Voronoi diagram and Delaunay triangulation

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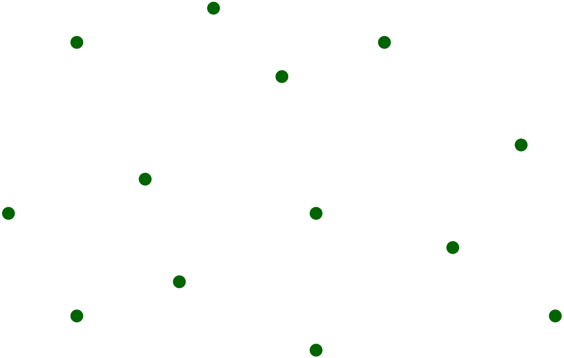
Outline

1. Voronoi diagram
2. Delaunay triangulation
3. Properties
   - Empty circle
   - Complexity
   - Min max angle
4. Algorithms and complexity
   - Incremental Delaunay
   - Further algorithms
5. (Generalizations and Representation)

Outline

Example and definition

Sites: $P := \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$



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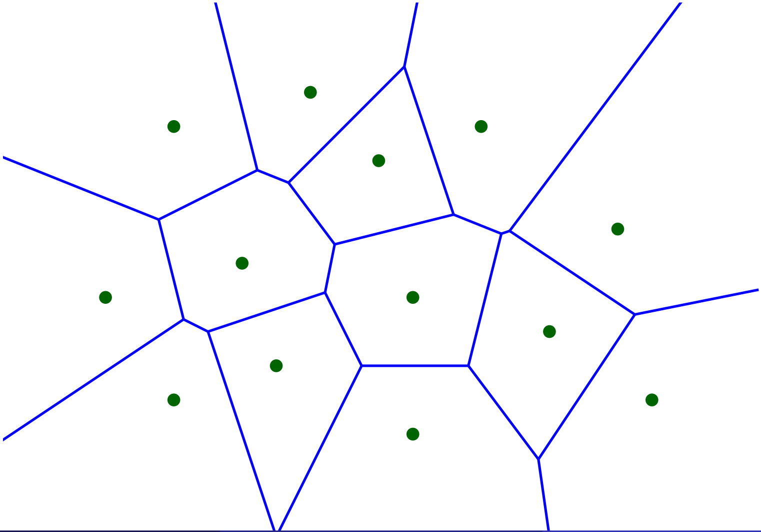
Sites: $P := \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$

Voronoi cell: $q \in V(p_i) \Leftrightarrow \text{dist}(q, p_i) \leq \text{dist}(q, p_j), \forall p_j \in P, j \neq i$



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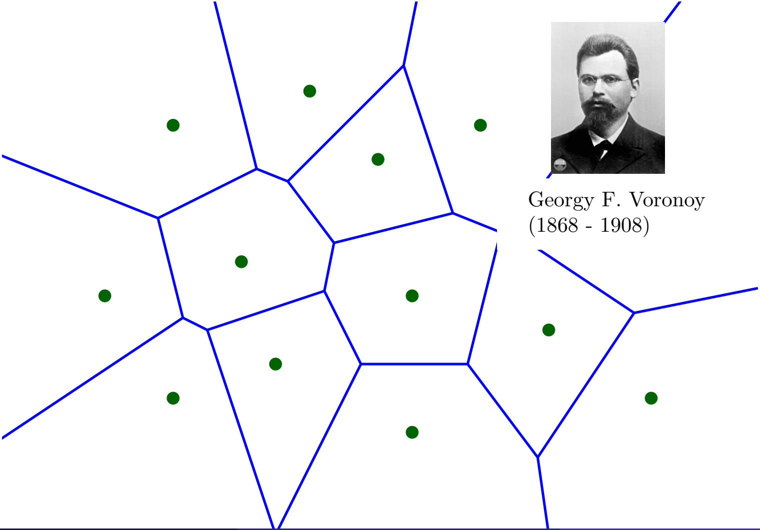
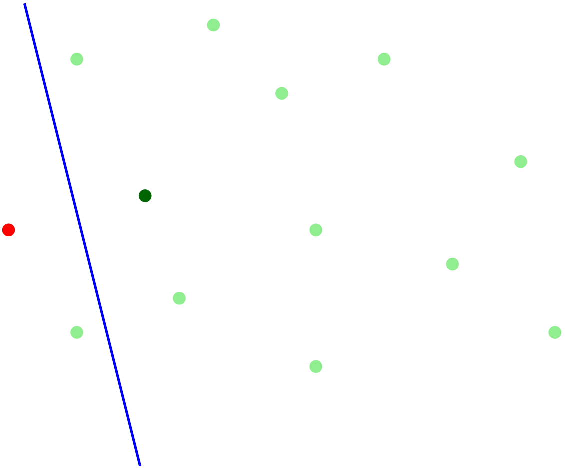


Georgy F. Voronoy
(1868 - 1908)

Faces (edges) of Voronoi diagram



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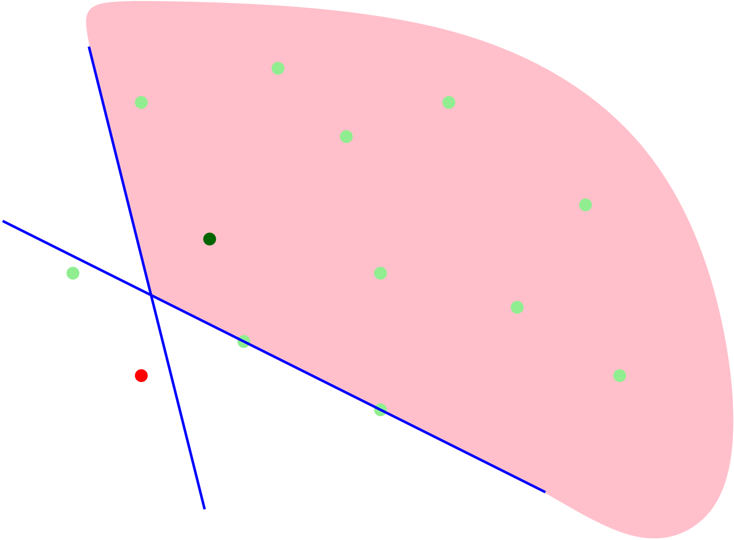

