Test Matrix Collection

(Non-Hermitian Eigenvalue Problems)

Work in Progress

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1 Introduction

The primary purpose of this collection is to provide a testbed for developing numerical algorithms for solving nonsymmetric eigenvalue problems. It is a part of ongoing numerical linear algebra test matrix collection project. In addition, as with many other existing collections of test matrices, our goal includes providing an easy access to "practical" eigenproblems for researchers and educators who are interested in the origins of large scale nonsymmetric eigenvalue problems, and in the development and testing of numerical algorithms.

In this document, we describe the mechanism for obtaining a copy of the the matrices and for using the collection. All test matrices currently included in the collection are documented in Appendices A and B.

2 How to obtain the collection

The complete collection will be available through direct electronic transfer, anonymous ftp and worldwide Web (HTML user interface). Currently, individual test matrices can be obtained from Zhaojun Bai (`bai@ms.uky.edu`) or David Day (`dday@ms.uky.edu`).

3 Matrix Formats

There are two formats in which to represent the sparse matrices in the collection, namely the standard sparse matrix format and the matrix-vector multiplication format.

3.1 Sparse column format

The matrices stored in data files are stored in sparse column format (i.e., Harwell-Boeing format). In this section sparse column format is summarized. See the User's Guide for the Harwell-Boeing Sparse Matrix Collection for details.

In any sparse matrix format only the non-zero entries of a matrix are stored. An entry is specified by its row index, column index, and value. In sparse column format a single integer array and a single floating point array are used to store the row indices and the values, respectively, for all columns. The data for each column are stored in consecutive locations, the columns are stored in order, and there is no space between columns. An other integer array points to the first entry in each column.

The sparse matrix A of order n and with nnz non-zero entries is represented by a floating point array $a(1:nnz)$ that stores the values, an integer array $ia(1:nnz)$ that stores the row indices, and an integer array $ja(1:n+1)$ with the property that $ja(k+1)-1$ points to the last element in column k.

3.2 Matrix-vector multiplication format

Many of the matrices in this collection are presented in a different format than the standard Harwell-Boeing format. Instead of storing the matrix in a data file, the matrix is represented as a subroutine that computes the matrix-vector multiplications Ax and A^Ty given x and y. The calling sequences for the FORTRAN and C subroutines for the matrix-vector multiplications Ax and A^Ty are described with A.

4 How to use the collection

4.1 Sparse column format

The matrices stored in Harwell-Boeing format are accessed exactly as matrices in the Harwell-Boeing collection. The matrices are read using code described in the Harwell-Boeing Users Guide and available in SPARSKIT.

4.2 Matrix-vector multiplication format

For the matrices stored in the matrix-vector multiplication format, these matrix-vector multiplications are computed directly by calling the corresponding subroutine.

4.3 Conversion

FORTRAN subroutines are available in SPARSKIT to convert the sparse column format to other sparse matrix storage formats.

4.4 Other utilities

The following subroutines are available for

1. computing the matrix-vector multiplications Ax and A^Tx where matrix A is stored in sparse column format.
2. writing in sparse column format when the matrix A is in matrix vector multiplications format.
3. converting Matlab sparse format to the sparse column format.

5 Related Work

- I. S. Duff, R. G. Grimes and J. G. Lewis, Sparse matrix test problems, ACM Trans. Math. Softw. 15, 1–14, 1989
- I. S. Duff, R. G. Grimes and J. G. Lewis, User's Guide for the Harwell-Boeing Sparse Matrix Collection (Release I), CERFACS, TR/PA/92/86, Oct. 1992. It is available via anonymous ftp: orion.cerfacs.fr, cd pub/harwell_boeing.

- Y. Saad, SPARSKIT: A basic tool kit for sparse matrix computation. It is available via anonymous ftp: ftp.cs.umn.edu, cd ...

Appendix A
Matrices in Standard Sparse Matrix Format

Title	Keys
Transient stability analysis (Airfoils)	AF23560
Bounded finline waveguide eigenmodes	BFWA62, BFWB62, BFWA398, BFWB398, BFWA782, BFWB782
Chuck	CK104, CK400, CK656
Quantum chemistry	QC324, QC2534
Square dielectric waveguide	DW256A, DW256B, DW1024, DW4096
MHD Alfven spectrum	MHDA416, MHDB416 MHDA1280, MHDB1280 (not included yet) MHDA3200, MHDB3200 (not included yet) MHDA4800, MHDB4800 (not included yet)
Quebec Hydro Power system	QH882
Compute system evaluation	LOP163
Robotic control	RBSA480, RBSB480
Tubular reactor model	TUB100, TUB1000
Reaction-diffusion Brusselator model	RDB968, RDB2048, RDB5000
Olmstead model	OLM100, OLM500, OLM1000, OLM2000, OLM5000
Symmetrical pipe Poiseuille Flow	SPPxxxx (not included yet)

The following data files are also available, but the corresponding matrices are described in Appendix B.

Title	Keys
Brusselator wave model	BWM200, BWM2000
Model eigenproblem of ODE	ODEA400, ODEB400
Model 2-D convection-diffusion operator	CDDE1, CDDE2, CDDE3, CDDE4, CDDE5
Markov chain modeling (random walk)	RW136, RW496, RW5151
Model PDE	PDE225, PDE2961

Title: Transient Stability Analysis of Navier-Stokes Solver
Key: AF23560

Source: A. Mahajan, NASA Lewis Research Center

Discipline: Computational fluid dynamics

Further details: This test matrix is from transient stability analysis of Navier-Stokes solvers, supplied by Dr. A. Mahajan at NASA Lewis Research Center. The order of the test matrix is 23,560. The eigenvalues and eigenvectors are associated with small perturbation analysis of a finite difference representation of the Navier-Stokes equations for flows over airfoils. Such eigensystem information is central to stability analysis of Navier-Stokes solvers, for determining the modal bahavior of fluid in a fluid-structure interaction problem and for development of reduced order models based on variational principles for Navier-Stokes solvers. A repensentative eigenvalue constellation is reported in the references, where the Lanczos procedure with no re-biorthogonalization is used. The number of Lanczos iteration is between 1,000 to 1,200.

Data files:

Filename/Key	Order	Number of entries
AF23560	23,560	484,256

References:

A. Mahajan, E. H. Dowell, and D. Bliss. Eigenvalue calculation procedure for an Euler/Navier-Stokes solver with applications to flows over airfoils. *J. of Comput. Phy.*, 97:398–413, 1991.

A. Mahajan, E. H. Dowell, and D. Bliss. *AIAA J.*, 29:555–xxx, 1991.

Title: Bounded Finline Dielectric Waveguide
Key: DWA62

Source: B. Shultz and S. Gedney, Unviersity of Kentucky

Discipline: Electrical engineering

Further details: Mellimeter wave technology has been applied in radar, communication, radiometry and intruements. Finline waveude is an example of a bounded waveguide which operates extremely well in the millimeter wave spectrum. The generalized eigenvalue problem

$$Ax = \lambda Bx$$

studied in Shultz's thesis is the discretization problem of the Maxwell's equation by the finite element method (see [Fernandez and Lu] and [Jin]) for finding the propagating modes and magentic field profiles of a rectangular waveguide filled with dielectric and PEC structures. The eigenvalues and corresponding eigenvectors of interest are the ones with positive real parts, which correspond to the propagation modes of a waveguide. The matrix A is non-symmetric and B is symmetric positive definite. Although only real data is collected here, in application, complex matrices may be involved.

Data files:

Filename/Key	Order	Number of entries
BFWA62	62	450
BFWB62	62	342
BFWA398	398	3678
BFWB398	398	2910
BFWA782	782	7514
BFWB782	782	5982

References:

B. Shultz, Bounded Waveguide Eigenmodes, Finite Element Method Solution, Masters Thesis, Department of Electrical Engineering, Unviersity of Kentucky, 1994.

F. A. Fernandez and Y. Lu, Variational finite element analysis of dielectric waveguide with non spurious solutions, Electron. Letter., 26(25):2125–2126, 1990.

J. Jin, The Finite Element Method in Electromagnetics, John Wiley & Sons, Inc., New York, 1993.

Title: Chuck Matrices
Key: CK104

Source: Chuck Siewert of North Carolina State.

