Volume computation, sampling and applications

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Volume computation problem

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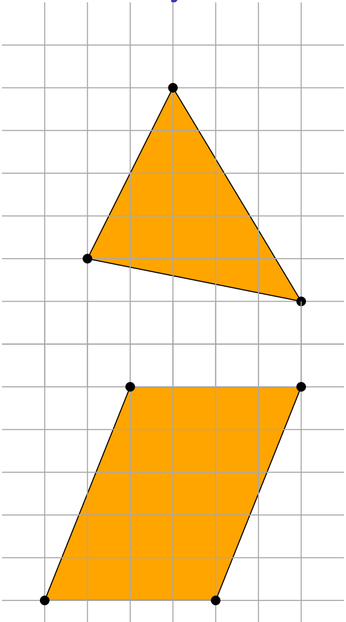
3. How large is d? e.g. $d = 2,\ 3,\ 50$

Easy cases: volume of elementary shapes



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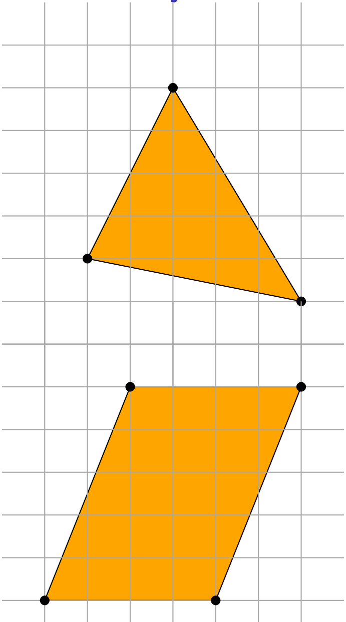

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 6 & 1 & 1 \end{vmatrix} / 2! = 11$$

$$\begin{vmatrix} 2 & 5 \\ 4 & 0 \end{vmatrix} = 20$$

Easy cases: planar polygons

A planar simple polygon with a positively oriented (counter clock wise) sequence of points $P_1, \ldots, P_n$, $P_i = (x_i, y_i), i = 1, \ldots, n$.

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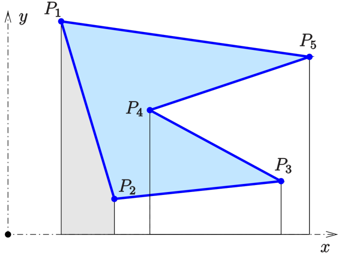

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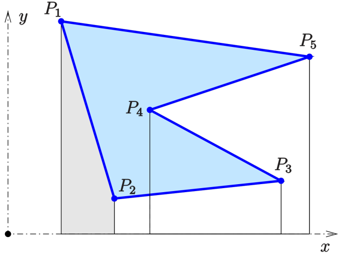
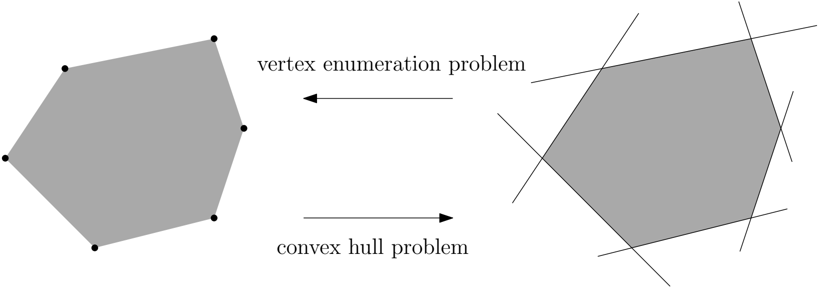


$$A = \frac{1}{2} \sum_{i=1}^{n} (y_i + y_{i+1})(x_i - x_{i+1})$$
$$= \frac{1}{2}\Big((y_1 + y_2)(x_1 - x_2) + \cdots + (y_n + y_1)(x_n - x_1)\Big)$$

Polytope Representations

A convex polytope $P \subseteq \mathbb{R}^d$ can be represented as the

1. convex hull of a pointset $\{p_1, \ldots, p_n\}$ (V-representation)
2. intersection of halfspaces $\{x \in \mathbb{R}^d \;:\; Ax \leq b\}$ (H-representation)



Faces of polytopes: vertices, edges, ..., facets

Volume computation problem (revisited)

Given P a convex polytope in $\mathbb{R}^d$ compute the volume of P.