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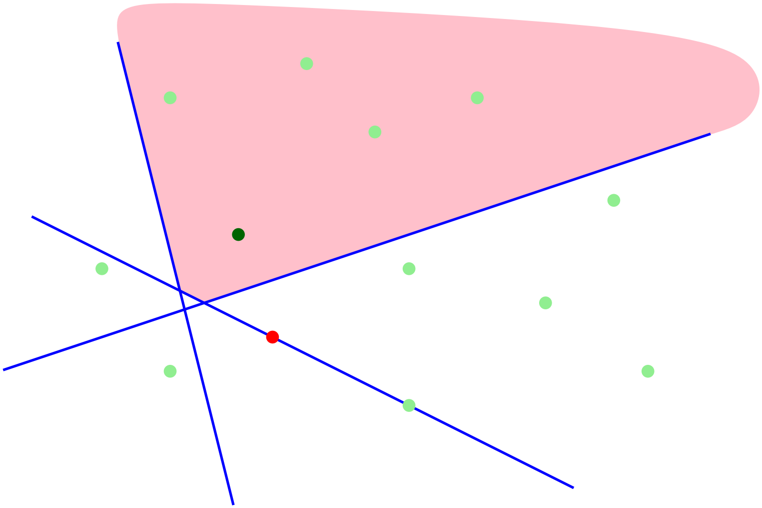

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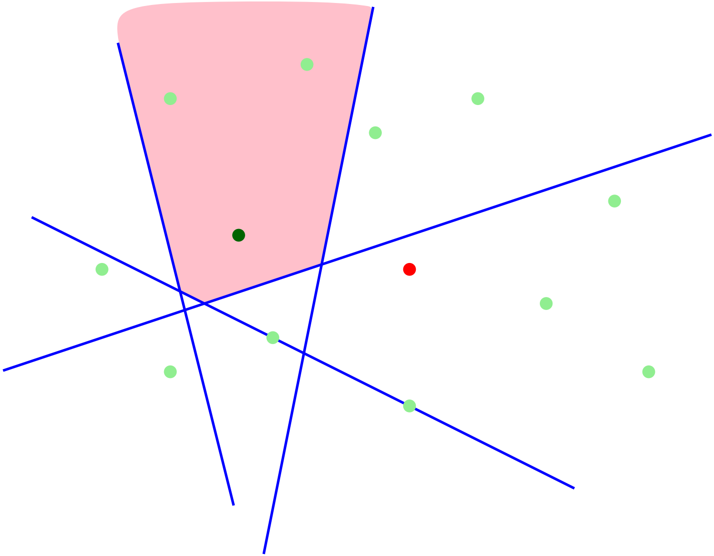

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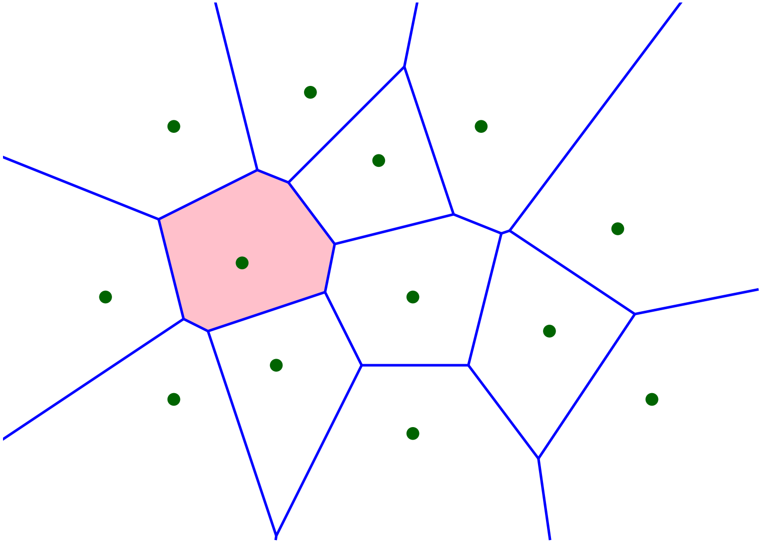
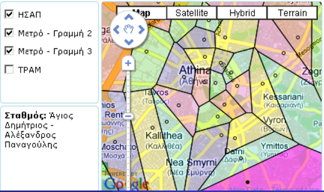


Voronoi diagram



Formalization

- sites: points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$.
- Voronoi cell/region $V(p_i)$ of site p_i:

$$q \in V(p_i) \Leftrightarrow \text{dist}(q, p_i) \leq \text{dist}(q, p_j), \forall p_j \in P, j \neq i.$$

- Voronoi edge is the common boundary of two adjacent cells.
- Voronoi vertex is the common boundary of 3 adjacent cells, or the intersection of ≥ 2 (hence ≥ 3) Voronoi edges.

Generically, of exactly 3 Voronoi edges.

Voronoi diagram of P = dual of Delaunay triangulation of P.