Discipline: unknown

Further details: According to the following reference, this set of test matrices is supplied by Chuck Siewert of North Carolina State. It is not clear what is the application background of these matrices. The objective is to compute those eigenvalues with magnitudes greater than 1. The matrices have several multiple eigenvalues and clusters of eigenvalues. The eigenvalues occur in clusters of order 4; each cluster consists of two pairs of very nearly multiple eigenvalues.

Data files:

Filename/Key	Order	Number of entries
CK104	104	992
CK400	400	2860
CK656	656	3884

References:

J. Cullum and R. A. Willoughby, A Practical Procedure for Computing Eigenvalues of Large Sparse Nonsymmetric Matrices, In Large Scale Eigenvalue Problem, J. Cullum and R. A. Willoughby eds. Elsevier Science Pub. North-Holland, 1986.

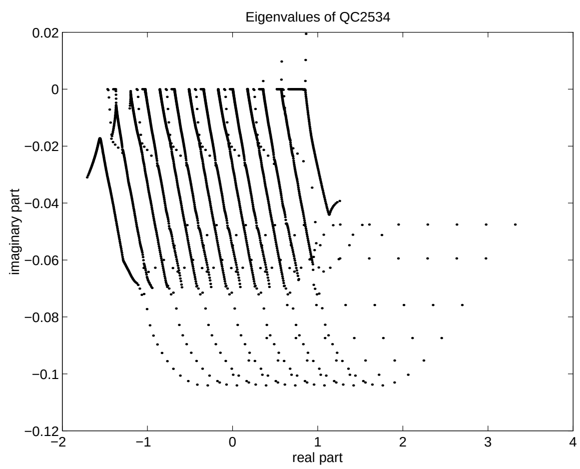
Title: Quantum Chemistry
Key: QC324

Source: S. Chu, Unviersity of Kansas

Discipline: Computational chemistry

Further details: The physical problem is to solve the eigenvalue problem for $\mathbf{H}^{2+}$ in an electro-magnetic field. The complex-scaling technique applied to the self-adjoint Hamiltonian from quantium mechanics yields a non-self-adjoint eigenvalue problem. To determine the stability of the system, the Floquet exponents are computed by the standard Fourier-grid method. The two matrices given here are complex symmetric.

The computational task is to determine the eigenvalues nearest to or above the real axis. The figure displays the eigenvalues computed by the QR algorithm for the matrix QC2534.



Data files:

Filename/Key	Order	Number of entries
QC324	324	26,730
QC2534	2534	463,360

References:

S.I. Chu, Journal of Chemical Physics, 94, 7901 – (1991)

Title: Square Dielectric Waveguide
Key: DW256A

Source: H. Dong, Unviersity of Minnesota

Discipline: Electrical engineering

Further details: Dielectric channel waveguide problems arise in many integrated circuit applications. Discretization of the governing Helmholtz equation for the magnetic field H,

$$\begin{aligned} \nabla^2 H_x + k^2 n^2(x,y) H_x &= \beta^2 H_x, \\ \nabla^2 H_y + k^2 n^2(x,y) H_y &= \beta^2 H_y, \end{aligned}$$

by finite differences leads to a nonsymmetric eigenvalue problem of the form

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} H_x \\ H_y \end{pmatrix} = \beta^2 \begin{pmatrix} B_{11} & \\ & B_{22} \end{pmatrix} \begin{pmatrix} H_x \\ H_y \end{pmatrix}$$

where C_{11} and C_{22} are five- or tri-diagonal matrices, C_{12} and C_{21} are (tri-) diagonal matrices, and B_{11} and B_{22} are nonsingular diagonal matrices. This generalized eigenvalue problem is reduced to a standard eigenvalue problem $Ax = x\lambda$, where $A = B^{-1}C$, since B is diagonal.

The computational task is to determine the right most eigenvalues and their corresponding eigenvectors. In some cases, there are eigenvalues with negative real part several orders of magnitude larger than the desired eigenvalues with positive real part. This problem presents a challenge to existing numerical methods.

Data files:

Filename/Key	Order	Number of entries
DW256A	256	2816
DW256B	256	2816
DW1024	1024	11264
DW4096	4096	45056

Note that DW256A and DW256B are the matrices of same order but different parameters. **Double check, the matrix sizes are double, as pointed by Michiel Kooper**

References:

H. Dong, A. Chronopoulos, J. Zou and A. Gopinath, Vectorial integrated finite-difference analysis of dielectric waveguides, private communication, 1993

A. Galick, T. Kerhoven and U. Ravaioli, Iterative solution of the eigenvalue problem for a dielectric waveguid, IEEE Trans. Microwave Theory Tech. vol. MTT-40, pp.699–705, 1992.

Title: Magnetohydrodynamics
Key: MHD416

Source: A. Booten, M. N. Kooper, H. A. van der Vorst, S. Poedts and J. P. Goedbloed, University of Utrecht, the Netherlands.

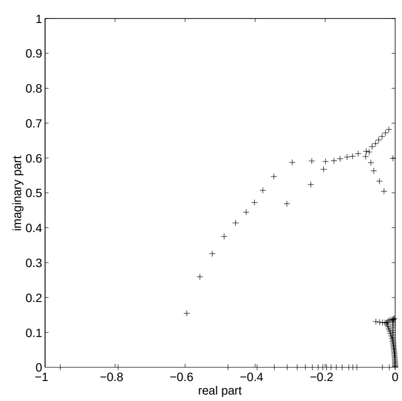
Discipline: Plasma Physics

Further details Large nonsymmetric generalized matrix eigenvalue problems ($Ax = \lambda Bx$) arise in the modal analysis of dissipative magnetohydrodynamics (MHD). The MHD system combines Maxwell's and fluid flow equations. The physical objective of these MHD systems is to derive nuclear energy from the fusion of light nuclei. The plasmas generated exhibit both the characteristics of an ordinary fluid and special features caused by the magnetic field. The study of linearised motion in MHD has contributed significantly to the understanding of resistive and nonadiabatics MHD plasma phenomena, such as plasma stability, wave propagation and hearing.

The MHD equations are solved by applying the Galerkin method in conjunction with finite elements, which leads to the generalized eigenvalue problem. The corresponding eigenproblem comprises complicated eigenvalue patterns having different orders of magnitude corresponding to the very different time scales of the behavior involved in the system. Great details are given in literature on the problem formulation, discretization, and numerical solution of the resulting generalized eigenvalue problem.

Kerner reports difficulties in numerically computing the eigenvalues of the Alfeven wave operator. The spectrum of this operator consists of three branches. He reports that the eigenvalues at the intersection of the branches are very hard to compute, which is similar to the spectra of the Orr-Sommerfeld equation.

The Alfven spectra of the test problem



Data files:

Filename/Key	Order	Number of entries
MHDA416	416	8562
MHDB416	416	2312
MHDA1280	1280	
MHDB1280	1280	
MHDA3200	3200	
MHDB3200	3200	
MHDA4800	4800	
MHDB4800	4800	

References:

W. Kerner, Large-scale complex eigenvalue problem, J. of Comp. Phy. 85:1–85, 1989.

J. Cullum, W. Kerner and R. Willoughby, A generalized nonsymmetric Lanczos procedure, Comp. Phys. Comm., 53:19–48, 1989

M.N. Kooper, H.A. van der Vorst, S. Poedts, and J.P. Goedbloed, Application of the implicitly updated Arnoldi methods with a complex shift and invert strategy in MHD, *Journal of Comp. Phys.*, **118**, 320-328, 1995.

J. G. L. Booten, P. M. Meijer, H. J. J. te Riele and H. A. van der Vorst, Parallel Arnoldi method for the construction of a Krylov subspace basis: an application in magnetohydrodynamics. In Proceedings of International Conference and Exhibition on High-Performance Computing and Networking, Munich, Germany, April, 1994. Vol.II: Networking and Tools, W. Gentzsch and U. Harms, eds., Lecture Notes in Computer Science 797, Springer-Verlag, Berlin, 1994.

