ΜΑΘΗΜΑ 9

ΕΠΑΝΑΛΗΨΗ

Σεπτ. 2012

Θ1) Λύση: Β1. 2013-2014/Ασκήσεις 6-7/Taylor of 1 variable.pdf. (παρόμοιο)

Θ2) $\vec{F}(x,y) = (x^2+y^2, e^{x^2}, x+y)$

$\vec{G}(x,y) = (\ln(x+y), xy, y)$

Να υπολογισθεί το διαφορικό της $\vec{F}\times\vec{G}=\vec{W}$

στο σημείο $(0,1)$.

Λύση:

$$\vec{W} = \vec{F}\times\vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x^2+y^2 & e^{x^2} & x+y \\ \ln(x+y) & xy & y \end{vmatrix} =$$

$$= (ye^{x^2} - xy(x+y))\vec{i} - (yx^2+y^3 - \ln(x+y)(x+y))\vec{j}$$
$$+ (x^3y + xy^3 - e^{x^2}\ln(x+y))\vec{k} =$$

$$= (ye^{x^2} - xy(x+y), yx+y^3 - \ln(x+y)(x+y), x^3y+xy^3-e^{x^2}\ln(x+y))$$

Είναι $\vec{w} = \vec{w}(x,y) = \left(w_1(x,y), w_2(x,y), w_3(x,y)\right)$

$$w_1(x,y) = y e^{x^2} - xy\,\eta\mu y$$

$$\left.\begin{aligned} \frac{\partial w_1}{\partial x} &= 2xy e^{x^2} - y\,\eta\mu y \\ \frac{\partial w_1}{\partial y} &= e^{x^2} - x\,\eta\mu y - xy\,\sigma\upsilon\nu y \end{aligned}\right\} \text{συνεχείς στο } R^2 \Rightarrow \exists\, dw_1(0,1)$$

$$dw_1(0,1)(h_1,h_2) = \left(\frac{\partial w_1}{\partial x}(0,1), \frac{\partial w_1}{\partial y}(0,1)\right) \cdot \underbrace{\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}}_{\vec{h}} =$$

$$= (-\eta\mu 1, 1) \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = -\eta\mu 1\, h_1 + h_2$$

$$w_2(x,y) = yx + y^3 - \ln(x+y)\,\eta\mu y$$

$$\left.\begin{aligned} \frac{\partial w_2}{\partial x} &= y - \eta\mu y \cdot \frac{1}{x+y} \\ \frac{\partial w_2}{\partial y} &= x + 3y^2 - \sigma\upsilon\nu y \ln(x+y) - \eta\mu y \frac{1}{x+y} \end{aligned}\right\} \text{συν. στο } R^2 \Rightarrow \exists\, dw_2(0,1)$$

$$dw_2(0,1)(h_1,h_2) = (1 - \eta\mu 1, 3 - \eta\mu 1)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} =$$

$$= (1 - \eta\mu 1) h_1 + (3 - \eta\mu 1) h_2$$

$$w_3(x,y) = x^3 y + x y^3 - e^{x^2} \ln(x+y)$$

$$\left.\begin{aligned} \frac{\partial w_3}{\partial x} &= 3yx^2 + y^3 - 2x e^{x^2} \ln(x+y) - e^{x^2}\frac{1}{x+y} \\ \frac{\partial w_3}{\partial y} &= x^3 + 3xy^2 - e^{x^2}\frac{1}{x+y} \end{aligned}\right\} \longrightarrow \text{συν. στο } R^2$$

$\exists$ το $\quad dw_3(0,1)(h_1,h_2) = (0,-1)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = -h_2$

$$d\vec{W}(0,1)(\vec{h}) = d\,\vec{F}\times\vec{G}(0,1)(\vec{h}) =$$

$$= \left(-4+1h_1 + h_2,\ (1-4+1)h_1 + (3-4+1)h_2,\ -h_2\right) =$$

$$= \begin{pmatrix} -4+1 & 1 \\ 1-4+1 & 3-4+1 \\ 0 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}}_{\vec{h}}$$

με

$$J_{\vec{w}}(0,1) = \begin{pmatrix} -4+1 & 1 \\ 1-4+1 & 3-4+1 \\ 0 & -1 \end{pmatrix} \in R^{3\times 2}$$

Θ2) 1η Πρόοδος 17/04/2010

$$\vec{r}(t) = \left(e^{t}\text{συν}\, t,\ e^{t}\text{ημ}\, t,\ e^{t}\right), \quad t \in \mathbb{R}$$

i) $x(t) = e^{t}\text{συν}\, t$

$y(t) = e^{t}\text{ημ}\, t$

$z(t) = e^{t}$

$$\sqrt{x^2(t) + y^2(t)} = \sqrt{e^{2t}\text{συν}^2 t + e^{2t}\text{ημ}^2 t} = \sqrt{(e^{t})^2} = e^{t} > 0$$

Άρα $\sqrt{x^2 + y^2} = z$, $z > 0$.

$$\vec{r}\,'(t) = \left(e^{t}\text{συν}\, t - e^{t}\text{ημ}\, t,\ e^{t}\text{ημ}\, t + e^{t}\text{συν}\, t,\ e^{t}\right)$$

$$\text{συν}\,\theta_t = \frac{\vec{r}(t) \cdot \vec{r}\,'(t)}{\|\vec{r}(t)\| \cdot \|\vec{r}\,'(t)\|} = \frac{2\cancel{e^{2t}}}{\sqrt{2}\cancel{e^{t}} \cdot \sqrt{3}\cancel{e^{t}}} = \sqrt{\frac{2}{3}} \ (\text{σταθ.})$$