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ChatGPT

Computing the volume of a convex polytope in $\mathbb{R}^d$ can be done using various methods depending on the representation of the polytope. Here's a general approach:

1. **Vertex Representation:** If the polytope is given by its vertices, you can use the formula for the volume of a convex polytope known as the "mixed volume." This formula generalizes the concept of volume to higher dimensions and is defined in terms of determinants.
2. **Half-space Representation:** If the polytope is given by the intersection of half-spaces (i.e., in the form $\{x \in \mathbb{R}^d : Ax \leq b\}$), then you can compute the volume using techniques like the Monte Carlo method or the Monte Carlo Markov Chain (MCMC) method. These methods involve sampling points within the polytope and estimating the volume based on the density of points.
3. **Convex Hull:** If you have a set of points representing the polytope, you can compute the convex hull of those points and then calculate the volume of the resulting convex hull.

Volume via triangulation

Algorithm: compute a triangulation of the input polytope, then sum up the volumes of simplices.

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Reference: Büeler, Enge, Fukuda - Exact Volume Computation for Polytopes: A Practical Study

Implementations

- `VINCI` [Bueler et al'00], `Latte` [deLoera et al], `Qhull` [Barber et al], `LRS` [Avis], `Normaliz` [Bruns et al]

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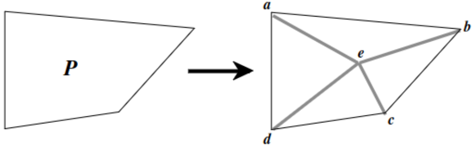

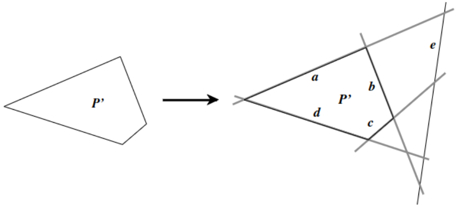

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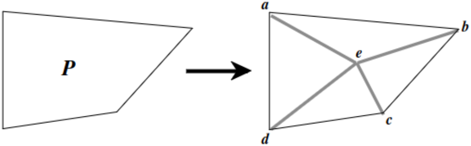
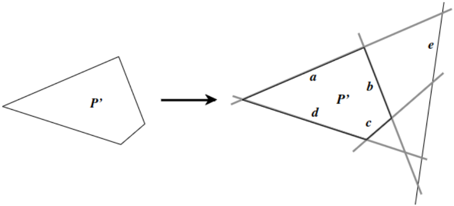
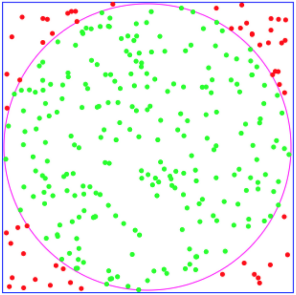




- cannot compute in high dimensions (e.g. > 15) in general

Volume via (naive) Monte Carlo

Rejections techniques (sample from bounding box)



Question: how to sample points from a cube?

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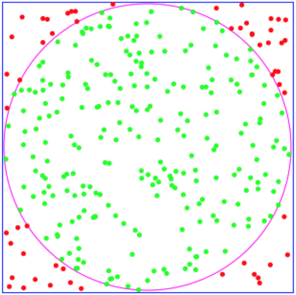


Question: how to sample points from a cube?

volume(unit cube) $= 1$
volume(unit ball) $\sim (c/d)^{d/2}$ –*drops exponentially with* d

Outline

Volumes, polytopes, applications

Algorithms and complexity

Applications

Randomized algorithms

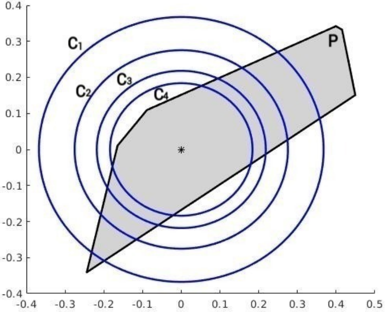
Volume algorithms parts

1. **M**ultiphase **M**onte **C**arlo (MMC)
   e.g. Sequence of balls, Annealing of functions
2. Sampling via geometric random walks
   e.g. grid-walk, ball-walk, hit-and-run, billiard walk