J. G. L. Booten, H. A. van der Vorst, P. M. Meijer and J. J. te Riele, A preconditioned Jacobi-Davidson method for solving large generalized eigenvalue problems, 1994

Z. Bai, D. Day and Q. Ye, ABLE: an adaptive block Lanczos method for non-Hermitian eigenvalue problems, Research Report 95-04, Department of Mathematics, University of Kentucky, May 1995

Title: Quebec Hydroelectric Power System
Key: QH882

Source: Deo Ndereyimana, Quebec, Canada

Discipline: Power systems simulations

Further details: QH882 represents the Hydro-Quebec power systems's small-signal model. In the application, one wants to compute all eigenvalues $a+ib$ in a box of the complex plane. Specifically, $a_{\min} < a < a_{\max}$ and in general, $a_{\min} = -300$ and $a_{\max} = 100$. $b_{\min} < b < b_{\max}$ and in general, $b_{\min} = 0$ and $b_{\max} = 2 \times 60\pi$.

Data files:

Filename/Key	Order	Number of entries
QH882	882	3354

References:

Deo Ndereyimana, private communication, 1994

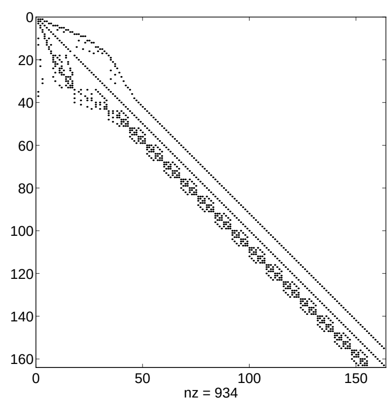
Title: LOPSI Stochastic Test Matrix
Key: LOP163

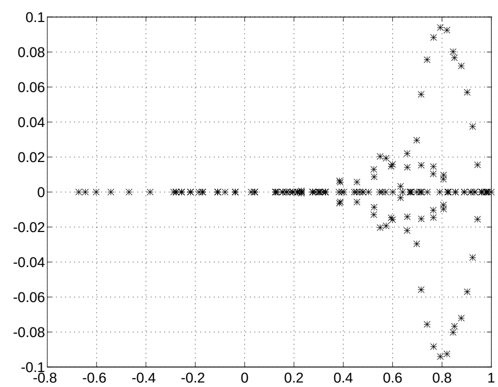
Source: W. J. Stewart, North Carolina State University and A. Jennings, Queen's University, Belfast, North Ireland.

Discipline: Analysis and evaluation of computer systems

Further details: This stochastic matrix is derived from the application of Markov modeling techniques to the analysis and evaluation of computer systems [?]. The matrix is a 163 by 163 stochastic matrix with 935 nonzero entries. The object is to compute a few dominant eigenvalues and their corresponding eigenvectors.

The following left figure shows the sparsity structure of the 163 by 163 stochastic matrix, and the right figure shows the eigenvalue distribution.





Data files:

Filename/Key	Order	Number of entries
LOP163	163	936

We note that the number of nonzero entries in our count is 936, however, the recorded count shown in the paper of Stewart and Jennings is 1207. The 15 dominant eigenvalues computed from the resulting matrix agree, however, with the results in the paper of Stewart and Jennings.

References:

W. J. Stewart, A comparison of numerical techniques in Markov modeling, Comm. ACM 21(2):144–152,1978.

W. J. Stewart and A. Jennings, A simultaneous iteration algorithm for real matrices, ACM Trans. Math. Soft. 7:184–198, 1981.

I. S. Duff and J. A. Scott, Computing selected eigenvalues of sparse unsymmetric matrices using subspace iteration, ACM TOMS ...

Title: Forward Kinematics for the Stewart platform of Robotics
Key: RBSA480

Source: J. Canny, I. Emris, H. Ren, UC Berkeley

Discipline: Robotic Control

Further details: The Stewart platform, also called left hand, is a *parallel manipulator* with six prismatic joints connecting two rigid bodies, or platforms. The base platform is considered fixed while the top platform, or end-effector, is moving in 3-dimensional space, controlled by the lenghts of joints. Parallel robots are especially useful when high stiffness and position precision are predominant requirements. The platform has one degree of freedom per joint; the position and orientation of the top platform is specified by six parameters, namely three for the orientation and three for the position in 3D space. The forward kinematics assumes that the leg lengths are known and the displacement of the top platform is to be found. The algrbraic problem reduces to the solution of a well-constrained system of polynomial equations. By using resultant (sparse resultant) method, solving the system of polynomial equations can reduce to solve the eigenproblem of a matrix pencil. An interesting user request for this problem is *only* the real eigenvalues and the corresponding eigenvectors are needed but empirically, most of the eigenvalues are complex.

Data files:

Filename/Key	Order	Number of entries
RBSA480	480	17088
RBSB480	480	17088

References:

D. Manocha and J. Canny, MultiPolynomial Resultant Algorithms, Computer Science Division Report, University of California, Berkeley.

I. Emiris, Sparse Elimination and Applications in Kinematics, Ph.D. thesis, Computer Science Division, University of California at Berkeley, 1993.

Title: Tubular Reactor Model
Subroutine name: TUB100

Source: K. Meerbergen and D. Roose, Katholieke Universiteit Leuven, Belgium

Discipline: computational fluid dynamics

Further details: The conservation of reactant and energy in a homogeneous tube of length L in dimensionless form is modeled by

$$\begin{aligned} \frac{L}{v}\frac{dy}{dt} &= -\frac{1}{Pe_m}\frac{\partial^2 y}{\partial X^2} + \frac{\partial y}{\partial X} + Dy\exp(\gamma - \gamma T^{-1}), \\ \frac{L}{v}\frac{dT}{dt} &= -\frac{1}{Pe_h}\frac{\partial^2 T}{\partial X^2} + \frac{\partial T}{\partial X} + \beta(T - T_0) - BDy\exp(\gamma - \gamma T^{-1}), \end{aligned}$$

where y and T represent concentration and temperature and $X \in [0.1]$ denote the spatial coordinate. Boundary conditions are $y'(0) = Pe_m y(0)$, $T'(0) = Pe_h T(0)$, $y'(1) = 0$ and $T'(1) = 0$. Central differences are used to discretize in space. For $x^T = [y_1, T_1, y_2, T_2, \ldots, y_{N/2}, T_{N/2}]$, the equations can be written as $\dot{x} = f(x)$. The parameters in the differential equation are set to $Pe_m = Pe_h = 5, B = 0.5, \gamma = 25, \beta = 3.5$ and $D = 0.2662$. One wants to compute the rightmost eigenvalues of the Jacobi matrix $A = \partial f / \partial x$. A is a banded matrix with bandwidth 5.

Data files:

Filename/Key	Order	Number of entries
TUB100	100	396
TUB1000	1000	3996

References:

R. F. Heinemann and A. B. Poore, Multiplicity, stability, and oscillatory dynamics of a tubular reactor, Chem. Eng. Sci. 36:1411-1419, 1981

T. J. Garratt, The numerical detection of Hopf bifurcations in large systems arising in fluid mechanics, PhD thesis, University of Bath, UK, 1991

K. Meerbergen and D. Roose, Matrix transformation for computing rightmost eigenvalues of large sparse nonsymmetric matrices, Report TW 206, Department of Computer Science, Katheolieke Universiteit Leuven, Belgium, 1994 (revised April 1995).

Title: Olmstead model
Subroutine name: OLM1000

Source: K. Meerbergen, Katholieke Universiteit Leuven, Belgium

Discipline: Hydrodynamics

Further details: Olmstead model represents the flow of a layer of viscoelastic fluid heated from below. The equations are

$$\begin{aligned} \frac{\partial u}{\partial t} &= (1-C)\frac{\partial^2 v}{\partial X^2} + C\frac{\partial^2 u}{\partial X^2} + Ru - u^3 \\ B\frac{\partial v}{\partial t} &= u - v \end{aligned}$$

with boundary conditions $u(0) = u(1) = 0$ and $v(0) = v(1) = 0$. u represents the speed of the fluid and v is related to viscoelastic forces. The equation was discretised with central differences with grid size $h = 1/(N/2)$. After discretization the equation can be written as $\dot{x} = f(x)$ with $x^T = [u_1, v_1, u_2, v_2, \ldots, u_{N/2}, v_{N/2}]$. One wants to compute the rightmost eigenvalues of the Jacobi matrix $A = \partial f/\partial x$ with parameters $B = 2$, $C = 0.1$ and $R = 4.7$.