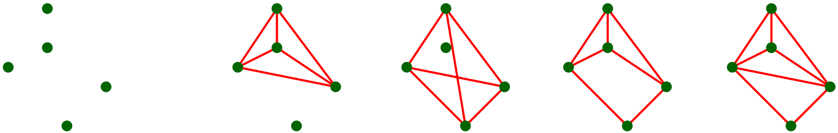
- Voronoi cell $\leftrightarrow$ vertex of Delaunay triangles: site
- neighboring cells (Voronoi edge) $\leftrightarrow$ Delaunay edge, defined by corresponding sites (line of Voronoi edge $\perp$ line of Delaunay edge)
- Voronoi vertex $\leftrightarrow$ Delaunay triangle.

Outline

Triangulation

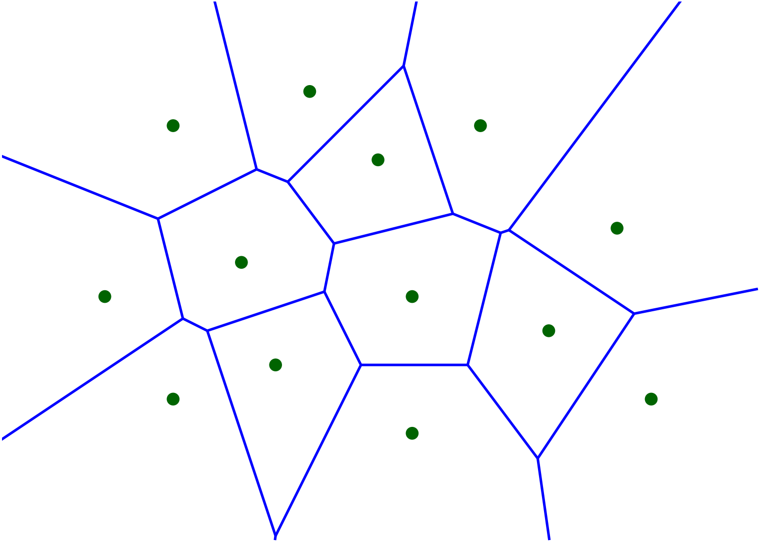
A triangulation of a pointset (sites) $P \subset \mathbb{R}^2$ is a collection of triplets from P, namely triangles, s.t.

- The union of the triangles covers the convex hull of P.
- Every pair of triangles intersect at a (possibly empty) common face ($\emptyset$, vertex, edge).
- Usually (CGAL): Set of triangle vertices $= P$.



Example: P, incomplete, invalid, subdivision, triangulation.

Delaunay Triangulation: dual of Voronoi diagram



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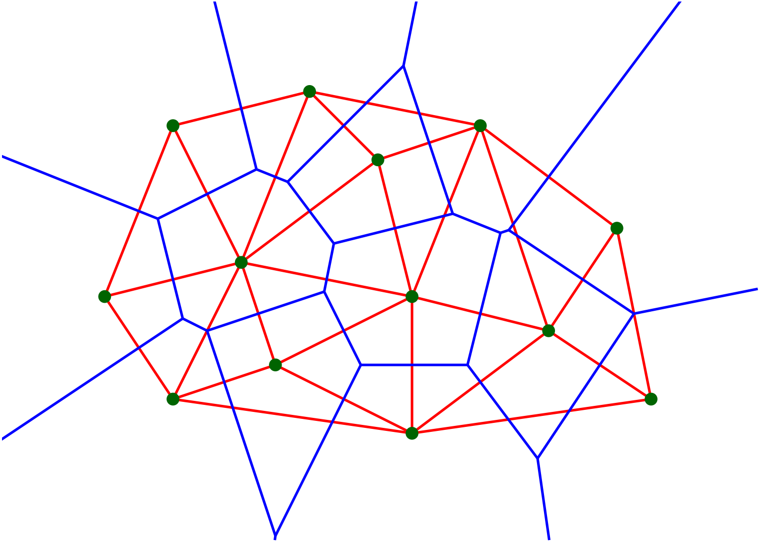

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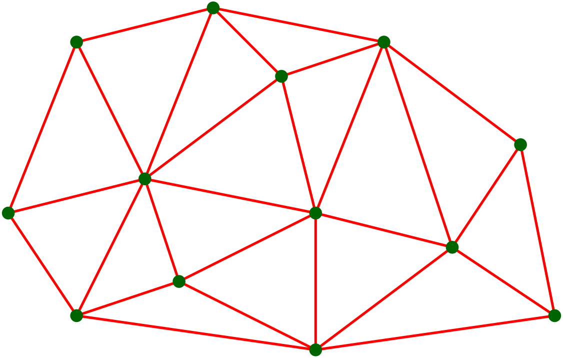

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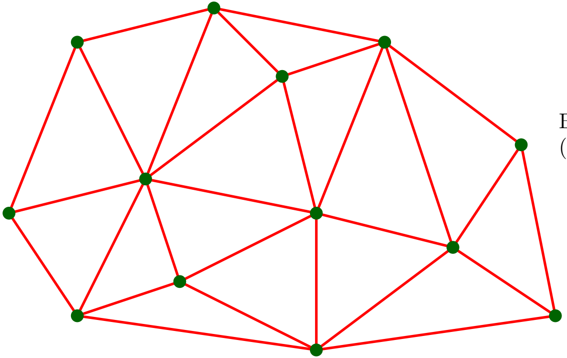
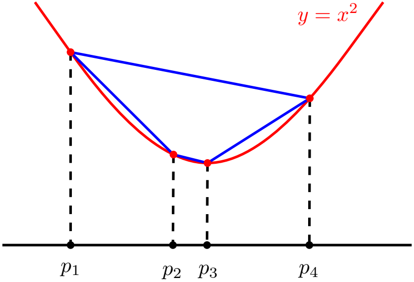


Boris N. Delaunay (1890 - 1980)

