ΠΑΡΑΓΩΓΟΙ ΑΝΩΤΕΡΗΣ ΤΑΞΗΣ

Έστω $A \subseteq \mathbb{R}^m$ ανοιχτό, $f: A \to \mathbb{R}^n$ συνάρτηση της οποίας υπάρχουν οι μερικές παράγωγοι

$$g_i := \frac{\partial f}{\partial x_i} : A \to \mathbb{R}^n \quad , \quad \forall i = 1, \dots, m.$$

Αν $\exists \; \dfrac{\partial g_i}{\partial x_j}$, συμβολίζεται με $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$. Ειδικά για $i=j$, γράφουμε $\dfrac{\partial^2 f}{\partial x_i^2}$. Αν υπάρχουν οι (δεύτερες) μερικές παράγωγοι $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ μπορούμε να ορίσουμε τις (τρίτες) μ.π. $\dfrac{\partial^3 f}{\partial x_k \partial x_j \partial x_i}$, κτλ.

Παράδ. $f(x,y) = x + y^2 + e^x \text{συν} y \implies$

$$\frac{\partial f}{\partial x} = 1 + e^x \text{συν} y \quad , \quad \frac{\partial f}{\partial y} = 2y - e^x \text{ημ} y,$$

$$\frac{\partial^2 f}{\partial x^2} = e^x \text{συν} y \quad , \quad \frac{\partial^2 f}{\partial y^2} = 2 - e^x \text{συν} y ,$$

$$\frac{\partial^2 f}{\partial y \partial x} = -e^x \text{ημ} y = \frac{\partial^2 f}{\partial x \partial y} .$$

ΘΕΩΡ. Έστω $f: A \subseteq \mathbb{R}^m \to \mathbb{R}^n$. Αν υπάρχουν οι $\dfrac{\partial^2 f}{\partial x \partial y}$ και $\dfrac{\partial^2 f}{\partial y \partial x}$ και είναι συνεχείς, τότε είναι ίσες.

ΣΥΜΒΟΛΙΣΜΟΣ

Συνηθίζουμε για ευκολία να γράφουμε f_x αντί $\frac{\partial f}{\partial x}$ και f_y αντί $\frac{\partial f}{\partial y}$, κλπ. Επίσης, για μεγαλύτερης τάξης παραγώγους γράφουμε, π.χ:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xxy} = \frac{\partial^3 f}{\partial y \partial x^2}$$

$$f_{xxyy} = \frac{\partial^4 f}{\partial y^2 \partial x^2} \quad \text{κλπ.}$$

ΟΡΣ Έστω $f: \mathbb{R}^2 \to \mathbb{R}$ διαφορίσιμη. Ονομάζουμε τελεστή Laplace της f την

$$\nabla^2 f := \nabla(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Η f λέγεται αρμονική $\Longleftrightarrow$ $\nabla^2 f = 0$.

Ασκ $F(s,t) = f(x(s,t), y(s,t))$ με $x(s,t) = s\,\text{συν}\,t$, $y(s,t) = s\,\text{ημ}\,t$. Νδο

(1) $\|\nabla f\|^2 = \left(\frac{\partial F}{\partial s}\right)^2 + \frac{1}{s^2}\left(\frac{\partial F}{\partial t}\right)^2,$

(2) $\nabla^2 f = \frac{\partial^2 F}{\partial s^2} + \frac{1}{s}\cdot\frac{\partial F}{\partial s} + \frac{1}{s^2}\cdot\frac{\partial^2 F}{\partial s^2}.$

(1) Παρατηρούμε ότι $F = f\circ g$, όπου $g(s,t) = (x(s,t), y(s,t))$.