Subroutine: FORTRAN calling sequences for forming matrix-vector Jx and $J^T x$

Data files:

Filename/Key	Order	Number of entries
OLM100	100	396
OLM500	500	1996
OLM1000	1000	3996
OLM2000	2000	7996
OLM5000	5000	19996

References:

W. E. Olmstead, W. E. Davis, S. H. Rosenblat and W. L. Kath, Bifurcation with memory, SIAM J. Appl. Math. 46:171–188, 1986

K. Meerbergen and A. Spence, A spectral transformation for finding complex eigenvalues of large sparse nonsymmetric matrices, Report TW 219, Department of Computer Science, Katheolieke Universiteit Leuven, Belgium, 1994

K. Meerbergen and D. Roose, Matrix transformation for computing rightmost eigenvalues of large sparse nonsymmetric matrices, Report TW 206, Department of Computer Science, Katheolieke Universiteit Leuven, Belgium, 1994 (revised April 1995).

Title: Reaction-diffusion Brusselator Model
Subroutine name: RDB968

Source: K. Meerbergen, Katholieke Universiteit Leuven, Belgium and A. Spence, University of Bath, UK.

Discipline: Chemical engineering

Further details: The equations

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{D_u}{L^2}\left(\frac{\partial^2 u}{\partial X^2}+\frac{\partial^2 u}{\partial Y^2}\right)-(B+1)u+u^2v+C\\ \frac{\partial v}{\partial t} &= \frac{D_v}{L^2}\left(\frac{\partial^2 v}{\partial X^2}+\frac{\partial^2 v}{\partial Y^2}\right)-u^2v+Bu\end{aligned}$$

for u and $v \in (0,1)\times(0,1)$ with homogeneous Dirichlet boundary conditions form a 2D reaction-diffusion model where u and v represent the concentrations of two reactions. The equatiosn are discretized with central differences with grid size $h_u = h_v = 1/(n+1)$ with $n = (N/2)^{1/2}$. For $x^T = [u_{1,1}, v_{1,1}, u_{1,2}, v_{1,2}, \ldots, u_{n,n}, v_{n,n}]$, the disretized equations can be written as $\dot{x} = f(x)$. One wants to compute the rightmost eigenvalues of the Jacobi matrix $A = \partial f/\partial x$. Where the parameters $B = 5.45, C = 2, D_u = 0.004, D_v = 0.008$ and $L = 1$.

Data files:

Filename/Key	Order	Number of entries
RDB968	968	5632
RDB2048	2048	12032
RDB5000	5000	29600

References:

B. D. Hassard, N. Kazarinoff and Y. H. Wan, Theory and Applications of Hopf Bifurcation, Cambridge University Press, Cambridge, 1981

K. Meerbergen and A. Spence, A spectral transformation for finding complex eigenvalues of large sparse nonsymmetric matrices, Report TW 219, Department of Computer Science, Katheolieke Universiteit Leuven, Belgium, 1994

Appendix B
Matrices in Matrix-Vector Multiplication Format

Title	Subroutine Names
Random Sparse Matrix	MVMRAN
Brusselator wave model in chemical reaction	MVMBWM
Model 2-D convection-diffusion problem	MVMMCD
Grcar matrix	MVMGRA
Ising model of ferromagnetic materials	MVMISI
Model eigenproblem of ODE	MVPODE
Model PDE	MVMPDE
Markov chain modeling (random walk)	MVMRW
Tolosa matrix	MVMTLS

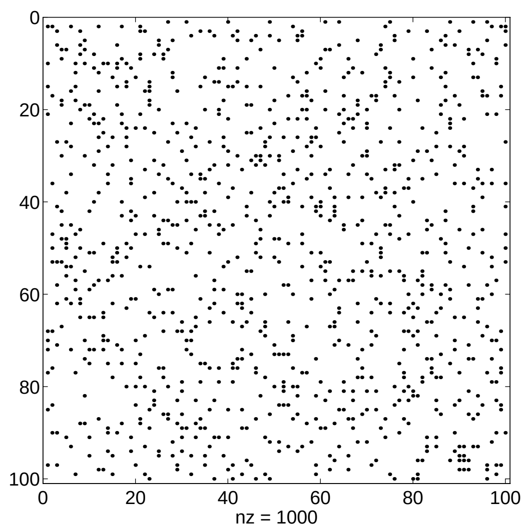
Title: Random Sparse Matrix
Subroutine name: MVMRAN

Source: J. Cullum, IBM T. J. Watson Research Center

Discipline: numerical linear algebra

Further details: Random matrices are always the favorite test matrices in an algorithm test suit. We provide a subroutine which generates a random sparse matrix. Users can specify the matrix size, the number of nonzero entries in each column, the distribution of random numbers, and the norm (scale) of the generated matrix.

The following figure shows an example of the sparsity pattern of a random 100 by 100 matrix with 10 nonzero entries in each column.



Subroutine: FORTRAN calling sequences for forming matrix-vector Jx and J^Tx is

```
SUBROUTINE MVMRAN( TRANS, N, X, Z )
```

with

`TRANS`	(input) CHARACTER*1 if `TRANS = 'N'`, computes the product Ax, if `TRANS = 'T'`, computes the product A^Tx,
`N`	(input) INTEGER The order of the matrix J.
`X`	(input) REAL array, dimension (`N`) contains the vector x.
`Z`	(output) REAL array, dimension (`N`) On return, `Z` contains the product Ax if `TRANS = 'N'`, or the product A^Tx if `TRANS = 'T'`,

In addition, the parameters k should be passed using a common block.

References:

J. Cullum, private communication, 1992

Title: Brusselator wave model in chemical reaction
Subroutine name: MVMBWM

Source: Y. Saad, University of Minnesota

Discipline: Chemical engineering

Further details: This problem models the concentration waves for reaction and transport interaction of chemical solutions in a tubular reactor. The concentrations $x(t,z)$ and $y(t,z)$ of two reacting and diffusing components are modeled by the system

$$\frac{\partial x}{\partial t} = \frac{\delta_1}{L^2}\frac{\partial^2 x}{\partial^2 z} + f(x,y), \quad (1)$$

$$\frac{\partial y}{\partial t} = \frac{\delta_2}{L^2}\frac{\partial^2 y}{\partial^2 z} + g(x,y), \quad (2)$$

with the initial conditions $x(0,z) = x_0(z)$, $y(0,z) = y_0(z)$ and the Dirichlet boundary conditions $x(t,0) = x(t,1) = x^*$, $y(t,0) = y(t,1) = y^*$, where $0 \le z \le 1$ is the space coordinate along the tube, and t is time. Raschman *et al* considered in particular the so-called Brusselator wave model in which

$$f(x,y) = \alpha - (\beta+1)x + x^2y, \quad g(x,y) = \beta x - x^2 y.$$

Then, the above system admits the trivial stationary solution $x^* = \alpha$, $y^* = \beta/\alpha$. In this problem one is primarily interested in the existence of stable periodic solutions to the system as the bifurcation parameter L varies. This occurs when the eigenvalues of largest real parts of the Jacobian of the right hand side of (1) and (2), evaluated at the steady station solution, is purely imaginary. For the purpose of verifying this fact numerically, one first needs to discretize the equations with respect to the variable z and compute the eigenvalues with largest real parts of the resulting discrete Jacobian.

If we discretize the interval $[0,1]$ using n interior points with the mesh size $h = 1/(m+1)$. Then the discretized Jacobian of the system is a 2×2 block matrix of the form

$$J = \begin{pmatrix} \tau_1 T + (\beta-1)I & \alpha^2 I \\ -\beta I & \tau_2 T - \alpha^2 I \end{pmatrix}$$

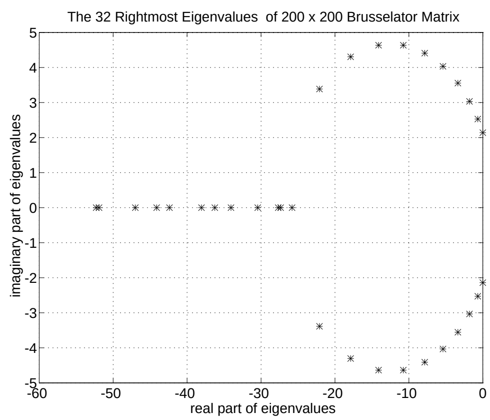
where $T = \text{tridiag}\{1,-2,1\}$, $\tau_1 = \frac{1}{h^2}\frac{\delta_1}{L^2}$ and $\tau_2 = \frac{1}{h^2}\frac{\delta_2}{L^2}$

