii) $$\frac{\partial^2 F}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial F}{\partial r}\right) = \frac{\partial}{\partial r}\left(f_x\,\sigma\upsilon\nu\varphi + f_y\,\eta\mu\varphi\right) =$$

$$= \frac{\partial f_x}{\partial x}\frac{\partial x}{\partial r}\sigma\upsilon\nu\varphi + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial r}\sigma\upsilon\nu\varphi + \frac{\partial f_y}{\partial x}\frac{\partial x}{\partial r}\eta\mu\varphi$$

$$+ \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial r}\eta\mu\varphi = f_{xx}\,\sigma\upsilon\nu^2\varphi + f_{xy}\,\eta\mu\varphi\,\sigma\upsilon\nu\varphi + f_{yx}\,\eta\mu\varphi\,\sigma\upsilon\nu\varphi + f_{yy}\,\eta\mu^2\varphi \quad (1)$$

$$\frac{1}{r}\frac{\partial F}{\partial r} = \frac{1}{r} f_x\,\sigma\upsilon\nu\varphi + \frac{1}{r} f_y\,\eta\mu\varphi \quad (2)$$

$$\frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2} = \frac{1}{r^2}\frac{\partial}{\partial \varphi}\left(\frac{\partial F}{\partial \varphi}\right) = \frac{1}{r^2}\frac{\partial}{\partial \varphi}\left(-f_x r\,\eta\mu\varphi + r f_y\,\sigma\upsilon\nu\varphi\right) =$$

$$= \frac{1}{r^2}\Big(-r f_{xx}\frac{\partial x}{\partial \varphi}\eta\mu\varphi - r f_{xy}\frac{\partial y}{\partial \varphi}\eta\mu\varphi - f_x r\,\sigma\upsilon\nu\varphi -$$

$$- f_y r\,\eta\mu\varphi + f_{yx}\frac{\partial x}{\partial \varphi} r\,\sigma\upsilon\nu\varphi + f_{yy}\frac{\partial y}{\partial \varphi} r\,\sigma\upsilon\nu\varphi\Big)$$

$$= f_{xx}\,\eta\mu^2\varphi - f_{xy}\,\eta\mu\varphi\,\sigma\upsilon\nu\varphi - \frac{1}{r} f_x\,\sigma\upsilon\nu\varphi - \frac{1}{r} f_y\,\eta\mu\varphi - f_{yx}\,\eta\mu\varphi\,\sigma\upsilon\nu\varphi + f_{yy}\,\sigma\upsilon\nu^2\varphi \quad (3)$$

Αθροίζω 1 + 2 + 3 $\Rightarrow$

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2} =$$

$$= f_{xx}\left(\cos^2\varphi + \sin^2\varphi\right) + f_{yy}\left(\sin^2\varphi + \cos^2\varphi\right) =$$

$$= f_{xx} + f_{yy} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \nabla^2 f.$$

Διαφορικό 2ης τάξης:

$$D_2 f(\vec{a})(\vec{h}) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})\, h_i h_j = \vec{h}^T Hf(\vec{a})\vec{h}$$

$$Hf(\vec{a}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}_{(\vec{a})}$$

Διαφορικό κ-τάξης:

$$D_k f(\vec{a})(\vec{h}) = \sum_{i_1,\dots,i_k=1}^{n} \frac{\partial^k f(\vec{a})}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_k}}\, h_{i_1} h_{i_2}\cdots h_{i_k}$$

2) Να υπολογισεί το διαφορικό $2^{ης}$ τάξης για $f : A(\subseteq \mathbb{R}^3) \to \mathbb{R}$ η οποία είναι C^2.

$f \equiv f(x,y,z)$

$$D_2 f(\vec{a})(\vec{h}) = \begin{pmatrix} h_1 & h_2 & h_3 \end{pmatrix} \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}_{(\vec{a})} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} =$$

$$= \begin{pmatrix} h_1 & h_2 & h_3 \end{pmatrix} \begin{pmatrix} f_{xx}(\vec{a})h_1 + f_{xy}(\vec{a})h_2 + f_{xz}(\vec{a})h_3 \\ f_{yx}(\vec{a})h_1 + f_{yy}(\vec{a})h_2 + f_{yz}(\vec{a})h_3 \\ f_{zx}(\vec{a})h_1 + f_{zy}(\vec{a})h_2 + f_{zz}(\vec{a})h_3 \end{pmatrix} =$$

$$= f_{xx}(\vec{a})h_1^2 + f_{yy}(\vec{a})h_2^2 + f_{zz}(\vec{a})h_3^2$$
$$+ 2 f_{xy}(\vec{a})h_1h_2 + 2 f_{xz}(\vec{a})h_1h_3$$
$$+ 2 f_{yz}(\vec{a})h_2h_3$$

