ΑΣΚΗΣΕΙΣ - ΜΑΘΗΜΑ 9

Κάθετο διάνυσμα - Εφαπτόμενο Επίπεδο

Να βρεθεί το κάθετο διάνυσμα της επιφάνειας S στο $\vec{x}_0 \in S$ και το εφαπτόμενο επίπεδο:

1) $S = \{(x,y,z) : z = x^2 + y^2\}$, στο $\vec{x}_0 = (1,1,2) \in S$

Α' τρόπος

$$f(x,y,z) = x^2 + y^2 - z \quad (F : R^3 \to R) \text{ με}$$

ισοσταθμική την $c=0$. $(f(x,y,z) = 0)$.

$$\vec{N}(1,1,2) = \nabla f(1,1,2) = (2x, 2y, -1)_{(1,1,2)} =$$

$$= (2, 2, -1)$$

Β' τρόπος

$$z = f(x,y) = x^2 + y^2$$

$$\vec{N}(1,1,2) = \left(\nabla f(x_0,y_0), -1\right) = \left(\frac{\partial f}{\partial x}(1,1), \frac{\partial f}{\partial y}(1,1), -1\right)$$

$$= (2, 2, -1)$$

Γ' τρόπος

$$S = \{ \vec{r}(a,\theta) = (a\,\text{συν}\theta, a\,\text{ημ}\theta, a^2),\ (a,\theta) \in [0,+\infty) \times [0,2\pi] \}$$

(Πολικός μετασχηματισμός)

$$(1,1,2) = \vec{r}\left(\sqrt{2}, \frac{\pi}{4}\right)$$

$$\Gamma_a : \quad \vec{r}_1\left(a, \frac{\pi}{4}\right) = \left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}, a^2\right)$$

$$\vec{r}_a\left(\sqrt{2}, \frac{\pi}{4}\right) = \frac{d}{da}\left(\vec{r}_1\left(a, \frac{\pi}{4}\right)\right)\Big|_{a=\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 2a\right)_{a=\sqrt{2}} =$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 2\sqrt{2}\right)$$

$$\Gamma_\theta : \quad \vec{r}_2(\sqrt{2}, \theta) = (\sqrt{2}\,\text{συν}\theta, \sqrt{2}\,\text{ημ}\theta, 2)$$

$$\vec{r}_\theta\left(\sqrt{2}, \frac{\pi}{4}\right) = \frac{d}{d\theta}\vec{r}_2(\sqrt{2}, \theta)\Big|_{\theta=\frac{\pi}{4}} = (-\sqrt{2}\,\text{ημ}\theta, \sqrt{2}\,\text{συν}\theta, 0)\Big|_{\theta=\frac{\pi}{4}} =$$

$$= (-1, 1, 0)$$

$$\vec{N}(1,1,2) = \vec{r}_a\left(\sqrt{2}, \frac{\pi}{4}\right) \times \vec{r}_\theta\left(\sqrt{2}, \frac{\pi}{4}\right) =$$

$$= \begin{vmatrix} i & j & k \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2\sqrt{2} \\ -1 & 1 & 0 \end{vmatrix} = -2\sqrt{2}\,\vec{i} - (0 + 2\sqrt{2})\,\vec{j} +$$

$$+ \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\vec{k} =$$

$$= \left(-2\sqrt{2}, -2\sqrt{2}, \sqrt{2}\right) = \sqrt{2}(-2,-2,1)$$

Παρατήρηση: Τα διανύσματα δεν είναι κανονικοποιημένα (δηλ. μοναδιαία). Αν θέλω ορίζω

$$\vec{N}'(1,1,2) = \frac{\vec{N}}{\|\vec{N}\|} = \frac{1}{3}(2,2,-1) = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$$

$$\|\vec{N}\| = \sqrt{2^2+2^2+1} = 3$$

Το <u>εφαπτόμενο επίπεδο</u> στην επιφάνεια S είναι:

$$\vec{N}(1,1,2)\cdot(\vec{x}-\vec{x}_0) = 0 \Leftrightarrow$$

$$(2,2,-1)\cdot(x-1, y-1, z-2) = 0 \Leftrightarrow$$

$$2(x-1) + 2(y-1) + 2 - z = 0 \Leftrightarrow$$

$$z = 2x + 2y - 2$$

2) Ομοίως για $S: x^2+y^2+z^2 = a^2,\ a>0$

στο $(x_0, y_0, z_0) \in S$

<u>Α' τρόπος</u>

$$f(x,y,z) = x^2+y^2+z^2-a^2$$

$$\nabla F(x_0,y_0,z_0) = (2x_0, 2y_0, 2z_0) \neq 0$$ διότι $a>0$.

