

PRACNIQUES

The Techniques Department is interested in publishing short descriptions of Techniques which improve the logistics of information processing. To quote from the policy statement, *Communications of the ACM* 1 (Jan. 1958), 5: "It is preferable that techniques contributed be factual and in successful usage, rather than speculative or theoretical. One of the major criteria for acceptance and the question one should answer before submitting any material is—Can the reader use this tomorrow?" Clear, concise statements of fairly well-known but rarely documented methods will contribute significantly to raising the general level of professional competence.—C.L.McC.

CLOSING OUT A PRINT TAPE

Some computer operating systems have a bad habit of inserting tape marks into the output stream willy-nilly. This often results in loss of output when a tape is scratched after its first file has been printed. Other operating systems, notably those recently distributed by IBM for use with the 7090, put tape marks on print tapes only on command, so that frequently a print tape is removed from the computer without any tape mark, and extraneous material is printed.

There is a simple procedure which eliminates these difficulties. Whenever a computer halt is imminent,

WRITE A TAPE MARK ON THE PRINT TAPE
AND BACKSPACE.

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A NOTE ON A SET OF TEST MATRICES FOR INVERSION

An additional set of test matrices to those presented by M. L. Pei [*Comm. ACM* 5, 10 (Oct., 1962)] and discussed by W. S. La Sor [*Comm. ACM* 6, 3 (Mar. 1963)] but having one further property is:

$$Q = q_{ij} = \begin{cases} 1 & \text{if } i = j \\ t & \text{if } i \neq j \end{cases}.$$

As in Pei's matrix, Q has an easily computed inverse:

$$Q^{-1} = q_{ij}^{-1} = \begin{cases} \frac{\frac{1}{t} \left(\frac{1}{t} + n - 2 \right)}{\frac{1}{t} \left(\frac{1}{t} + n - 2 \right) - (n - 1)}, & i = j, \\ -\frac{1}{t} \\ \frac{\frac{1}{t} \left(\frac{1}{t} + n - 2 \right) - (n - 1)}{\frac{1}{t} \left(\frac{1}{t} + n - 2 \right) - (n - 1)}, & i \neq j. \end{cases}$$

Using La Sor's argument it is easily shown that:

$$|Q| = t^n \left(\frac{1}{t} - 1 \right)^{n-1} \left(\frac{1}{t} + n - 1 \right)$$

and

$$\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = 1 - t, \quad \lambda_n = 1 + nt - t,$$

where λ_i is an eigenvalue of Q .

The additional property of Q which makes it useful for certain

classes of problems (e.g., a montecarlo inversion) is:

$$\max_i |\lambda_i(I-Q)| \leq |t(n-1)|,$$

where $\lambda_i(I-Q)$ is an eigenvalue of the matrix $I - Q$. Indeed, the eigenvalues of $I - Q$ are:

$$\lambda_1(I-Q) = \lambda_2(I-Q) = \dots = \lambda_{n-1}(I-Q) = t, \quad \lambda_n(I-Q) = t(1-n)$$

e.g., by varying t appropriately one might observe how the rate of convergence of a Monte Carlo inversion changes.

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PEI MATRIX EIGENVECTORS

M. L. Pei [*Comm. ACM* 5, 10 (Oct. 1962)] gave an explicit inverse for a matrix of the form $M + \delta I$, where M is an n -square matrix of ones and δ is a nonzero parameter. The eigenvalues of the Pei matrix were given by W. S. LaSor [*Comm. ACM* 6, 3 (Mar. 1963)]. The eigenvectors may be obtained by considering the system $(M + \delta I)\bar{x} = \lambda\bar{x}$, the j th equation of which is

$$S + \delta x_j = \lambda x_j, \quad (1)$$

where S denotes $\sum_{i=1}^n x_i$. On summing the equations for $j = 1, 2, \dots, n$, we obtain $nS + \delta S = \lambda S$. From this we conclude that (a) $S = 0$ or (b) $\lambda = n + \delta$.

Corresponding to case (a) we can construct $n - 1$ orthogonal eigenvectors \bar{V}_k , $k = 2, 3, \dots, n$, with all components zero except the first and the k th, which are 1, -1 respectively. These vectors satisfy (1) for all j , provided $\lambda = \delta$. The eigenvector for case (b) is uniquely determined in direction by the requirement that it must be orthogonal to all the \bar{V}_k . In summary therefore, we find that: δ is an $(n-1)$ -fold eigenvalue whose corresponding eigenvectors have the property that the algebraic sum of their components is zero. The remaining eigenvalue is $n\delta$; the components of the corresponding eigenvector are all equal.