Notes:

- MMC (1) at each phase solves a sampling problem (2)
- geometric random walks are (most of the times) Marcov chains where each "event" is a d-dimensional point
- Algorithmic complexity is polynomial in d [Dyer, Frieze, Kannan'91]

Multiphase Monte Carlo



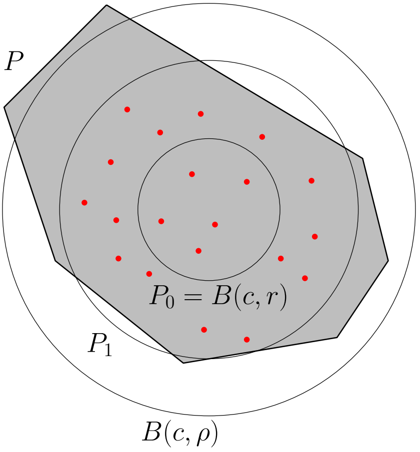
- Sequence of convex bodies $C_1 \supseteq \cdots \supseteq C_m$ intersecting P, then:

$$\text{vol}(P) = \text{vol}(P_m)\frac{\text{vol}(P_{m-1})}{\text{vol}(P_m)} \cdots \frac{\text{vol}(P_1)}{\text{vol}(P_2)}\frac{\text{vol}(P)}{\text{vol}(P_1)}$$

where $P_i = C_i \cap P$ for $i = 1, \ldots, m$.

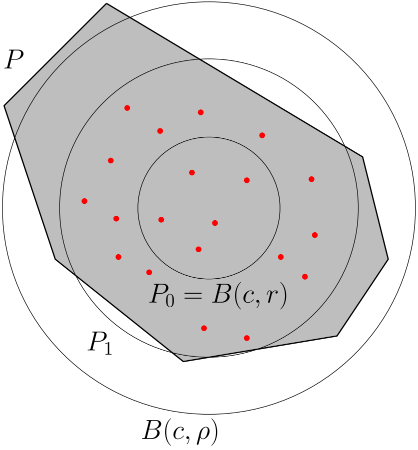
- Estimate ratios by sampling.

Multiphase Monte Carlo



- Sequence of k cocentric balls,
  $B_0 = B(c,r) \subseteq P \subseteq B(c,\rho) = B_k$
- Set $P_i = P \cap B_i$
- Estimate $\frac{vol(P_1)}{vol(P_0)}, \frac{vol(P_2)}{vol(P_1)} \dots$ via sampling
- $vol(P) = vol(P_0) \prod_{i=1}^{k} \frac{vol(P_i)}{vol(P_{i-1})}$
- How large is k?

Multiphase Monte Carlo



- $B(c, 2^{i/d})$, $i = \alpha, \alpha+1, \ldots, \beta$,
  $\alpha = \lfloor d \log r \rfloor$, $\beta = \lceil d \log \rho \rceil$
- $P_i := P \cap B(c, 2^{i/d}),\ i = \alpha, \alpha+1, \ldots, \beta$
  $P_\alpha = B(c, 2^{\alpha/d}) \subseteq B(c, r)$
- $k = d \log(\rho/r)$ where ρ/r is the "sandwitching ratio"

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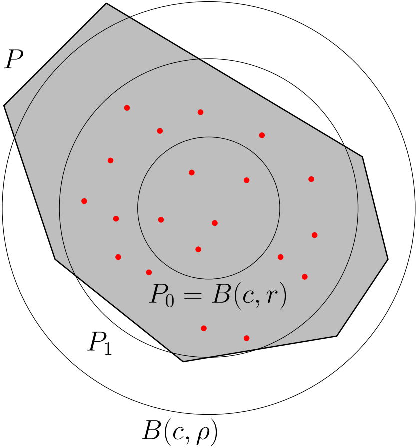