The exact eigenvalues of J are known since there exists a quadratic relation between the eigenvalues of the matrix A and those of the classical difference matrix tridiag$\{1,-2,1\}$. The following is the Matlab M-file for computing the $2m$ eigenvalues of J:

```
h = 1/(m+1); tau1  = delta1/(h*L)^2; tau2  = delta2/(h*L)^2;
for j=1:m,
   eigofT(j) = -2*(1- cos(pi*j*h) );  % eigenvalues of T
end;
for j=1:m,
   coeff(1) = 1;
   coeff(2) = alpha^2 - (beta - 1) - (tau1+tau2)*eigofT(j);
```

```
    coeff(3) = beta*alpha^2 + tau1*tau2*eigofT(j)^2 + ...
               tau2*(beta-1)*eigofT(j) - ...
               alpha^2*tau1*eigofT(j) - alpha^2*(beta-1);
    d = roots(coeff);
    Jeig(j) = d(1);  Jeig(m+j) = d(2); % eigenvalues of J
end;
```

The following figure shows the 32 rightmost eigenvalue distribution of 200 by 200 of the matrix J ($m = 100$) corresponding to the set of parameters $\delta_1 = 0.008$, $\delta_2 = \frac{1}{2}\delta_1 = 0.004$, $\alpha = 2$, $\beta = 5.45, L \approx 0.51302$ as used in Saad's book.



Subroutine: FORTRAN calling sequences for forming matrix-vector Jx and J^Tx is

```
SUBROUTINE MVMBWM( TRANS, N, X, Z )
```

with

- `TRANS` (input) CHARACTER*1
  if `TRANS = 'N'`, computes the product Jx,
  if `TRANS = 'T'`, computes the product J^Tx,
- `N` (input) INTEGER
  The order of the matrix J.
- `X` (input) REAL array, dimension (`N`)
  contains the vector x.
- `Z` (output) REAL array, dimension (`N`)
  On return, `Z` contains the product Jx if `TRANS = 'N'`, or the product J^Tx if `TRANS = 'T'`,

In addition, the parameters $\delta_1, \delta_2, L, \alpha$ and β should be passed using a common block.

Data files:

Filename/Key	Order	Number of entries
BW200	200	796
BW2000	2000	7996

References:

P. Raschman, M. Kubicek and M. Maros, Waves in distributed chemical systems: experiments and computations, P. J. Holmes ed., New Approaches to Nonlinear Problems in Dynamics - Proceedings of the Asilomar Conference Ground, Pacific Grove, California 1979. The Engineering Foundation, SIAM, pp.271–288, 1980.

Y. Saad, Numerical Methods for Large Eigenvalue Problems, Halsted Press, Div. of John Wiley & Sons, Inc., New York, 1992

Title: Model 2-D Convection Diffusion Problem
Subroutine name: MVMMCD

Source: not sure

Discipline: computational fluid dynamics

Further details: This test matrix is from the following constant-coefficient convection-diffusion equation, which is widely used in literature for testing and analyzing numerical methods for the solution of linear system of equations. The equation reads

$$\begin{aligned} -\Delta u + 2p_1 u_x + 2p_2 u_y - p_3 u &= f \quad \text{in} \quad \Omega \\ u &= g \quad \text{on} \quad \partial\Omega \end{aligned}$$

where Ω is the unit square $\{(x,y) \in \mathbf{R}^2, 0 \le x, y \le 1\}$, p_1, p_2 and p_3 are positive constants. Discretization by finite differences with a 5-point stencil on a uniform $m \times m$ grid gives rise to a sparse linear system of equations

$$Au = b$$

where A is of order $n = m^2$ and u and b are now vectors of size n. If the grid points are numbered using the rowwise natural ordering, then A is a block tridiagonal matrix of the form

$$A = \begin{pmatrix} T & (\beta+1)I & & & \\ (-\beta+1)I & T & (\beta+1)I & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & (\beta+1)I \\ & & & (-\beta+1)I & T \end{pmatrix}$$

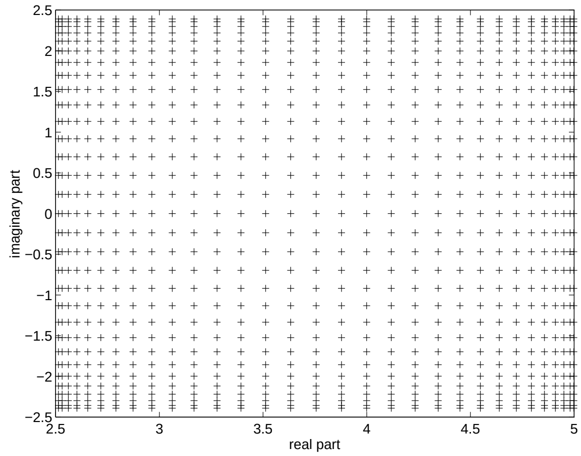
with

$$T = \begin{pmatrix} 4-\sigma & \gamma-1 & & & \\ -\gamma-1 & 4-\sigma & \gamma-1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \gamma-1 \\ & & & -\gamma-1 & 4-\sigma \end{pmatrix},$$

where $\beta = p_1 h, \gamma = p_2 h, \sigma = p_3 h^2$ and $h = 1/(n+1)$. The exact eigenvalues of A are known and are given by

$$\lambda_{kl} = 4 - \sigma + 2(1-\beta^2)^{1/2} \cos\frac{k\pi}{n+1} + 2(1-\gamma^2)^{1/2} \cos\frac{l\pi}{n+1}$$

for $k, l = 1, 2, \ldots, n$. The following figure shows the eigenvalue distribution of 961 by 961 convection diffusion matrix with $p_1 = 25, p_2 = 50$ and $p_3 = 250$.



We note that, in practice, the choice of the mesh size h and the coefficients p_1, p_2 and p_3 have to satisfy certain conditions for the discretization to be stable. We refer the reader to the references for the discussion of this issue.

Subroutine: FORTRAN calling sequences for forming matrix-vector Ax and x^TA is

```
SUBROUTINE MVMMCD( TRANS, N, X, Z )
```

with

`TRANS`	(input) CHARACTER*1 if `TRANS = 'N'`, computes the product Ax, if `TRANS = 'T'`, computes the product A^Tx,
`N`	(input) INTEGER The order of the matrix C.
`X`	(input) REAL array, dimension (`N`) contains the vector x.
`Z`	(output) REAL array, dimension (`N`) On return, `Z` contains the product Ax if `TRANS = 'N'`, or the product A^Tx if `TRANS = 'T'`,

In addition, the parameters p_1, p_2 and p_3 should be passed to the subroutine using a common block.

Data files:

Elman and Streit tested preconditioners for linear systems on six convection-diffusion matrices arising on a 31 by 31 grid. Note that when p_1 and p_2 are large compared to the grid size, the local error in the discretization is significant. Also note that when p_1 and p_2 are large the solution, u, forms boundary layers which are not practical to resolve using regular grids. The matrices correspond to the following choices of p_1,p_2 and p_3.

Filename/Key	Order	Number of entries	(p_1, p_2, p_3)
CDDE1	961	4681	(1,2,30)
CDDE2	961	4681	(25,50,30)
CDDE3	961	4681	(1,2,80)
CDDE4	961	4681	(25,50,80)
CDDE5	961	4681	(1,2,250)
CDDE6	961	4681	(25,50,250)

References:

H. C. Elman and R. L Streit, Polynomial iteration for nonsymmetric indefinite linear systems, Lec. Notes in Math. Vol. 1230, J. P. Hennart, ed. Numerical Analysis Proceedings, Gauuajsato, Mexico, Springe Verlag, 1984.

Y. Saad, Variations on Arnoldi's method for computing eigenelements of large unsymmetric matrices, Lin. Alg. Appl. 34:269–295, 1980.

Z. Bai and G. W. Stewart, SRRIT – A FORTRAN subroutine to calculator the dominant invariant subspaces of a nonsymmetric matrix, Comp. Sci. Dept. Tech. Rep. TR-2908, Univ. of Maryland, MD, April 1992, (submitted to ACM TOMS).

Z. Jia, Some numerical methods for large unsymmetric eigenproblems, Ph.D. thesis, The faculty of Mathematics, University of Bielefeld, Germany, Feb. 1994.

R. B. Lehoucq, Analysis and implementation of an implicitly Restarted Arnoldi Iteration, Ph.D thesis, Department of Computational and Applied Methematics, Rice University, TR95-13, May 1995.