Άρα (σελ. 44):

$$\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}\cdot συνt + \frac{\partial f}{\partial y}\cdot ημt,$$

$$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(-s\,ημt) + \frac{\partial f}{\partial y}\cdot s\,συνt$$

Υπολογίζουμε:

$$\left(\frac{\partial F}{\partial s}\right)^2 + \frac{1}{s^2}\left(\frac{\partial F}{\partial t}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 συν^2t + \left(\frac{\partial f}{\partial y}\right)^2 ημ^2t +$$

$$+ 2\,\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot συνt\cdot ημt + \frac{1}{s^2}\left[\left(\frac{\partial f}{\partial x}\right)^2\cdot s^2 ημ^2t +\right.$$

$$\left. + \left(\frac{\partial f}{\partial y}\right)^2\cdot s^2\cdot συν^2t - 2\,\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot s^2\cdot ημt\cdot συνt\right] =$$

$$= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \langle \nabla f, \nabla f\rangle = \|\nabla f\|^2.$$

(2) Η $\frac{\partial f}{\partial x}$ είναι μια συνάρτηση, έστω h, των x,y

άρα $\frac{\partial f}{\partial x} = h(x,y) = h\circ g(s,t)$. Ομοίως $\frac{\partial f}{\partial y} = w\circ g(s,t)$.

Εφαρμόζοντας πάλι τον τύπο της σελ. 44,

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial h}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial h}{\partial y}\cdot\frac{\partial y}{\partial t} =$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\cdot(-s\,ημt) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cdot s\,συνt =$$

$$= -s\,ημt\cdot\frac{\partial^2 f}{\partial x^2} + s\,συνt\,\frac{\partial^2 f}{\partial x\partial y},$$

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} =$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\cdot(-s\eta\mu t) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\cdot s\cdot\sigma\upsilon\nu t =$$

$$= -s\eta\mu t\,\frac{\partial^2 f}{\partial x\partial y} + s\sigma\upsilon\nu t\,\frac{\partial^2 f}{\partial y^2}.$$

$$\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial h}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial h}{\partial y}\cdot\frac{\partial y}{\partial s} =$$

$$= \frac{\partial^2 f}{\partial x^2}\cdot\sigma\upsilon\nu t + \frac{\partial^2 f}{\partial x\partial y}\cdot\eta\mu t$$

$$\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} =$$

$$= \frac{\partial^2 f}{\partial x\partial y}\cdot\sigma\upsilon\nu t + \frac{\partial^2 f}{\partial y^2}\cdot\eta\mu t.$$

Οπότε:

$$\frac{\partial^2 F}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\cdot\sigma\upsilon\nu t + \frac{\partial f}{\partial y}\cdot\eta\mu t\right) =$$

$$= \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\right)\cdot\sigma\upsilon\nu t + \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial y}\right)\cdot\eta\mu t =$$

$$= \frac{\partial^2 f}{\partial x^2}\sigma\upsilon\nu^2 t + 2\eta\mu t\,\sigma\upsilon\nu t\,\frac{\partial^2 f}{\partial x\partial y} + \eta\mu^2 t\,\frac{\partial^2 f}{\partial y^2}$$

$$\frac{1}{s}\frac{\partial F}{\partial s} = \frac{1}{s}\cdot\frac{\partial f}{\partial x}\cdot\sigma\upsilon\nu t + \frac{1}{s}\cdot\frac{\partial f}{\partial y}\cdot\eta\mu t$$

$$\frac{1}{s^2}\cdot\frac{\partial^2 F}{\partial t^2} = \frac{1}{s^2}\cdot\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial t}\right) = \frac{1}{s^2}\cdot\frac{\partial}{\partial t}\left(-s\eta\mu t\cdot\frac{\partial f}{\partial x} + s\sigma\upsilon\nu t\,\frac{\partial f}{\partial y}\right) =$$

$$= \cdots = -\frac{1}{s}\sigma\upsilon\nu t\,\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2}\eta\mu^2 t - 2\eta\mu t\,\sigma\upsilon\nu t\,\frac{\partial^2 f}{\partial x\partial y}$$

$$-\frac{1}{s}\cdot\eta\mu t\,\frac{\partial f}{\partial y} + \sigma\upsilon\nu^2 t\,\frac{\partial^2 f}{\partial y^2}.$$

Αθροίζοντας έχουμε το ζητούμενο.