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NOTE ON STOCHASTIC MATRICES

There are systems which consist of a number of states, any one of which may be reached, in due course, from any other. In addition, the path through the system, of anything entering it, can be described probabilistically.

Such systems are sometimes represented by trees, or state diagrams, and often more compactly by transition matrices. If the matrix is regular stochastic, there is the advantage that the fixed point can be found, and also an equilibrium condition of the system thus described.

If, however, each state s_i has associated with it some time t_i so that any element entering s_i remains in that state for time t_i , the description in the last paragraph becomes less satisfying. In fact, the above transition matrix cannot be used if, for example, it was desired to investigate the behavior of the system from some starting situation (vector) for a certain number of units of time. For now any state s_i becomes as many states as there are units of time in t_i , with the probability 1 of an element remaining in s_i during all units of time except the last one.

A matrix showing all intermediate states can become inordinately large. However, a simple relation between the large and reduced matrices will be shown, which will give one the choice of using one or the other as the need arises.

Consider the $r \times r$ stochastic matrix, $\mathbf{P} = (p_{ij})$, where $\sum_{j=1}^r p_{ij} = 1$, $i = 1, 2, \dots, r$, and $p_{ij} \geq 0$, and suppose that there exists a $1 \times r$ matrix (vector) $\mathbf{a}' = (a_1, a_2, \dots, a_r)$ such that $\mathbf{a}'\mathbf{P} = \mathbf{a}'$, $a_1 + a_2 + \dots + a_r = 1$.

Let us enlarge the matrix P by replacing the diagonal element p_{ii} by the $n_i \times n_i$ matrix

$$\mathbf{P}_{ii} = \begin{pmatrix} 0100 \dots\dots 0 \\ 0010 \dots\dots 0 \\ 0001 \dots\dots 0 \\ \dots\dots\dots \\ 0000 \dots\dots 1 \\ p_{ii}000 \dots\dots 0 \end{pmatrix}$$

and element p_{ij} by the $n_i \times n_j$ matrix

$$\mathbf{P}_{ij} = \begin{pmatrix} 000 \dots\dots 0 \\ 000 \dots\dots 0 \\ 000 \dots\dots 0 \\ \dots\dots\dots \\ 000 \dots\dots 0 \\ P_{ij}00 \dots\dots 0 \end{pmatrix}$$

Now set $n = n_1 + n_2 + n_3 + \dots + n_r$ and denote the enlarged $n \times n$ matrix by \mathbf{Q} , that is, $\mathbf{Q} = (\mathbf{P}_{ij})$. Note that \mathbf{Q} is clearly also a stochastic matrix.

Under the conditions stated above there exists a $1 \times n$ matrix (vector) $\mathbf{b}' = (b_1, b_2, \dots, b_n)$ such that $\mathbf{b}'\mathbf{Q} = \mathbf{b}'$, $b_1 + b_2 + \dots + b_n = 1$, and $b_i = a_i / (n_1 a_1 + n_2 a_2 + \dots + n_r a_r)$ where $n_1 + \dots + n_{i-1} + 1 \leq i \leq n_1 + \dots + n_{i-1} + n_i$, $i = 1, 2, \dots, r$.

For convenience in presentation, let us illustrate the method of proof for the case $r = 3$, $n_1 = 4$, $n_2 = 3$, $n_3 = 2$. The equations $\mathbf{a}'\mathbf{P} = \mathbf{a}'$ are:

$$\begin{aligned} a_1 p_{11} + a_2 p_{21} + a_3 p_{31} &= a_1 \\ a_1 p_{12} + a_2 p_{22} + a_3 p_{32} &= a_2 \\ a_1 p_{13} + a_2 p_{23} + a_3 p_{33} &= a_3 \end{aligned}$$

and the equations $\mathbf{b}'\mathbf{Q} = \mathbf{b}'$ are:

$$\begin{aligned} b_4 p_{11} + b_7 p_{21} + b_9 p_{31} &= b_1 \\ b_1 &= b_2 \\ b_2 &= b_3 \\ b_3 &= b_4 \\ b_4 p_{12} + b_7 p_{22} + b_9 p_{32} &= b_5 \\ b_5 &= b_6 \\ b_6 &= b_7 \\ b_4 p_{13} + b_7 p_{23} + b_9 p_{33} &= b_8 \\ b_8 &= b_9 \end{aligned}$$

It is clear that $b_1 = b_2 = b_3 = b_4$, $b_5 = b_6 = b_7$ and $b_8 = b_9$.