Β' τρόπος

$S: \vec{r}(\theta,\varphi) = (a\,\text{συν}\theta\,\text{ημ}\varphi,\ a\,\text{ημ}\theta\,\text{ημ}\varphi,\ a\,\text{συν}\varphi),$

$(\theta,\varphi) \in [0,2\pi]\times[0,\pi].$

(Σφαιρικός μετασχηματισμός)

$$\vec{r}_\theta(\theta_0,\varphi_0) = \frac{d}{d\theta}\left(\vec{r}(\theta,\varphi_0)\right)\Big|_{\theta=\theta_0} = \left(-a\,\text{ημ}\theta\,\text{ημ}\varphi_0,\ a\,\text{συν}\theta\,\text{ημ}\varphi_0,\ 0\right)\Big|_{\theta=\theta_0} =$$

$$= \left(-a\,\text{ημ}\theta_0\,\text{ημ}\varphi_0,\ a\,\text{συν}\theta_0\,\text{ημ}\varphi_0,\ 0\right)$$

$$\vec{r}_\varphi(\theta_0,\varphi_0) = \frac{d}{d\varphi}\left(\vec{r}(\theta_0,\varphi)\right)\Big|_{\varphi=\varphi_0} = \left(a\,\text{συν}\theta_0\,\text{συν}\varphi,\ a\,\text{ημ}\theta_0\,\text{συν}\varphi,\ -a\,\text{ημ}\varphi\right)\Big|_{\varphi=\varphi_0}$$

$$= \left(a\,\text{συν}\theta_0\,\text{συν}\varphi_0,\ a\,\text{ημ}\theta_0\,\text{συν}\varphi_0,\ -a\,\text{ημ}\varphi_0\right)$$

$$\vec{N} = \vec{r}_\theta(\theta_0,\varphi_0) \times \vec{r}_\varphi(\theta_0,\varphi_0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\,\text{ημ}\theta_0\,\text{ημ}\varphi_0 & a\,\text{συν}\theta_0\,\text{ημ}\varphi_0 & 0 \\ a\,\text{συν}\theta_0\,\text{συν}\varphi_0 & a\,\text{ημ}\theta_0\,\text{συν}\varphi_0 & -a\,\text{ημ}\varphi_0 \end{vmatrix}$$

$$= \left(-a^2\,\text{συν}\theta_0\,\text{ημ}^2\varphi_0\right)\vec{i} - \left(a^2\,\text{ημ}\theta_0\,\text{ημ}^2\varphi_0\right)\vec{j} +$$

$$+ \left(-a^2\,\text{ημ}^2\theta_0\,\text{ημ}\varphi_0\,\text{συν}\varphi_0 - a^2\,\text{συν}^2\theta_0\,\text{ημ}\varphi_0\,\text{συν}\varphi_0\right)\vec{k} =$$

$$= \left(-a^2\,\text{συν}\theta_0\,\text{ημ}^2\varphi_0,\ -a^2\,\text{ημ}\theta_0\,\text{ημ}^2\varphi_0,\ -a^2\,\text{ημ}\varphi_0\,\text{συν}\varphi_0\right) =$$

$$= -a\,\text{ημ}\varphi_0\left(a\,\text{συν}\theta_0\,\text{ημ}\varphi_0,\ a\,\text{ημ}\theta_0\,\text{ημ}\varphi_0,\ a\,\text{συν}\varphi_0\right) =$$

$$= -a\,\eta\mu\,\varphi_0\,\vec{r}(\theta_0,\varphi_0) \quad \left(= c(x_0,y_0,z_0) \text{ σταθερό}\right)$$

Εξ. εφαπτόμενου επιπέδου:

$$\nabla f(x_0,y_0,z_0)\cdot(\vec{x}-\vec{x_0})=0$$

$$(2x_0,2y_0,2z_0)\cdot(x-x_0,\,y-y_0,\,z-z_0)=0$$

$$2x_0(x-x_0)+2y_0(y-y_0)+2z_0(z-z_0)=0$$

$$2x_0 x+2y_0 y+2z_0\cdot z=2(x_0^2+y_0^2+z_0^2)$$

$$x_0 x+y_0 y+z_0\cdot z=a^2$$