Delaunay triangulation: projection from parabola

Definition/Construction of Delaunay triangulation:

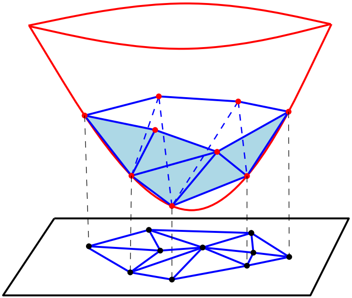
- Lift sites $p = (x) \in \mathbb{R}$ to $\widehat{p} = (x, x^2) \in \mathbb{R}^2$
- Compute the convex hull of the lifted points
- Project the lower hull to $\mathbb{R}$

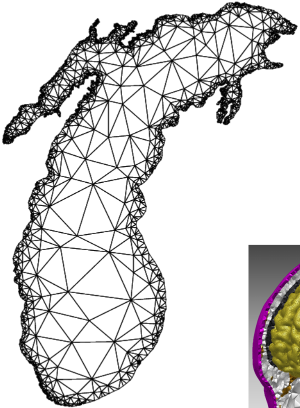


Delaunay triangulation: going a bit higher...

Definition/Construction of Delaunay triangulation:

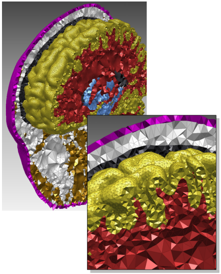
- Lift sites $p = (x, y) \in \mathbb{R}^2$ to $\widehat{p} = (x, y, x^2 + y^2) \in \mathbb{R}^3$
- Compute the convex hull of the lifted points
- Project the lower hull to $\mathbb{R}^2$: arbitrarily triangulate lower facets that are polygons (not triangles)

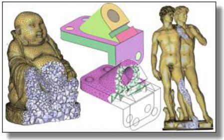


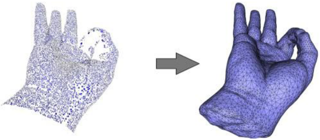


Applications

Nearest Neighbors
Reconstruction
Meshing







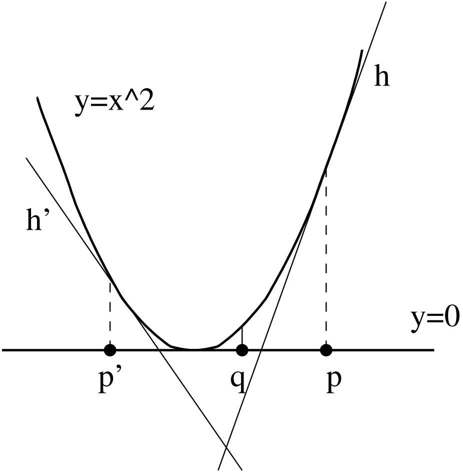
Voronoi by Lift & Project

Lifting:
– Consider the paraboloid $x_3 = x_1^2 + x_2^2$.
– For every site p, consider its lifted image $\widehat{p}$ on the parabola.
– Given $\widehat{p}$, $\exists$ unique (hyper)plane tangent to the parabola at $\widehat{p}$.

Project:
– For every (hyper)plane, consider the halfspace above.
– The intersection of halfspaces is a (unbounded) convex polytope
– Its Lower Hull projects bijectively to the Voronoi diagram.

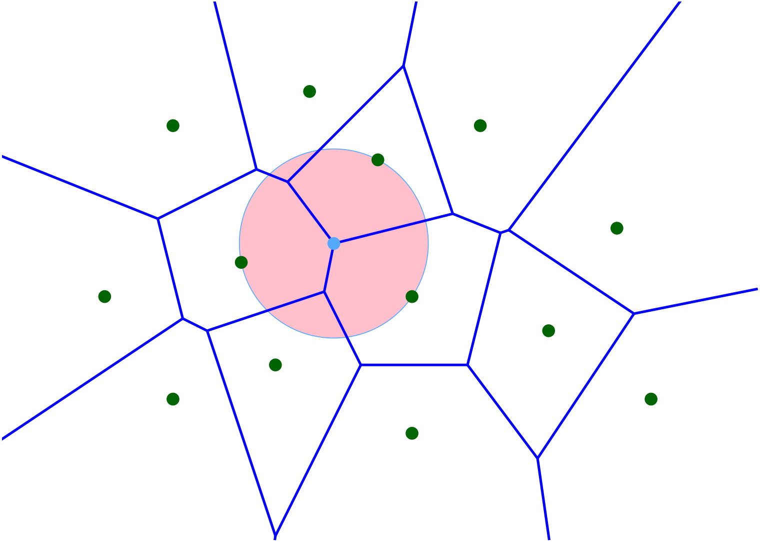
Proof:
– Let $E : x_1^2 + x_2^2 - x_3 = 0$ be the paraboloid equation.
– $\nabla E(a) = \left(\frac{\partial E}{\partial x_1}, \frac{\partial E}{\partial x_2}, \frac{\partial E}{\partial x_3}\right)_a = (2a_1, 2a_2, -1)$.
– Point $x \in$ plane $h(x) \Leftrightarrow (x-a) \cdot \nabla E(a) = 0 \Leftrightarrow$
$2a_1(x_1 - a_1) + 2a_2(x_2 - a_2) - (x_3 - a_3) = 0$, which is h's equation.

