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Composition of Post classes and normal forms of Boolean functions

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Abstract

The class composition $\mathscr{C} \circ \mathscr{K}$ of Boolean clones, being the set of composite functions $f(g_1, \ldots, g_n)$ with $f \in \mathscr{C}, g_1, \ldots, g_n \in \mathscr{K}$, is investigated. This composition $\mathscr{C} \circ \mathscr{K}$ is either the join $\mathscr{C} \vee \mathscr{K}$ in the Post Lattice or it is not a clone, and all pairs of clones $\mathscr{C}, \mathscr{K}$ are classified accordingly.

Factorizations of the clone Ω of all Boolean functions as a composition of minimal clones are described and seen to correspond to normal form representations of Boolean functions. The median normal form, arising from the factorization of Ω with the clone *SM* of self-dual monotone functions as the leftmost composition factor, is compared in terms of complexity with the well-known DNF, CNF, and Zhegalkin (Reed–Muller) polynomial representations, and it is shown to provide a more efficient normal form representation.

MSC: 06E30; 08A70

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1. Introduction and notation

Let $\mathbb{B} = \{0, 1\}$. A *Boolean function* is a map $f : \mathbb{B}^n \to \mathbb{B}$, for some positive integer n called the *arity* of f. Because we only discuss Boolean functions, we refer to them simply as *functions.* A *class* of functions is a subset $\mathscr{C} \subseteq \bigcup_{n \geqslant 1} \mathbb{B}^{\mathbb{B}^n}$. For a fixed arity n, the n different *projection maps* $(a_1, \ldots, a_n) \mapsto a_i$, $1 \leqslant i \leqslant n$, are also called *variables*, denoted $x_1, \ldots, x_n$, where the arity is clear from the context.

If f is an n-ary function and $g_1, \ldots, g_n$ are all m-ary functions, then the *composition* $f(g_1, \ldots, g_n)$ is an m-ary function, and its value on $(a_1, \ldots, a_m) \in \mathbb{B}^m$ is $f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$. Let $\mathscr{I}$ and $\mathscr{J}$ be classes of functions. The *composition of* $\mathscr{I}$ *with* $\mathscr{J}$, denoted $\mathscr{I} \circ \mathscr{J}$ (or sometimes, when the context is clear, just $\mathscr{I}\mathscr{J}$),

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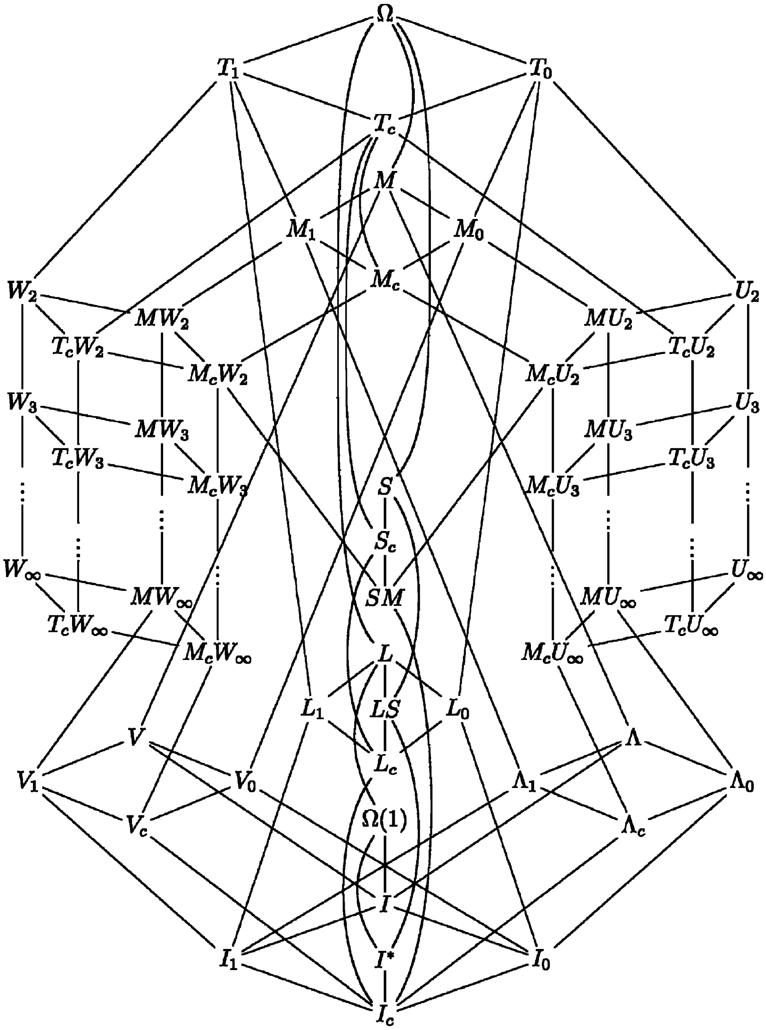