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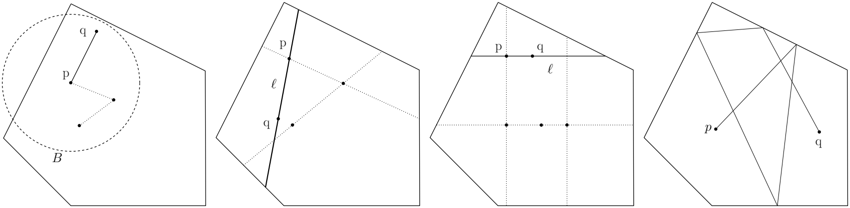
Using sampling the polytope can be transformed into "near isotropic position" such that $\rho/r = O(d)$ [Lovász et al.'97]

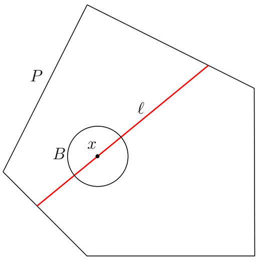
How we sample uniformly?

For arbitrary polytopes we need *random walks*

- Ball walk
- Random directions hit and run (rdhr)
- Cooridnate directions hit and run (cdhr)
- Billiard walk



Hit and run (random directions)



1. line ℓ through x, uniform on $B(x, 1)$
2. set x to be a uniform disrtibuted point on $P \cap \ell$

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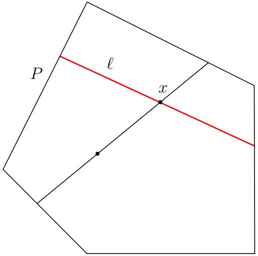

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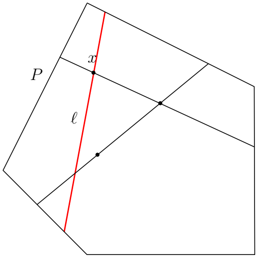

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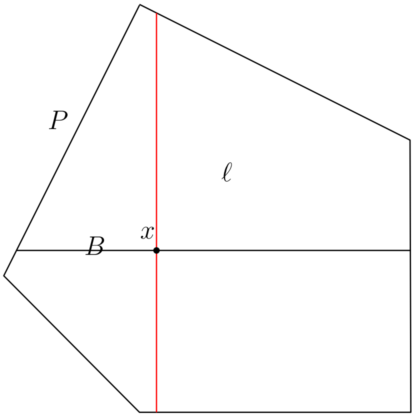

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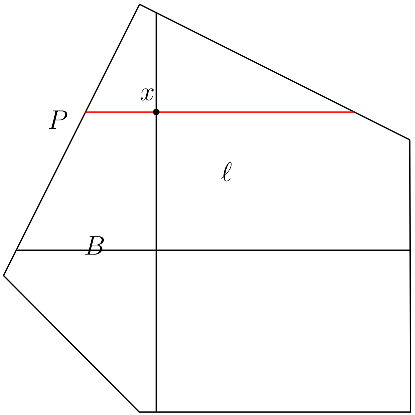

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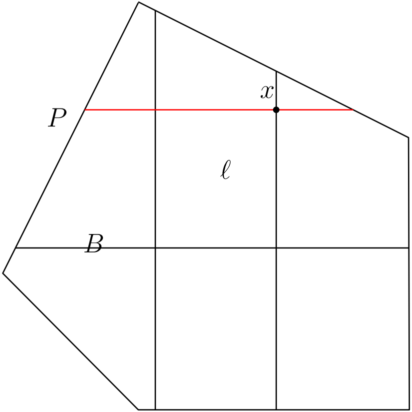

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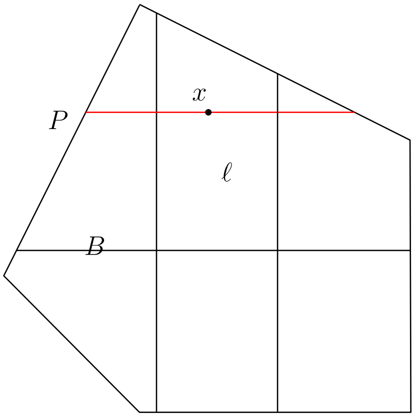
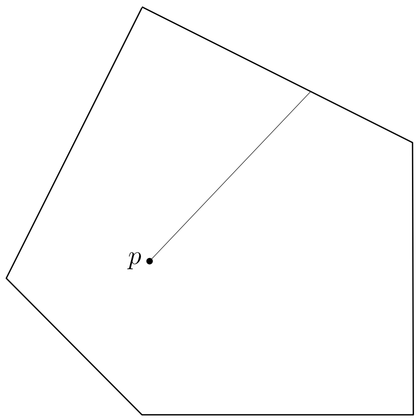