Title: Grcar Matrix
Subroutine name: MVMGRA

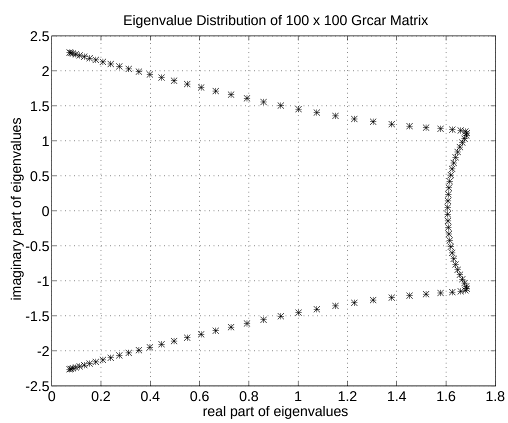
Source: J. Grcar, Sandia National Lab.

Discipline: numerical linear algebra

Further details: An $n \times n$ Grcar matrix is a nonsymmetric Toeplitz matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & & & & \\ -1 & 1 & 1 & 1 & 1 & & & \\ & -1 & 1 & 1 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & -1 & 1 & 1 & 1 & 1 \\ & & & & -1 & 1 & 1 & 1 \\ & & & & & -1 & 1 & 1 \\ & & & & & & -1 & 1 \end{pmatrix}$$

The matrix has sensitive eigenvalues. The following figure shows the eigenvalue distribution of 100 by 100 Grcar matrix.



Subroutine: FORTRAN calling sequences for forming matrix-vector Ax and $A^T x$ is

```
SUBROUTINE MVMGRA( TRANS, N, X, Z )
```

with

`TRANS` (input) CHARACTER*1
if `TRANS = 'N'`, computes the product Ax,
if `TRANS = 'T'`, computes the product $A^T x$,

`N` (input) INTEGER
The order of the matrix J.

`X` (input) REAL array, dimension (`N`)
contains the vector x.

Z (output) REAL array, dimension (N)
On return, Z contains the product Ax if TRANS = 'N', or the product $A^T x$ if TRANS = 'T',

References:

J. Grcar, Operator coefficient methods for linear equations, Sandia National Lab. Rep. SAND89-8691, Nov. 1989.

N.M. Nachtigal, L. Reichel and L.N. Trefethen, A hybrid GMRES algorithm for nonsymmetric linear systems, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 796-825.

N. J. Higham, A collection of test matrices in MATLAB, ACM Trans. Math. Softw. 17:289–305, 1991. Second edition, ...

Title: Ising model for ferromagnetic materials
Subroutine name: MVMISI

Source: B. Friedman

Discipline: Material science

Further details: This test matrix is from the analysis of the Ising model for ferromagnetic materials. The matrix A is the product of the two $2m$ by $2m$ matrices K and L,

$$A = KL$$

where

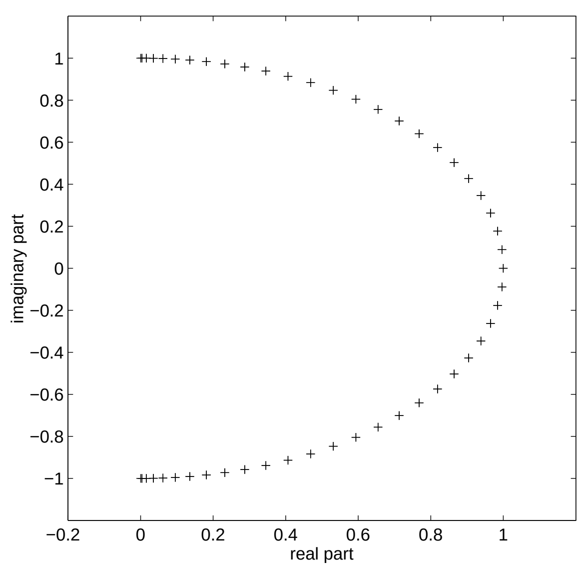
$$K = \begin{pmatrix} E & & & & \\ & E & & & \\ & & \ddots & & \\ & & & E & \\ & & & & E \end{pmatrix}, \qquad L = \begin{pmatrix} \cos\beta & & & & -\sin\beta \\ & F & & & \\ & & \ddots & & \\ & & & F & \\ \sin\beta & & & & \cos\beta \end{pmatrix}$$

$$E = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \qquad F = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}.$$

It can be shown that the eigenvalues of A are the $2m$ numbers that are obtained by computing the eigenvalues of the m 2 by 2 matrices

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\theta^k \sin\beta \\ \theta^{m-k}\sin\beta & \cos\beta \end{pmatrix}$$

for $k = 1, 2, \ldots, m$. The following figure shows the eigenvalue distribution of 100 by 100 Ising matrix with $\alpha = \pi/4$ and $\beta = \pi/4$



SUBROUTINE: FORTRAN calling sequences (heading) for forming matrix-vector Ax and $x^T A$.

```
SUBROUTINE MVMISI( TRANS, N, X, Z )
```

with

- `TRANS` (input) CHARACTER*1
  if `TRANS = 'N'`, computes the product Ax,
  if `TRANS = 'T'`, computes the product $A^T x$,
- `N` (input) INTEGER
  The order of the matrix A.
- `X` (input) REAL array, dimension (`N`)
  contains the vector x.
- `Z` (output) REAL array, dimension (`N`)
  On return, `Z` contains the product Ax if `TRANS = 'N'`, or the product $A^T x$ if `TRANS = 'T'`,

In addition, the parameters α and β should be passed using a common block.

Data files: not available, need to find out what are the values of parameters and how large of the matrix size.

References:

B. Kaufman, Crystal statistics II, Phys. Rev. 76, pp.1232, 1949.

B. Friedman, Eigenvalues of Composite matrices, Proc. Cambridge Philos. Soc. 57, pp.37-49, 1961

M. Marcus and H. Minc, A survey of Matrix Theory and matrix inequalities, Dover edition, New York, 1992.

Afternotes:

This Ising model was proposed to explain properties of ferromagnets but since then it has found application to topics in chemistry and biology as well as physics. For any reader unfamiliar with the model an excellent introduction is [B. A. Cipra, An Introduction to the Ising Model, American Mathematical Monthly, 94:937–959, 1987].

A numerical method for approximating the leading eigenvalues of 2D Ising models using a transfer matrix of order 2^n with $n = 30$ is reported in [B. Parlett and W. Heng, The Method of Minimal Representations in 2D Ising Model Calculations, PAM-549, University of California, Berkeley, May 1992].

We plan to include the transfer matrix in the future version of this collection.

Title: Model Eigenvalue Problem of ODE
Subroutine name: MVMODE

Source: G. W. Stewart, University of Maryland

Discipline: Model eigenvalue problem of ordinary differential equation.

Further details: Consider the following eigenvalue problem of an ordinary differential equation

$$y'' + \mu^2 y = 0 \tag{3}$$

with the boundary conditions

$$y(0) = 0 \quad \text{and} \quad y'(0) + \gamma y'(1) = 0, \quad 0 < \gamma < 1.$$

It can be shown that the eigenvalues μ are given by

$$\mu = i \cosh^{-1}(-\gamma^{-1}),$$

which are complex. Using the basic complex analysis, one obtains that

$$\mu^2 = ((2k+1)^2\pi^2 - \ln^2 \sigma) - i(2(2k+1)\pi \ln \sigma),$$

for $k = 0, \pm 1, \pm 2, \ldots$, where $\sigma = 1/\gamma \pm \sqrt{1/\gamma^2 - 1}$.

The eigenproblem of (3) can be approximated by finite differences as follows. Let y_i denote the approximate solution at the point $x_i = i/(n+1)$ $(i = 0, 1, \ldots, n)$. Replacing the derivatives in (3) with three point difference operators, we obtain the following $(n+1)$ by $(n+1)$ generalized matrix eigenvalue problem for $y = (y_1, y_2, \ldots, y_{n+1})^{\mathrm{T}}$:

$$Ay = -\mu^2 By, \tag{4}$$

where

$$A = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ 4 & -1 & \cdots & \gamma & -4\gamma & 3\gamma \end{pmatrix}$$

and $B = h^2 \mathrm{diag}(1, 1, \ldots, 1, 0)$. Problem (4) can be recast as the standard eigenvalue problem

$$Cy = -\frac{1}{\mu^2} y,$$

where $C = A^{-1}B$.