Lift & Project in 1D



Outline

Main Delaunay property: empty circle/sphere



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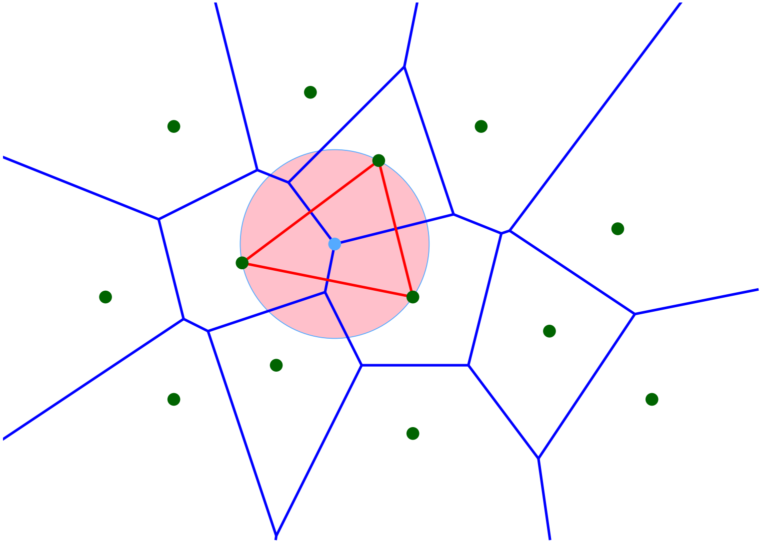

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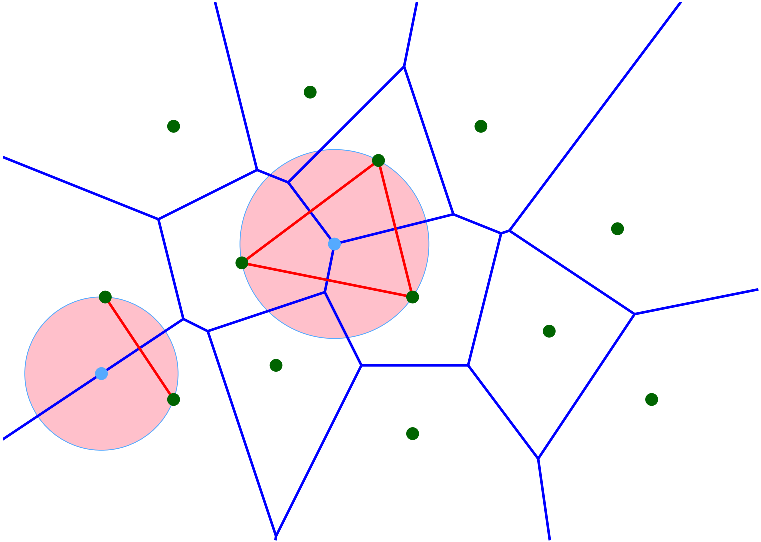
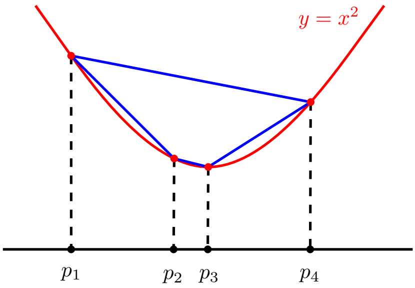


Main Delaunay property: 1 picture proof

Thm (in $\mathbb{R}$): $S(p_1, p_2)$ is a Delaunay segment $\Leftrightarrow$ its interior contains no p_i.

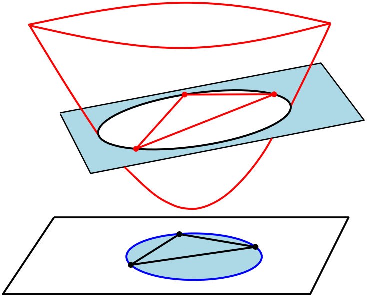
Proof. Delaunay segment $\Leftrightarrow$ $(\widehat{p_1}, \widehat{p_2})$ edge of the Lower Hull
$\Leftrightarrow$ no $\widehat{p_i}$ "below" $(\widehat{p_1}, \widehat{p_2})$ on the parabola
$\Leftrightarrow$ no p_i inside the segment (p_1, p_2).



Main Delaunay property: 1 picture proof

Thm (in $\mathbb{R}^2$): $T(p_1, p_2, p_3)$ is a Delaunay triangle $\Leftrightarrow$ the interior of the circle through p_1, p_2, p_3 (enclosing circle) contains no p_i.

Proof. Circle(p_1, p_2, p_3) contains no p_i in interior
$\Leftrightarrow$ plane of lifted $\widehat{p}_1, \widehat{p}_2, \widehat{p}_3$ leaves all lifted $\widehat{p}_i$ on same halfspace
$\Leftrightarrow$ CCW$(\widehat{p}_1, \widehat{p}_2, \widehat{p}_3, \widehat{p}_i)$ of same sign for all i.
Suffices to prove: p_i lies on Circle(p_1, p_2, p_3)
$\Leftrightarrow$ $\widehat{p}_i$ lies on plane of $\widehat{p}_1, \widehat{p}_2, \widehat{p}_3$ $\Leftrightarrow$ CCW$(\widehat{p}_1, \widehat{p}_2, \widehat{p}_3, \widehat{p}_i) = 0$.



Predicate InCircle

Given points p, q, r, $s \in \mathbb{R}^2$, point $s = (s_x, s_y)$ lies inside the circle through p, q, $r \Leftrightarrow$

$$\det \begin{pmatrix} p_x & p_y & p_x^2 + p_y^2 & 1 \\ q_x & q_y & q_x^2 + q_y^2 & 1 \\ r_x & r_y & r_x^2 + r_y^2 & 1 \\ s_x & s_y & s_x^2 + s_y^2 & 1 \end{pmatrix} > 0,$$

assuming p, q, r in clockwise order (otherwise det < 0).