Billiard walk



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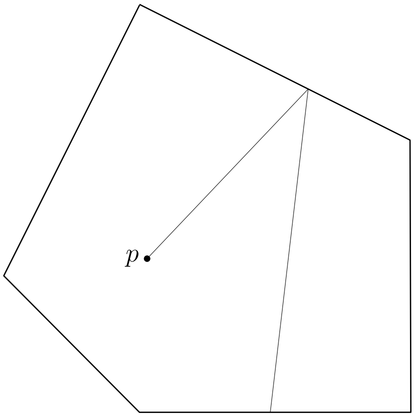

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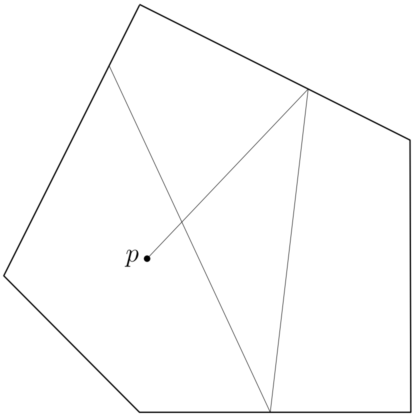

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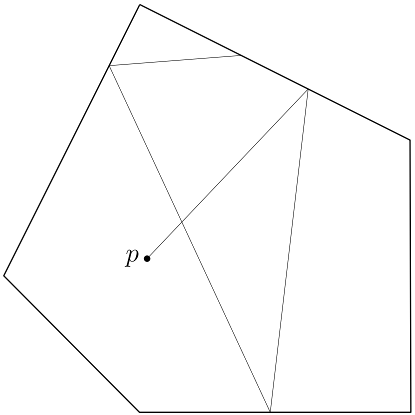

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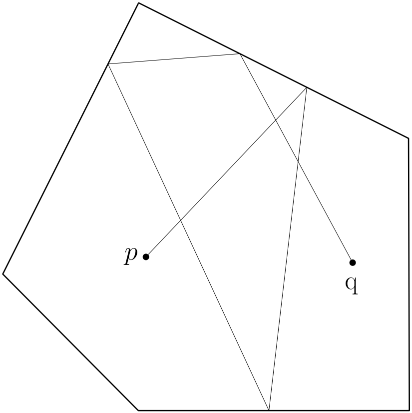
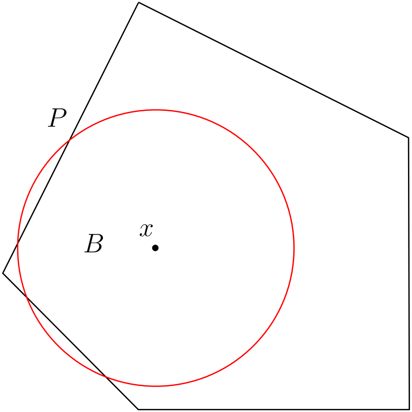


Two important parameters: number of reflections, total length

Ball walk



One important parameter: radius of the walk

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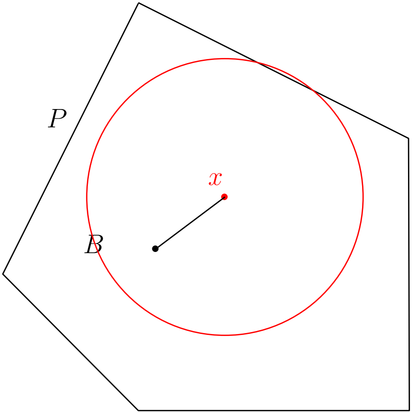