Subroutine: The matrix-vector products $z = Cx$ can be formed by solving the linear system $Az = Bx$ for z using the banded Gaussian elimination.

The matrix-vector products $z^T = x^T C$ can be formed by solving the linear system $A^T w = x$ for w using the banded Gaussian elimination and then compute $z^T = w^T B$.

FORTRAN calling sequences for forming matrix-vector Cx and $x^T C$ is

```
SUBROUTINE MVMODE( TRANS, N, X, Z )
```

with

TRANS	(input) CHARACTER*1 if `TRANS = 'N'`, computes the product Cx, if `TRANS = 'T'`, computes the product $C^T x$,
N	(input) INTEGER The order of the matrix C.
X	(input) REAL array, dimension (N) contains the vector x.
Z	(output) REAL array, dimension (N) On return, Z contains the product Cx if `TRANS = 'N'`, or the product $C^T x$ if `TRANS = 'T'`,

In addition, the parameter γ should be passed to the subroutine as a common variable.

Data Files : In the data files, $\gamma = 1/100$.

Data files: In the data files, $\gamma = 1/100$.

Filename/Key	Order	Number of entries
ODEPA400	400	1201
ODEPB400	400	399

References:

G. W. Stewart, SRRIT – A FORTRAN subroutine to calculator the dominant invariant subspaces of a real matrix, Comp. Sci. Dept. Tech. Rep. TR-514, Univ. of Maryland, College Park, Nov. 1978.

Z. Bai and G. W. Stewart, SRRIT – A FORTRAN subroutine to calculator the dominant invariant subspaces of a nonsymmetric matrix, Comp. Sci. Dept. Tech. Rep. TR-2908, Univ. of Maryland, MD, April 1992, (submitted to ACM TOMS).

Title: Partial differential equation
Subroutine name: MVMPDE

Source: H. Elman, University of Maryland

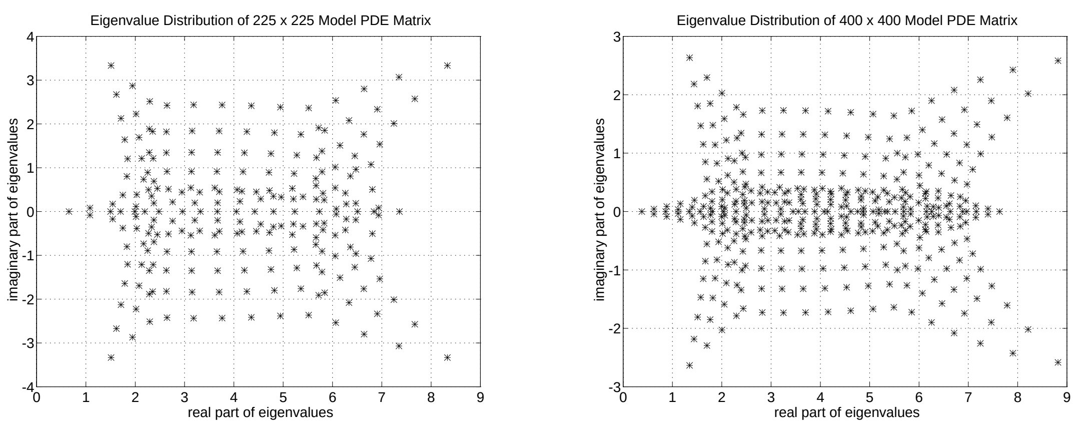
Discipline: partial differential equation

Further details: This test matrix is obtained from the finite difference discretization of the following two-dimensional partial differential operator

$$\mathcal{L}u = -\frac{\partial}{\partial x}\left(e^{-xy}\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(e^{xy}\frac{\partial u}{\partial y}\right) + \beta(x+y)\frac{\partial u}{\partial x} + \beta\frac{\partial}{\partial x}((x+y)u) + \gamma(x+y)\frac{\partial u}{\partial y} + \gamma\frac{\partial}{\partial y}((x+y)u) + \frac{1}{1+x+y}u$$

on the unit square $(0,1)\times(0,1)$, where $\beta, \gamma \in \mathbb{R}$ are parameters used to control the degree of nonormality of the matrices generated. We discretize the operator using central differences on an $m \times m$ grid with mesh size $h = 1/(m+1)$. This leads to a nonsymmetric matrix of order $n = m^2$. It is suggested to use values of β and γ between 0 and 250. The object is to estimate those eigenvalues with the largest real parts and to determine whether or not there are significant gaps in the spectrum.

The following figure shows the eigenvalue distribution for orders 225 ($m = 15$) and 400 ($m = 20$) of the matrix with $\beta = 20$ and $\gamma = 0$.



Subroutine: FORTRAN calling sequences for forming matrix-vector Ax and x^TA is

```
SUBROUTINE MVMPDE( TRANS, N, X, Z )
```

with

`TRANS` (input) CHARACTER*1
if `TRANS = 'N'`, computes the product Ax,
if `TRANS = 'T'`, computes the product A^Tx,

`N` (input) INTEGER
The order of the matrix C.

X (input) REAL array, dimension (N)
contains the vector x.

Z (output) REAL array, dimension (N)
On return, Z contains the product Ax if TRANS = 'N', or the product $A^T x$ if TRANS = 'T',

In addition, the parameters β and γ should be passed using a common block.

Data files:

Filename/Key	Order	Number of entries
PDE225	225	1065
PDE900	900	4380
PDE2961	2961	

References:

H. Elman, Iterative Methods for Large Sparse Nonsymmetric Systems of Linear Equations. PhD thesis, Yale University, New Haven, CT, 1982.

J. Cullum and R. A. Willoughby, A Practical Procedure for Computing Eigenvalues of Large Sparse Nonsymmetric Matrices, In Large Scale Eigenvalue Problem, J. Cullum and R. A. Willoughby eds. Elsevier Science Pub. North-Holland, 1986.

R. W. Freund, M. H. Gutknecht, and N. M. Nachtigal. An implementation of the look-ahead Lanczos algorithm for non-Hermitian matrices. *SIAM J. Sci. Comput.*, 14:137–158, 1993.

Title: Markov Chain Transition Matrix
Subroutine name: MVMRWK

Source: G. W. Stewart, University of Maryland

Discipline: Probability theory and its applications

Further details: Consider a random walk on an $(m+1)\times(m+1)$ triangular grid, illustrated below for $m = 6$.

6	•						
5	•	•					
4	•	•	•				
3	•	•	•	•			
2	•	•	•	•	•		
1	•	•	•	•	•	•	
0	•	•	•	•	•	•	•
j/i	0	1	2	3	4	5	6

The points of the grid are labeled $(j,i), (i = 0,\ldots,m, j = 0,\ldots,m-i)$. From the point (j,i), a transition may take place to one of the four adjacent points $(j+1,i), (j,i+1), (j-1,i), (j,i-1)$. The probability of jumping to either of the nodes $(j-1,i)$ or $(j,i-1)$ is

$$pd(j,i) = \frac{j+i}{m} \tag{5}$$

with the probability being split equally between the two nodes when both nodes are on the grid. The probability of jumping to either of the nodes $(j+1,i)$ or $(j,i+1)$ is

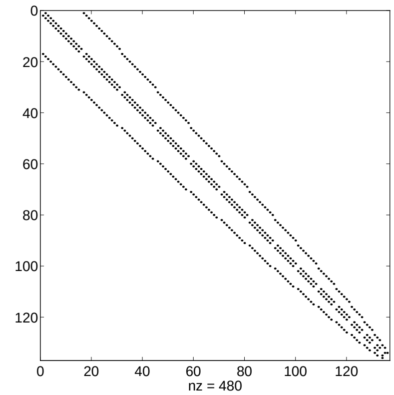
$$pu(j,i) = 1 - pd(j,i). \tag{6}$$

with the probability again being split when both nodes are on the grid.

If the $(m+1)(m+2)/2$ nodes (j,i) are numbered $1,2,\ldots,(m+1)(m+2)/2$ in some fashion, then the random walk can be expressed as a finite Markov chain whose transition matrix A of order $n = (m+1)(m+2)/2$ consisting of the probabilities a_{kl} of jumping from node l to node k (A is actually the transpose of the usual transition matrix; see [Feller]).