Were we to set $b_1 = a_1$, $b_5 = a_2$ and $b_8 = a_3$ and make like substitutions for their equals, the vector would still be transformed into itself. However, the sum of the elements of the vector would be larger than 1. The correction is made by multiplying each element of the vector by the reciprocal of the sum, or $1/(4a_1 + 3a_2 + 2a_3)$.

The foregoing was very useful directly and as a check, in a practical problem of some importance. It is hoped that it may be to other users. The writer would be interested to learn whether this point has been clearly covered anywhere; he has not been able to find anything on it. This is the kind of thing that has become important or practical only since the development of large scale computing facilities.

Incidentally, many observations in matrix theory are originally arrived at by other means; in this case, through common algebra.

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A SEMI-ITERATIVE PROCESS FOR EVALUATING ARCTANGENTS

The technique of obtaining arctangents by inverse interpolation [1] is a relatively long process not suitable for a subroutine. The Taylor series expansion for arguments less than unity converges rather slowly for those near unity. The method of small increments of the argument [2] is again inconvenient for a subroutine. A more rapid series expansion in terms of Chebyshev polynomial [3] is given in terms of a new argument, which is less than 0.1989. This method requires the storage of π , $\sqrt{2} - 1$ and seven coefficients and is perhaps widely used. However, for multiple precision not only the coefficients have to be evaluated to the precision desired, but more must be used. Therefore the following alternative method may prove to be convenient and efficient.

First, the argument will be decreased to a desired value by an iterative scheme. Without loss of generality the argument A may be taken in the range $0 \leq A \leq 1$. Consider

$$\tan^{-1} A = 2 \tan^{-1} A_1, \quad A = \tan(2 \tan^{-1} A_1) = \frac{2A_1}{1 - A_1^2} \quad (1a)$$

Thus,

$$A_1 = \frac{A}{1 + \sqrt{1 + A^2}} \quad (1b)^1$$

If the process is repeated n times,

$$\tan^{-1} A = 2^n \tan^{-1} A_n \quad (2)$$

$$A_n = \frac{A_{n-1}}{1 + \sqrt{1 + A_{n-1}^2}}, \quad A_0 = A. \quad (3)$$

Since $A_n \leq A_{n-1}/2 \leq A_0/2^n \leq 1/2^n$, the argument is quickly reduced.

A pure iterative scheme as described with $\tan^{-1} A_\infty = A_\infty$ is inefficient to obtain an answer, as for each iteration the square root routine must be employed and it only converges as $1/2^n$. The next step is then to apply truncated Taylor's series² to the arctangent of the reduced argument A_n . The truncation error can be easily estimated. One has

$$\tan^{-1} A = 2^n \sum_{m=1}^M \frac{A_n^{2m-1}}{2m-1} \quad (4)$$

of which the percentage error is less than

$$E = \frac{A_n^{2M}}{(2M+1) \left(1 - \frac{1}{3} \cdot \frac{1}{2^{2M}}\right)} < \frac{1}{2^{2Mn} (2M+1) \left(1 - \frac{1}{3} \cdot \frac{1}{2^{2M}}\right)} \quad (5)$$

For single precision with percentage error less than 10^{-9} , say, $n = 3$, $M = 5$ is sufficient, as $E < \frac{1}{2^{30}} \left(\frac{192}{11 \times 191}\right)$, $2^{30} = 1024$.

For double precision with percentage error less than 10^{-17} , say, $n = 4$, $M = 7$ is sufficient as $E < \frac{1}{2^{56}} \left(\frac{768}{15 \times 767}\right)$, $10^{-17} < 2^{-57}$.

¹This expression is preferable to the alternative form $[\sqrt{(1+A^2)}-1]/A$, as the latter may lose some significant figure for small values of A .

²"Telescoped" Taylor series can be used if so desired [4], but a few more storage spaces are required than for a simple truncated series.