Lemma. InCircle$(p, q, r, s) = 0 \Leftrightarrow \exists$ circle through p, q, r, s.
Proof. InCircle$(p, q, r, s) = 0 \Leftrightarrow CCW(\widehat{p}, \widehat{q}, \widehat{r}, \widehat{s}) = 0$

Delaunay faces

Theorem. Let P be a set of sites $\in \mathbb{R}^2$:

(i) Sites $p_i, p_j, p_k \in P$ are vertices of a Delaunay triangle $\Leftrightarrow$ the circle through p_i, p_j, p_k contains no site of P in its interior.

(ii) Sites $p_i, p_j \in P$ form an edge of the Delaunay triangulation $\Leftrightarrow$ there is a closed disc C that contains p_i, p_j on its boundary and does not contain any other site of P.

Outline

Triangulations of planar pointsets

Thm. Let P be set of n points in $\mathbb{R}^2$, not all colinear, $k = \#$points on boundary of CH(P). Any triangulation of P has $2n - 2 - k$ triangles and $3n - 3 - k$ edges.

Proof.

- f: #facets (except ∞)
- e: #edges
- n: #vertices

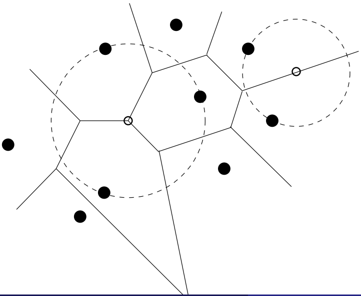
1. Euler: $n - e + (f + 1) - 1 = 1$; for d-polytope: $\sum_{i=0}^{d}(-1)^i f_i = 1$
2. Any planar triangulation: total degree $= 3f + k \ = 2e$.

Properties of Voronoi diagram

Lemma. $|V| \leq 2n-5$, $|E| \leq 3n-6$, $n = |P|$,
by Euler's theorem for planar graphs: $|V| - |E| + n - 1 = 1$.

Max Empty Circle $C_P(q)$ centered at q: no interior site $p_i \in P$.
Lem: $q \in \mathbb{R}^2$ is Voronoi vertex $\Leftrightarrow$ $C(q)$ has ≥ 3 sites on perimeter

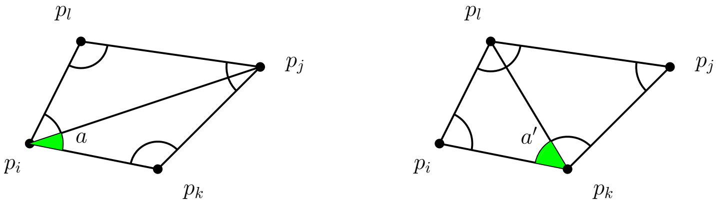
Any perpendicular bisector of segment (p_i, p_j) defines a Voronoi edge $\Leftrightarrow$
$\exists$ q on bisector s.t. $C(q)$ has only p_i, p_j on perimeter



Outline

Delaunay maximizes the smallest angle

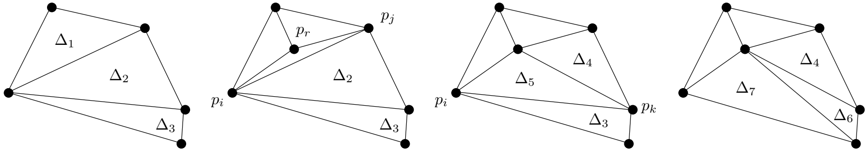
Let T be a triangulation with m triangles.
Sort the $3m$ angles: $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{3m}$. $T_a := \{a_1, a_2, \ldots, a_{3m}\}$.
Edge $e = (p_i, p_j)$ is illegal $\Leftrightarrow \min_{1 \leqslant i \leqslant 6} a_i < \min_{1 \leqslant i \leqslant 6} a'_i$.



T' obtained from T by flipping illegal e, then $T'_a >_{lex} T_a$.

Flips yield triangulation without illegal edges.
The algorithm terminates (angles decrease), but is $O(n^2)$.

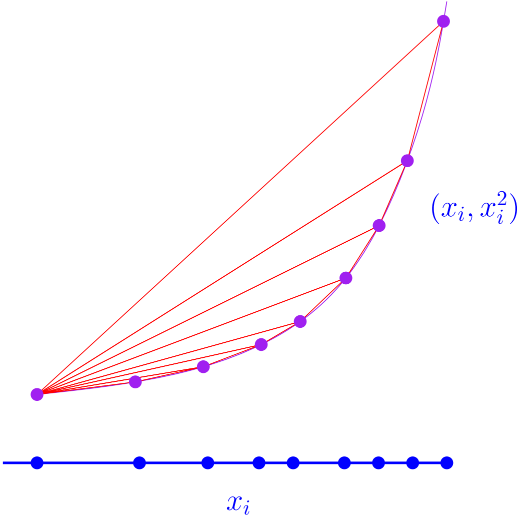
Insertion by flips



Outline

Lower bound

$\Omega(n \log n)$ by reduction from sorting



Delaunay triangulation

Theorem.

Let P be a set of points $\in \mathbb{R}^2$. A triangulation $\mathcal{T}$ of P has no illegal edge $\Leftrightarrow \mathcal{T}$ is a Delaunay triangulation of P.

Cor. Constructing the Delaunay triangulation is a fast (optimal) way of maximizing the min angle.