We are interested in the steady state probabilities of the chain, which is ordinarily the appropriately scaled eigenvector corresponding to the eigenvalue unity. However, if we number the diagonals on the grid that are parallel to the hypotenuse by $0,1,2,\ldots,n$, then an individual on an even diagonal can only jump to an odd diagonal, and vice versa. This means that the chain is cyclic with period two. Computationally it means that A has an eigenvalue of -1 as well as 1.

The following plot shows the sparsity pattern of the resulted random walk matrix of order 136 (i.e. $m = 15$).



Subroutine: To calculate the ith element of the vector Ax one need only regard the components of q as the average number of individuals at the nodes of the grid and use the probabilities (5) and (6) to calculate how many individuals will be at node i after the next transition.

FORTRAN calling sequences (heading) for forming matrix-vector Ax and $x^T A$ is

```
SUBROUTINE MVMRWK( TRANS, N, X, Z )
```

with

`TRANS`	(input) CHARACTER*1 if `TRANS = 'N'`, computes the product Ax, if `TRANS = 'T'`, computes the product $A^T x$,
`N`	(input) INTEGER The order of the matrix C.
`X`	(input) REAL array, dimension (`N`) contains the vector x.
`Z`	(output) REAL array, dimension (`N`) On return, `Z` contains the product Ax if `TRANS = 'N'`, or the product $A^T x$ if `TRANS = 'T'`,

Data files:

Filename/Key	Order	Number of entries
RW136	136	479
RW496	496	1859
RW5151	5151	20199

References:

W. Feller, An introduction to probability theory and its applications, John Wiley, New York, 1961

G. W. Stewart, SRRIT – A FORTRAN subroutine to calculator the dominant invariant subspaces of a real matrix, Comp. Sci. Dept. Tech. Rep. TR-514, Univ. of Maryland, College Park, Nov. 1978.

Y. Saad, Numerical methods for large eigenvalue problems. Halsted Press, Div. of John Wiley & Sons, Inc., New York, 1992.

I. S. Duff and J. A. Scott, Computing selected eigenvalues of sparse unsymmetric matrices using subspace iteration, ACM TOMS 19:137-159, 1993

Z. Bai and G. W. Stewart, SRRIT – A FORTRAN subroutine to calculator the dominant invariant subspaces of a nonsymmetric matrix, Comp. Sci. Dept. Tech. Rep. TR-2908, Univ. of Maryland, MD, April 1992 (submitted to ACM TOMS).

Title: Tolosa Matrix
Subroutine name: MVMTLS

Source: S. Godet-Thobie, CERFACS and C. Bès, Aerospatiale, France

Discipline: Aeroelasticity

Future details: The Tolosa matrix arises in the stability analysis of a model of an airplane in flight. The interesting modes of this system are described by complex eigenvalues whose imaginary parts lie in a prescribed frequency range. The task is to compute the eigenvalues with largest imaginary parts. The problem has been analyzed at CERFACS (Centre Europeen de Recherche et de Formation Avancee en Calcul Scientifique) in cooperation with the Aerospatiale Aircraft Division[1].

The matrix is a sparse 5×5 block matrix of order $n = 90 + 5k$. In practice, k is around 10^4. When $n = 90$, each block is of dimension 18×18 and

$$A = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ X_1 & X_2 & X_3 & X_4 & X_5 \\ 0 & I & L_1 & 0 & 0 \\ 0 & I & 0 & L_2 & 0 \\ 0 & I & 0 & 0 & L_3 \end{pmatrix}$$

where $L_i = \beta_i I$, $i = 1, 2, 3$, and X_i and β_i are given data. In general

$$A = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ Y_1 & Y_2 & Y_3 & Y_4 & Y_5 \\ 0 & I & L_1 & 0 & 0 \\ 0 & I & 0 & L_2 & 0 \\ 0 & I & 0 & 0 & L_3 \end{pmatrix}$$

where

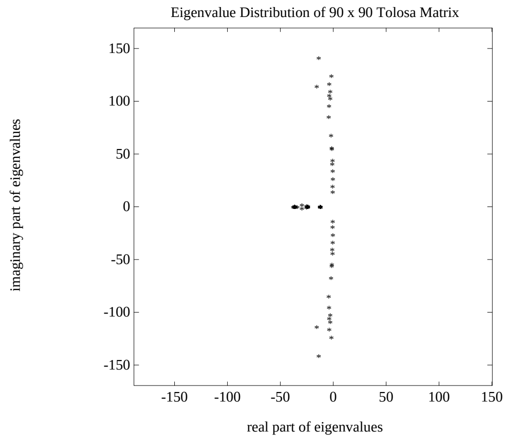
$$\begin{aligned} Y_1 &= \begin{pmatrix} X_1 & 0 \\ 0 & \mathrm{diag}(x_i) \end{pmatrix}, \quad x_i = -\omega_i^2, i = 1, \ldots, m - 18, \\ Y_2 &= \begin{pmatrix} X_2 & 0 \\ 0 & \mathrm{diag}(y_i) \end{pmatrix}, \quad y_i = -2\alpha_i \omega_i, \quad i = 1, \ldots, m - 18, \\ Y_k &= \begin{pmatrix} X_k & 0 \\ 0 & 0 \end{pmatrix}, \quad k = 3, 4, 5, \end{aligned}$$

and

$$\omega_i = 150 + 6i, \qquad \alpha_i = c_1 i + c_2, \qquad c_1 = \frac{.299}{n/5 - 18}, \qquad c_2 = 0.001 - c_1.$$

The following fugures show the eigenvalue distribution of Tolosa matrices of orders 90 and 340.

[1] Tolosa is the latin name of Toulouse, France, the location of CERFACS.



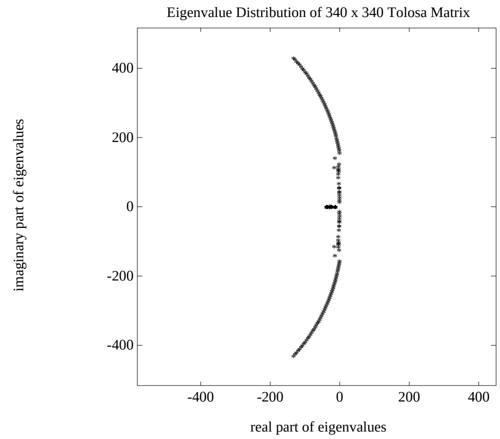


Figure 1: Eigenvalue distribution of 90 by 90 and 340 by 340 Tolosa matrices

Subroutine: It is necessary to read a small data file in the main program to initialize the data structure for this matrix. See the sample main program.

The subroutines OPTOL and OPTOLT can used to compute Aq and p^TA, respectively. The data file *toldat* is required to use this matrix.

```
.
      PROGRAM  TEST
      INTEGER          N, I, LDA
      PARAMETER        ( LDA = 500 )
      DOUBLE PRECISION P( LDA ), Q( LDA ), AQ( LDA ), PA( LDA )
c
      DOUBLE PRECISION ZERO, ONE
      PARAMETER        ( ZERO = 0.0D+0, ONE = 1.0D+0 )
      DOUBLE PRECISION BLK( 18,90 )
      COMMON           /OPSTIF/ BLK( 18,90 )
c
c     The common block "OPSTIF" is used to store the data and used
c     in the subroutines OPTOL and OPTOLT
c
c     Read matrix size
c
      READ(*,*)N
c
c     Data check
c
      IF( N.LT.90 .OR. MOD( N-90,5 ).NE.0 )THEN
         WRITE( *,111 )
         GO TO 20
      ELSE IF( N.GT.LDA )THEN
         WRITE( *,112 )
         GO TO 20
      END IF
```

```
c
c     Read in the data structure for matrix
c
      CALL READB( BLK )
c
c     For the purpose of illustration, P := (Q := ONES(N,1) ).
c
      DO 10 I = 1, N
         P( I ) = ONE
         Q( I ) = ONE
  10  CONTINUE
c
c     Compute A*Q and P^t*A
c
      CALL OPTOL( N, Q, AQ )
      CALL OPTOLT( N, P, PA )
c
  20  CONTINUE
c
 111  FORMAT( 'Error: N must be 90 + 5k' )
 112  FORMAT( 'Error: N must not be larger than LDA' )
c
      STOP
      END
```

Data files

References

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