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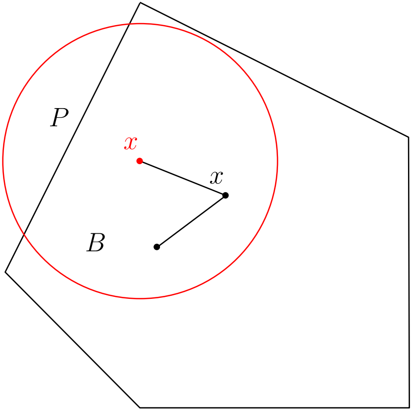

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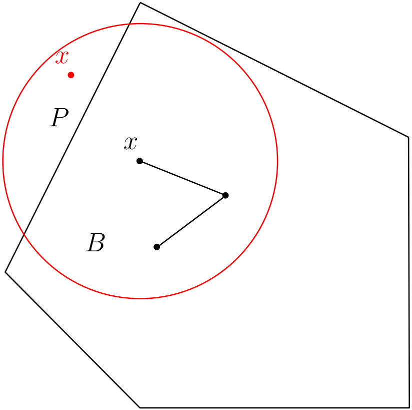

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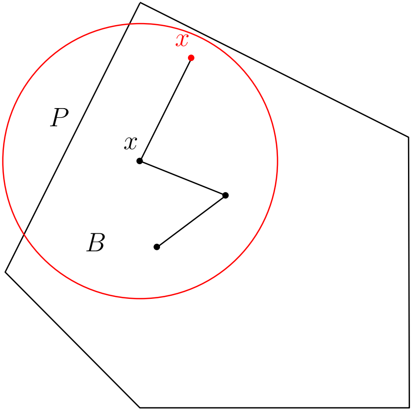

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Questions on random walks

- What is the representation of the polytope needed for each walk?
- How many steps needed to reach the target distribution?

Explicit Polytope Representations

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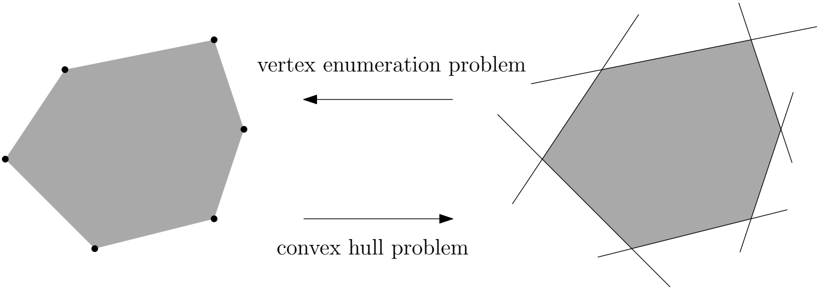
2. intersection of halfspaces $\{h_1, \ldots, h_n\}$ (H-representation)



Faces of polytopes: vertices, edges, . . . , facets

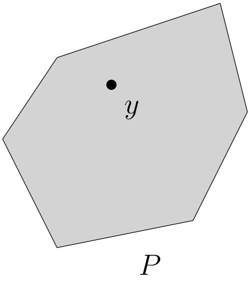
Implicit Polytope Representation (Oracles)

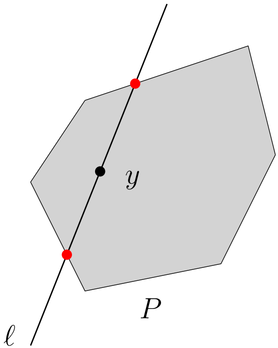
Membership oracle

Given point $y \in \mathbb{R}^d$, return yes if $y \in P$ otherwise return no.

Boundary oracle

Given point $y \in P$ and line ℓ goes through y return the points $\ell \cap \partial P$





Complexity [KannanLS'97]

Assuming $B(c,1) \subseteq P \subseteq B(c,\rho)$, the volume algorithm returns an estimation of vol(P), which lies between $(1-\epsilon)$vol(P) and $(1+\epsilon)$vol(P) with probability $\geq 3/4$, making

$$O^*(d^5)$$

oracle calls, where ρ is the radius of a bounding ball for P.

Techniques:
Isotropic sandwitching: $O^*(\sqrt{d})$ and ball walk.