ανεξ. του $t \in \mathbb{R}$

$$\|\vec{r}(t)\| = \sqrt{e^{2t}(\text{συν}^2 t + \text{ημ}^2 t + 1)} = \sqrt{2}\, e^{t}$$

$$\|\vec{r}\,'(t)\| = \sqrt{e^{2t}\left((\text{συν}\, t - \text{ημ}\, t)^2 + (\text{συν}\, t + \text{ημ}\, t)^2 + 1\right)} = \sqrt{3}\, e^{t}$$

και

$$\vec{r}(t) \cdot \vec{r}\,'(t) = e^{2t}\text{συν}^2 t - \cancel{e^{2t}\text{ημ}\, t\, \text{συν}\, t} + e^{2t}\text{ημ}^2 t + \cancel{e^{2t}\text{ημ}\, t\, \text{συν}\, t} + e^{2t} = 2e^{2t}$$

Θ.4) Πρόοδος 2010 (17/04)

$\varphi: \mathbb{R} \to \mathbb{R}$ διαφορ. Αλυσ. παρ/ση για $f = \varphi \circ h$

i) $h: \mathbb{R}^2 \to \mathbb{R}$ διαφορ.

$$\begin{tikzcd}
(x,y) \arrow[r, "h"] \arrow[dr, "f"'] & h(x,y)=t \arrow[d, "\varphi"] \\
& \varphi(t)
\end{tikzcd}$$

$f \equiv f(x,y)$ και έστω $(x_0,y_0) \in \mathbb{R}^2$ με $t_0 = h(x_0,y_0) \in \mathbb{R}$

Ισχύει ότι $\nabla(\varphi \circ h)_{(x_0,y_0)} = \varphi'(t_0) \cdot \nabla h_{(x_0,y_0)}$

$$\nabla h_{(x_0,y_0)} = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right)\Bigg|_{(x_0,y_0)} = \left(\frac{\partial h}{\partial x}(x_0,y_0), \frac{\partial h}{\partial y}(x_0,y_0)\right)$$

$$\frac{\partial f_{(x_0,y_0)}}{\partial x} = \frac{d\varphi}{dt}\Bigg|_{t=t_0} \cdot \frac{\partial h}{\partial x}(x_0,y_0) = \varphi'(t_0) \cdot h_x(x_0,y_0)$$

$$\frac{\partial f}{\partial y}(x_0,y_0) = \frac{d\varphi}{dt}\Bigg|_{t=t_0} \cdot \frac{\partial h}{\partial y}(x_0,y_0) = \varphi'(t_0) \cdot h_y(x_0,y_0)$$

ii) $f(x,y) = \varphi(x^2+y^2)$

Έστω $h(x,y) = x^2+y^2 \quad (h: \mathbb{R}^2 \to \mathbb{R})$

διαφορίσιμη με

$$h_x = \frac{\partial h}{\partial x} = 2x \,, \quad \frac{\partial h}{\partial y} = 2y = h_y$$

Από i)

$$f_x = \varphi'(x^2+y^2)\cdot 2x \quad , \quad f_y = \varphi'(x^2+y^2)\cdot 2y$$

Άρα

$$x f_y = 2xy\cdot\varphi'(x^2+y^2) = y f_x .$$

iii) $g(x,y) = \varphi(e^{xy})$

$h(x,y) = e^{xy}$ $\quad h: R^2 \to R$ διαφορ. με

$$\frac{\partial h}{\partial x} = y e^{xy} \quad , \quad \frac{\partial h}{\partial y} = x e^{xy}$$

Από i)

$$g_x = \varphi'(e^{xy})\cdot y e^{xy} \quad , \quad g_y = \varphi'(e^{xy})\, x e^{xy}$$

Άρα

$$x g_x = \underline{xy} e^{xy} \underline{\varphi'(e^{xy})} = y g_y$$

Θ. 6/ Πρόοδος (17/04/10)

i) $S_1 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z=9\}$ στο $(1,2,4)$

$F(x,y,z) = x^2+y^2+z-9$ με ισοσταθμική

$c=0$ $\quad (F(x,y,z)=0)$.

$$\vec{N}(1,2,4) = \nabla F(x,y,z) = (2x, 2y, 1)\Big|_{(1,2,4)} = (2,4,1)$$

εφ. επιπέδου: $\vec{N}(1,2,4)\cdot(\vec{x}-(1,2,4)) = 0$

$$(2,4,1)\cdot(x-1, y-2, z-4) = 0$$

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$2(x-1) + 4(y-2) + z - 4 = 0$$

$$2x + 4y + z = 14$$

ii) $S_2 = \{(x,y,z)\in\mathbb{R}^3 : z = x\cos y - ye^x\}$ στο $(0,0,0)$

$$z = x\cos y - ye^x = f(x,y).$$

$$\vec{N}(0,0,0) = (\nabla f(x_0,y_0), -1) = (\cos y - ye^x, -x\sin y - e^x, -1)\Big|_{(0,0)} = (1,-1,-1)$$

$$\frac{\partial f}{\partial x} = \cos y - ye^x$$

$$\frac{\partial f}{\partial y} = -x\sin y - e^x$$

Εξ. επιπέδου:

$$\vec{N}(0,0,0)\cdot(\vec{x}-\vec{0})=0 \qquad \text{με} \quad \vec{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$(1,-1,-1)\cdot(x,y,z)=0$$

$$x-y-z=0.$$