Ασκήσεις:

Να υπολογιστεί το διαφορικό 2ης τάξης για τις παρακάτω συναρτήσεις στο $\vec{a}$.

i) $f(x,y,z) = e^{xyz}$, $\vec{a} = (1,0,1)$

ii) $f(x,y,z) = x^3 + 2y^2 z + z^3 x$, $\vec{a} = (1,1,-1)$

iii) $f(x,y) = xe^y + y \cdot ημx$, $\vec{a} = (1,0)$

iv) $f(x,y) = \ln(xy) + xe^{y^2}$, $\vec{a} = (1,2)$

Λύση

i) Υπολογίζω τον Hessian πίνακα των μερ. παραγώγων 2ης τάξης.

$$f_x = \frac{\partial f}{\partial x} = yz e^{xyz}, \qquad f_{xx} = \frac{\partial f_x}{\partial x} = y^2 z^2 e^{xyz}$$

$$f_y = \frac{\partial f}{\partial y} = xz e^{xyz}, \qquad f_{yy} = \frac{\partial f_y}{\partial y} = x^2 z^2 e^{xyz}$$

$$f_z = \frac{\partial f}{\partial z} = xy e^{xyz}, \qquad f_{zz} = \frac{\partial f_z}{\partial z} = x^2 y^2 e^{xyz}$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = z e^{xyz} + xyz^2 e^{xyz} = f_{yx}$$

$$f_{yz} = \frac{\partial f_y}{\partial z} = x e^{xyz} + x^2 yz e^{xyz} = f_{zy}$$

$$f_{xz} = \frac{\partial f_x}{\partial z} = y e^{xyz} + xy^2 z e^{xyz} = f_{zx}$$

Εφόσον οι μερικές παράγ. είναι συνεχείς

Άρα $\vec{a} = (1,0,1)$

$$Hf(\vec{a}) = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}_{(\vec{a})} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Άρα $D_2 f(\vec{a})(\vec{h}) = (h_1 \ h_2 \ h_3) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} =$

$$= (h_1 \ \ h_2 \ \ h_3) \begin{pmatrix} h_2 \\ h_1 + h_2 + h_3 \\ h_2 \end{pmatrix} =$$

$$= h_1 h_2 + h_2(h_1 + h_2 + h_3) + h_3 h_2 =$$

$$= 2h_1h_2 + 2h_2h_3 + h_2^2$$

ii) Ομοίως (ως Άσκηση)

iii) $$Hf(\vec{a}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}_{(\vec{a})}$$

$$f_x = \frac{\partial f}{\partial x} = e^y + y\,\text{συν}\,x, \quad f_{xx} = \frac{\partial f_x}{\partial x} = -y\,\text{ημ}\,x$$

$$f_y = \frac{\partial f}{\partial y} = x e^y + \text{ημ}\,x, \quad f_{yy} = x e^y$$

$$f_{xy} = e^y + \text{συν}\,x = f_{yx} \quad (\text{μερ. παράγωγοι συνεχείς})$$

$\vec{a} = (1,0)$

$$Hf(\vec{a}) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}_{(\vec{a})} = \begin{pmatrix} 0 & \text{συν}1+1 \\ \text{συν}1+1 & 1 \end{pmatrix}$$

$$D_2 f(\vec{a})(\vec{h}) = (h_1 \ \ h_2) \begin{pmatrix} 0 & \text{συν}1+1 \\ \text{συν}1+1 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} =$$

$$= (h_1 \ \ h_2) \begin{pmatrix} (\text{συν}1+1)h_2 \\ (\text{συν}1+1)h_1 + h_2 \end{pmatrix} =$$

$$= (\sigma w 1 + 1) h_1 h_2 + h_2 \left[(\sigma w 1 + 1) h_1 + h_2 \right] =$$

$$= h_2^2 + 2(\sigma w 1 + 1) h_1 h_2$$

iv) Ομοίως (Άσκηση)