Fig. 1. Post Lattice.

is defined as

$$\mathscr{I} \circ \mathscr{J} = \{f(g_1, \ldots, g_n) : n, m \geqslant 1,\ \ f\ n\text{-ary in } \mathscr{I},\ \ g_1, \ldots, g_n\ m\text{-ary in } \mathscr{J}\}.$$

A *clone* is a class $\mathscr{C}$ of functions that contains all projections and satisfies $\mathscr{C}\mathscr{C} \subseteq \mathscr{C}$ (or equivalently, $\mathscr{C}\mathscr{C} = \mathscr{C}$).

The clones of Boolean functions, originally described by Post [14] (see [16,18,22]) for shorter recent proofs), form an algebraic lattice, where the lattice operations are the following: meet is the intersection, join is the smallest clone that contains the union. The greatest element is the clone Ω of all Boolean functions; the least element is the clone I_{c} of all projections. These clones and the lattice are often called the Post classes and the Post Lattice, respectively. For the nomenclature of the Post classes, see Appendix. The Post Lattice is illustrated in Fig. 1.

The set $\mathbb{B}^n$ is a Boolean (distributive and complemented) lattice of 2^n elements under the component-wise order of vectors. We will write simply $\mathbf{a} \preccurlyeq \mathbf{b}$ to denote comparison in this lattice. The *complement* of a vector $\mathbf{a} = (a_1, \ldots, a_n)$ is defined as $\overline{\mathbf{a}} = (1 - a_1, \ldots, 1 - a_n)$. We denote $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{1} = (1, \ldots, 1)$. Vectors are also called *points.*

The set $\mathbb{B}^{\mathbb{B}^n}$ is a Boolean lattice of 2^{2^n} elements under the point-wise ordering of functions. Both $\mathbb{B}^n$ and $\mathbb{B}^{\mathbb{B}^n}$ are vector spaces over the two-element field GF(2) $= \mathbb{B}$.

For a function f, the *dual* of f is defined as $f^{\mathrm{d}}(\mathbf{a}) = \overline{f(\overline{\mathbf{a}})}$ for all $\mathbf{a}$. For a class $\mathscr{C}$, the *dual* of $\mathscr{C}$ is defined as $\mathscr{C}^{\mathrm{d}} = \{f^{\mathrm{d}} : f \in \mathscr{C}\}$. The dual of a clone is a clone, and it is well-known that dualization gives the only nontrivial order-automorphism of the Post Lattice.

We also denote by **0** and **1** the constant functions of any arity having value 0 and 1, respectively, everywhere. We denote the ternary majority function $x_1x_2 + x_1x_3 + x_2x_3$ by μ and the ternary triple sum $x_1 + x_2 + x_3$ by τ.

We say that f is a *subfunction* of g if $f \in \{g\} \circ I_{\mathrm{c}}$. The subfunction relation is a preorder (i.e., a reflexive and transitive relation) on the set of functions. We say that functions f and g are *equivalent*, denoted $f \equiv g$, if they are subfunctions of each other. This is commonly described as follows: functions f and g are equivalent if there is a function that can be obtained from both f and g by repeated cylindrification and permutation of variables. For any functions $f_1, \ldots, f_n$, we denote $[f_1, \ldots, f_n] = \{f : f \equiv f_i \text{ for some } i = 1, \ldots, n\}$.

The plan of the paper is as follows. In Section 2 we restate a lemma on the associativity of function class composition, and state and prove a general sufficient condition for the composition of clones to be a clone. In Section 3 we completely classify those pairs of Boolean clones whose class composition is a clone (Section 3.1), and those pairs whose composition is not a clone (Section 3.2), and we summarize this classification in the two theorems of Section 3.3—from the direct point of view of compositions in Theorem 2 and from the reverse point of view of decompositions or factorizations of a given clone in Theorem 3. In Section 4 we consider certain factorizations of the clone Ω of all Boolean functions which correspond to normal form representations of functions such as disjunctive normal form (DNF), conjunctive normal form (CNF), and Zhegalkin or Reed–Muller polynomial representations. We conclude by showing that the representation using the ternary median (majority) function, based on the factorization of Ω with the clone *SM* of self-dual monotone functions as the leftmost factor, is asymptotically more efficient than the more classical DNF, CNF, and polynomial representations which use Boolean lattice or Boolean ring operations.