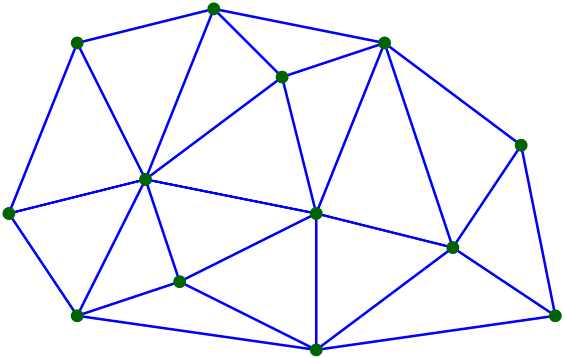
Algorithms in $\mathbb{R}^2$:

– Lift, CH3, project the lower hull: $O(n \log n)$
– Incremental algorithm: $O(n \log n)$ exp., $O(n^2)$ worst
– Voronoi diagram (Fortune's sweep): $O(n \log n)$
– Divide + Conquer: $O(n \log n)$

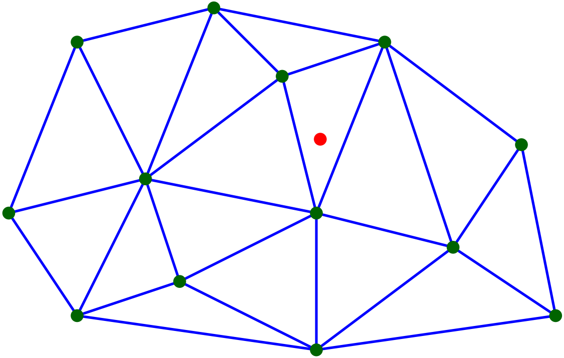
See Voronoi algo's below.