Runtime steps

- generates $d \log d$ balls
- generate $N = 400\epsilon^{-2} d \log d$ random points in each ball $\cap P$
- each point is computed after $O^*(d^3)$ random walk steps

State-of-the-art

Theory:

Authors-Year	Complexity (oracle steps)	Algorithm
[Dyer, Frieze, Kannan'91]	$O^*(d^{23})$	Seq. of balls + grid walk
[Kannan, Lovasz, Simonovits'97]	$O^*(d^5)$	Seq. of balls + ball walk + isotropy
[Lovasz, Vempala'03]	$O^*(d^4)$	Annealing + hit-and-run
[Cousins, Vempala'15]	$O^*(d^3)$	Gaussian cooling (* well-rounded)
[Lee, Vempala'18]	$O^*(Fd^{\frac{2}{3}})$	Hamiltonian walk (** H-polytopes)

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Software:

1. [Emiris, F'14] Sequence of balls + coordinate hit-and-run
2. [Cousins, Vempala'16] Gaussian cooling + hit-and-run
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Notes:

- (2) is (theory + practice) faster than (1)
- (1),(2) efficient only for H-polytopes
- (3) efficient also for V-,Z-polytope, non-linear convex bodies
- C++ implementation of (2) $\times 10$ faster than original (MATLAB)

Problem complexity

Input: Polytope $P := \{x \in \mathbb{R}^d \mid Ax \leq b\}$ $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$

Output: Volume of P

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- #P-hard for vertex and for halfspace repres. [DyerFrieze'88]

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- open if both vertex (V-rep) & halfspace (H-rep) representation is available
- no deterministic poly-time algorithm can compute the volume with less than exponential relative error [Elekes'86]

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Complexity

- #P-hard for vertex and for halfspace repres. [DyerFrieze'88]
- open if both vertex (V-rep) & halfspace (H-rep) representation is available
- no deterministic poly-time algorithm can compute the volume with less than exponential relative error [Elekes'86]
- randomized poly-time approximation of volume of a convex body with high probability and arbitrarily small relative error [DyerFriezeKannan'91]
  $O^*(d^{23}) \to O^*(m^2 d^{\omega-1/3})$ [LeeVempala'18],
  $O^*(md^{4.5} + md^4)$ [MangoubiVishnoi'19]

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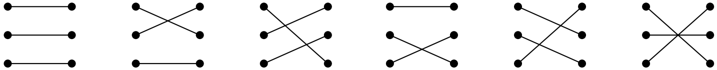
- 

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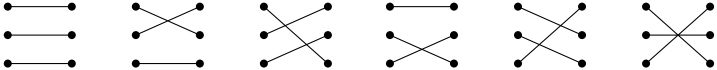
- 
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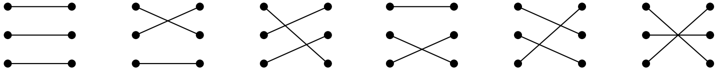
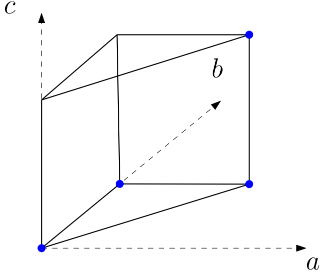
- 
- # faces of B_3: 6, 15, 18, 9; $\mathrm{vol}(B_3) = 9/8$
- there exist formulas for the volume [deLoera et al '07] but values only known for $n \leq 10$ after 1yr of parallel computing [Beck et al '03]

Volumes and counting

- Given n elements & partial order; order polytope $P_O \subseteq [0,1]^n$ coordinates of points satisfies the partial order



a, b, c

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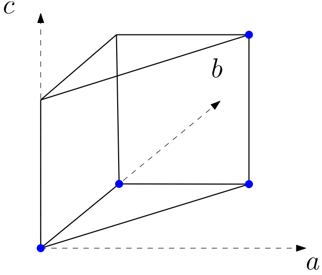

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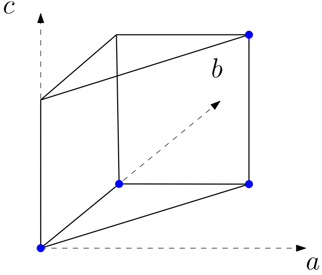