2. General rules and auxiliary theorems

Let us restate the Associativity Lemma of [4], particularized to Boolean functions.

Lemma 1 (*Associativity Lemma*). *Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be classes of Boolean functions*:

(i) $(\mathscr{A}\mathscr{B})\mathscr{C} \subseteq \mathscr{A}(\mathscr{B}\mathscr{C})$,
(ii) *if* $\mathscr{B} \circ I_{\mathrm{c}} \subseteq \mathscr{B}$, *then* $(\mathscr{A}\mathscr{B})\mathscr{C} = \mathscr{A}(\mathscr{B}\mathscr{C})$.

Let $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}$ be clones. The following hold and will be used repeatedly:

- Associativity: $\mathscr{C}_1(\mathscr{C}_2\mathscr{C}_3) = (\mathscr{C}_1\mathscr{C}_2)\mathscr{C}_3$.
- $\mathscr{C}_1 \cup \mathscr{C}_2 \subseteq \mathscr{C}_1\mathscr{C}_2 \subseteq \mathscr{C}_1 \vee \mathscr{C}_2$. The class composition $\mathscr{C}_1\mathscr{C}_2$ is a clone if and only if $\mathscr{C}_1\mathscr{C}_2 = \mathscr{C}_1 \vee \mathscr{C}_2$.
- If $\mathscr{C}_1 \subseteq \mathscr{C}_2$, then $\mathscr{C}_1\mathscr{C}_2 = \mathscr{C}_2$ and $\mathscr{C}_2\mathscr{C}_1 = \mathscr{C}_2$.
- If $\mathscr{C}_1 \subseteq \mathscr{C}_2$ and $\mathscr{C}_3 \subseteq \mathscr{C}_4$, then $\mathscr{C}_1\mathscr{C}_3 \subseteq \mathscr{C}_2\mathscr{C}_4$.
- If $\mathscr{C}_1\mathscr{C}_2 = \mathscr{C}$, $\mathscr{C}_1 \subseteq \mathscr{C}_3 \subseteq \mathscr{C}$, and $\mathscr{C}_2 \subseteq \mathscr{C}_4 \subseteq \mathscr{C}$, then $\mathscr{C}_3\mathscr{C}_4 = \mathscr{C}$.
- If $\mathscr{C}_1\mathscr{C}_2$ is not a clone, i.e., $\mathscr{C}_1\mathscr{C}_2 \neq \mathscr{C}_1 \vee \mathscr{C}_2$, and $\mathscr{C}_3 \subseteq \mathscr{C}_1$, $\mathscr{C}_4 \subseteq \mathscr{C}_2$, $\mathscr{C}_3 \vee \mathscr{C}_4 = \mathscr{C}_1 \vee \mathscr{C}_2$, then $\mathscr{C}_3\mathscr{C}_4$ is not a clone, i.e., $\mathscr{C}_3\mathscr{C}_4 \neq \mathscr{C}_3 \vee \mathscr{C}_4 = \mathscr{C}_1 \vee \mathscr{C}_2$.
- Duality: $\mathscr{C}_1^{\mathrm{d}}\mathscr{C}_2^{\mathrm{d}} = (\mathscr{C}_1\mathscr{C}_2)^{\mathrm{d}}$.
- If $\mathscr{C}_1\mathscr{C}_2 = \mathscr{C}_2\mathscr{C}_1$, then $\mathscr{C}_1\mathscr{C}_2$ is a clone.

As we shall see, it is not always true that $\mathscr{C}_1\mathscr{C}_2 = \mathscr{C}_2\mathscr{C}_1$. If $\mathscr{C}_1\mathscr{C}_2 \neq \mathscr{C}_2\mathscr{C}_1$, then either one of $\mathscr{C}_1\mathscr{C}_2$ and $\mathscr{C}_2\mathscr{C}_1$ is a clone while the other is not, or neither is a clone.

Theorem 1. *Let $\mathscr{G}$ and $\mathscr{H}$ be classes of Boolean functions*, *and let $\mathscr{C}$ and $\mathscr{K}$ be the clones generated by $\mathscr{G}$ and $\mathscr{H}$*, *respectively. If $(\mathscr{G} \cup I_{\mathrm{c}}) \circ I_{\mathrm{c}} \subseteq \mathscr{G}$*, *$(\mathscr{H} \cup I_{\mathrm{c}}) \circ I_{\mathrm{c}} \subseteq \mathscr{H}$*, *and $\mathscr{G}\mathscr{H} \subseteq \mathscr{K}\mathscr{G}$*, *then $\mathscr{K}\mathscr{C}$ is a clone.*

Proof. We first prove two Claims.

Claim 1. $\mathscr{G}\mathscr{K} \subseteq \mathscr{K}\mathscr{C}$.