Outline

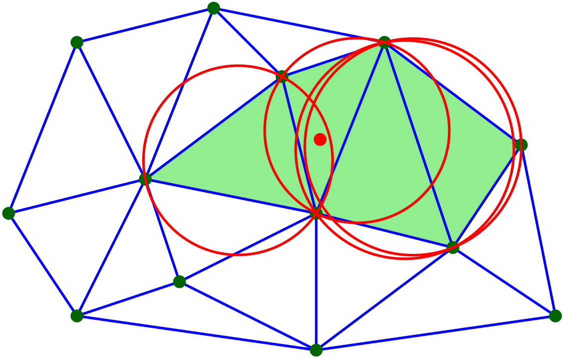
Incremental Delaunay



Incremental Delaunay

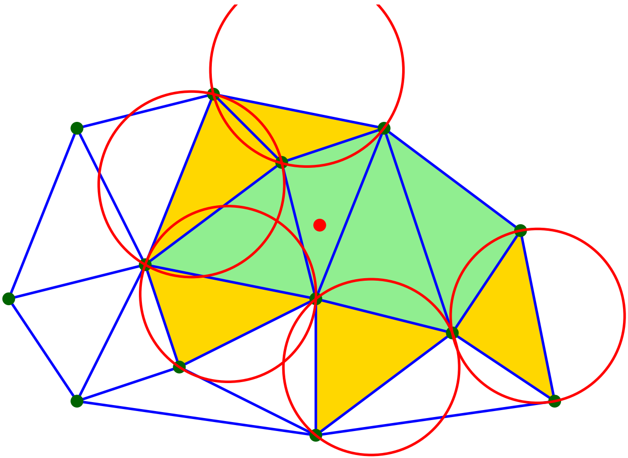


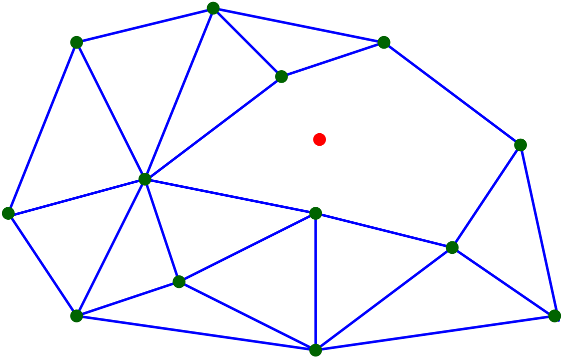
Incremental Delaunay



Find triangles in conflict

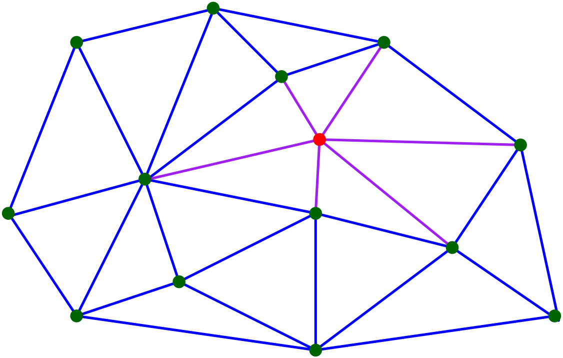
Incremental Delaunay





Delete triangles in conflict

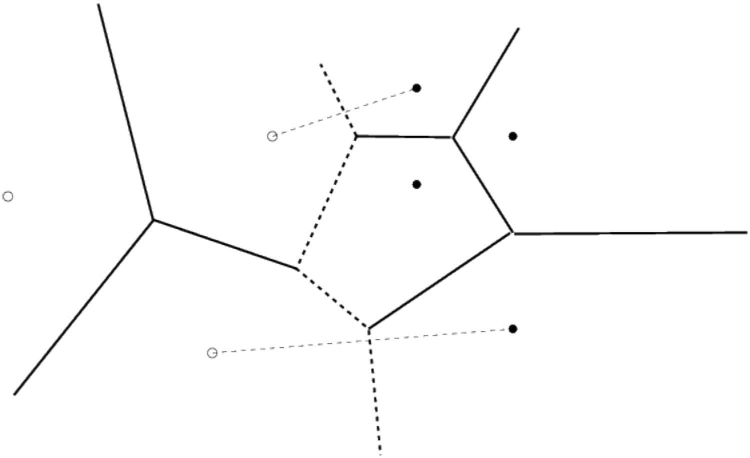
Incremental Delaunay



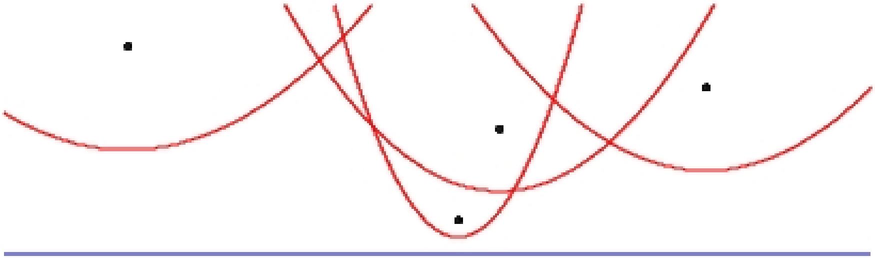
Triangulate hole

Outline

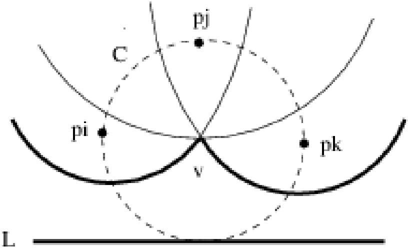
Divide & Conquer

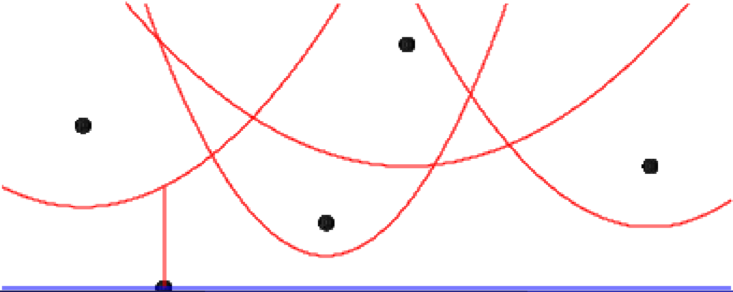


Fortune's sweep



Vertex, and Site events





Outline

General dimension polytopes

Faces of a polytope are polytopes forming its extreme elements.
A facet of a d-dimensional polytope is $(d-1)$-dimensional face:

- The facets of a segment are vertices (0-faces).
- The facets of a polygon are edges (1-faces)
- The facets of a 3-polyhedron are polygons.
- The facets of a 4d polytope are 3d polytopes.

General dimension triangulation

A triangulation of a pointset (sites) $P \subset \mathbb{R}^d$ is a collection of $(d+1)$-tuples from P, namely simplices, s.t.

- The union of the simplices covers the convex hull of P.
- Every pair of simplices intersect at a (possibly empty) common face.
- Usually: Set of simplex vertices $= P$.
- Delaunay: no site lies in the circum-hypersphere inscribing any simplex of the triangulation.

In 3d, two simplices may intersect at: $\emptyset$, vertex, edge, facet.

The triangulation is unique for generic inputs, i.e. no $d+2$ sites lie on same hypersphere, i.e. every $d+1$ sites define unique simplex.
A Delaunay facet belongs to: exactly one simplex iff it belongs to CH(P), otherwise belongs to exactly two (neighboring) simplices.

Complexity in general dimension

- Delaunay triangulation in $\mathbb{R}^d \simeq$ convex hull in $\mathbb{R}^{d+1}$.
- Convex Hull of n points in $\mathbb{R}^d$ is $\Theta(n \log n + n^{\lfloor d/2 \rfloor})$
  Hence d-Del $= \Theta(n \log n + n^{\lceil d/2 \rceil})$
- Lower bound [McMullen] on space Complexity
- optimal deterministic [Chazelle], randomized [Seidel] algorithms

Optimal algorithms by lift/project: $\mathbb{R}^2$: $\Theta(n \log n)$, $\mathbb{R}^3$: $\Theta(n^2)$.

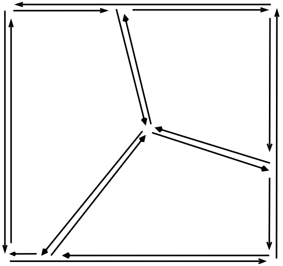
Generalized constructions

In $\mathbb{R}^2$: Various geometric graphs defined on P are subgraphs of $\mathcal{DT}(P)$, e.g. Euclidean minimum spanning tree (EMST) of P.

Delaunay triangulation $\mathcal{DT}(P)$ of pointset $P \subset \mathbb{R}^d$: triangulation s.t. no site in P lies in the hypersphere inscribing any simplex of $\mathcal{DT}(P)$.

- $\mathcal{DT}(P)$ contains d-dimensional simplices.
- hypersphere = circum-hypersphere of simplex.
- $\mathcal{DT}(P)$ is unique for generic inputs, i.e. no $d+2$ sites lie on the same hypersphere, i.e. every $d+1$ sites define unique Delaunay "triangle".
- $\mathbb{R}^d$: Delaunay facet belongs to exactly one simplex $\Leftrightarrow$ belongs to CH(P)

Plane Decomposition Representation



- Doubly Connected Edge List (DCEL)

– stores: vertices, edges and cells (faces);
– (undirected) edge: 2 twin (directed) half-edges

- Space complexity: $O(|V| + |E| + n)$,

$|V| = \#$vertices, $|E| = \#$edges, $n = \#$input sites.
– v: $O(1)$: coordinates, pointer to half-edge where v is starting.
– half-e $O(1)$: start v, right cell, pointer next/previous/twin half-e

- Operations:

– Given cell c, edge $e \subset c$, find (neighboring) cell c': $e \subset c'$: $O(1)$
– Given cell, print every edge of cell: $O(|E|)$.
– Given vertex v find all incident edges: $O(\#\text{neighbors})$.