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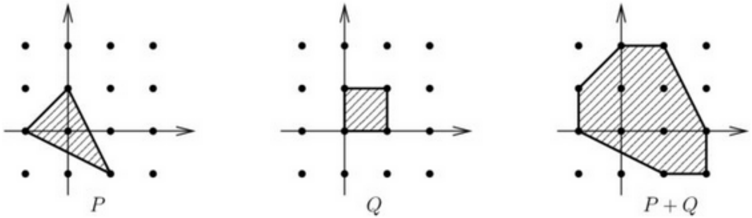
- # linear extensions $=$ volume of order polytope $\cdot\, n!$ [Stanley'86]
- Counting linear extensions is #P-hard [Brightwell'91]

Minkowski sum

The Minkowski sum of two convex sets P and Q is:

$$P + Q = \{p + q \mid p \in P, q \in Q\}$$



Volume of zonotopes (sums of segments) is used to test methods for order reduction which is important in several areas: autonomous driving, human-robot collaboration and smart grids

Mixed volume

Let $P_1, P_2, \ldots, P_d$ be polytopes in $\mathbb{R}^d$ then the mixed volume is

$$M(P_1, \ldots, P_d) = \sum_{I \subseteq \{1,2,\ldots,d\}} (-1)^{(d-|I|)} \cdot \mathrm{Vol}(\sum_{i \in I} P_i)$$

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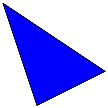
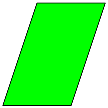
Example

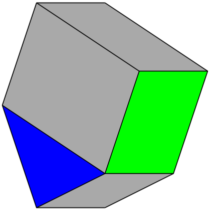
For $d = 2$: $M(P_1, P_2) = \mathrm{Vol}(P_1 + P_2) - \mathrm{Vol}(P_1) - \mathrm{Vol}(P_2)$



P_1



P_2



$P_1 + P_2$

Applications

Computing integrals for AI

- In Weighted Model Integration (WMI), given is a SMT formula and a weight function, then we want to compute the weight of the SMT formula.
- e.g. SMT formula:

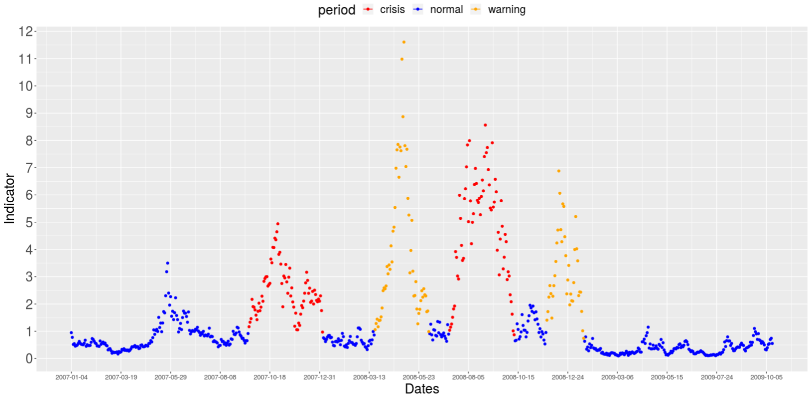
$$(A \mathbin{\&} (X > 20) \mid (X > 30)) \mathbin{\&} (X < 40)$$

Boolean formula + comparison operations. Let X has a weight function of $w(X) = X^2$ and $w(A) = 0.3$.

- WMI answers the question of the weight of this formula i.e. integration of a weight function over convex sets.
- [P.Z.D. Martires et al.2019]

Applications in finance

When is the next financial crisis?



Cales, Chalkis, Emiris, Fisikopoulos - Practical volume computation of structured convex bodies, and an application to modeling portfolio dependencies and financial crises, SoCG 2018

Software

1. Main library is volesti (C++):
   https://github.com/GeomScale/volesti
2. Two interfaces available: Python (https://github.com/GeomScale/dingo) and R (https://github.com/GeomScale/Rvolesti)
3. Google summer of code "internships" are available every year (applications in Spring, work on Summer)
4. Project topics for Google summer of code 2024:
   https://github.com/GeomScale/gsoc24/wiki/table-of-proposed-coding-projects
5. How to participate:
   https://github.com/GeomScale/gsoc24/wiki