To prove this claim, denote by $\mathscr{H}^i$, $i \geqslant 1$, the composition $\mathscr{H} \cdots \mathscr{H}$ of the class $\mathscr{H}$ with itself i times. (Parentheses are not necessary, because $\mathscr{H} \circ I_{\mathrm{c}} \subseteq \mathscr{H}$.) Observe that $\mathscr{H}^i \subseteq \mathscr{H}^{i+1}$ for every $i \geqslant 1$, and

$$\mathscr{K} = \bigcup_i \mathscr{H}^i, \quad \mathscr{G}\mathscr{K} = \bigcup_i \mathscr{G}\mathscr{H}^i, \quad \mathscr{K}\mathscr{G} = \bigcup_i \mathscr{H}^i\mathscr{G}.$$

Thus it is sufficient to show, by induction on i, that $\mathscr{G}\mathscr{H}^i \subseteq \mathscr{K}\mathscr{G}$ for all $i \geqslant 1$. For $i = 1$, this is part of the hypothesis in the statement of the theorem. The inductive step is accomplished by assuming $\mathscr{G}\mathscr{H}^i \subseteq \mathscr{K}\mathscr{G}$ and observing that $\mathscr{G}\mathscr{H}^{i+1} = \mathscr{G}(\mathscr{H}^i\mathscr{H}) = (\mathscr{G}\mathscr{H}^i)\mathscr{H}$, because $\mathscr{H} \circ I_{\mathrm{c}} \subseteq \mathscr{H}$ and $\mathscr{H}^i \circ I_{\mathrm{c}} \subseteq \mathscr{H}^i$, and

$$(\mathscr{G}\mathscr{H}^i)\mathscr{H} \subseteq (\mathscr{K}\mathscr{G})\mathscr{H} = \left(\bigcup_j \mathscr{H}^j\mathscr{G}\right)\mathscr{H} = \bigcup_j (\mathscr{H}^j\mathscr{G})\mathscr{H}$$

by the inductive hypothesis. For every j, $(\mathscr{H}^j\mathscr{G})\mathscr{H} \subseteq \mathscr{H}^j(\mathscr{G}\mathscr{H})$ by the Associativity Lemma, and

$$\mathscr{H}^j(\mathscr{G}\mathscr{H}) \subseteq \mathscr{H}^j\left(\bigcup_i \mathscr{H}^i\mathscr{G}\right)$$

by the hypothesis of the theorem. Furthermore, since every $\mathscr{H}^i$ contains all projections and $\mathscr{H} = \mathscr{H}^1 \subseteq \mathscr{H}^2 \subseteq \cdots$,

$$\mathscr{H}^j\left(\bigcup_i \mathscr{H}^i\mathscr{G}\right) \subseteq \bigcup_i (\mathscr{H}^j\mathscr{H}^i)\mathscr{G} \subseteq \mathscr{K}\mathscr{G},$$

which shows that $\mathscr{G}\mathscr{H}^{i+1} \subseteq \mathscr{K}\mathscr{G}$, completing the proof of Claim 1.

Claim 2. $\mathscr{C}\mathscr{K} \subseteq \mathscr{K}\mathscr{C}$.

Let $(\mathscr{G} \cup I_{\mathrm{c}})^i$, $i \geqslant 1$, denote the composition $(\mathscr{G} \cup I_{\mathrm{c}}) \cdots (\mathscr{G} \cup I_{\mathrm{c}})$ of $\mathscr{G} \cup I_{\mathrm{c}}$ with itself i times. (We can omit parentheses by the Associativity Lemma.) We have $(\mathscr{G} \cup I_{\mathrm{c}})^i \subseteq (\mathscr{G} \cup I_{\mathrm{c}})^{i+1}$ for all $i \geqslant 1$ and

$$\mathscr{C} = \bigcup_i (\mathscr{G} \cup I_{\mathrm{c}})^i, \quad \mathscr{C}\mathscr{K} = \bigcup_i (\mathscr{G} \cup I_{\mathrm{c}})^i\mathscr{K}.$$

We show by induction on i that $(\mathscr{G} \cup I_{\mathrm{c}})^i\mathscr{K} \subseteq \mathscr{K}\mathscr{C}$. For $i = 1$ this is true because

$$(\mathscr{G} \cup I_{\mathrm{c}})\mathscr{K} = \mathscr{G}\mathscr{K} \cup I_{\mathrm{c}} \circ \mathscr{K} = \mathscr{G}\mathscr{K} \cup \mathscr{K} \subseteq \mathscr{K}\mathscr{C} \cup \mathscr{K}\mathscr{C} = \mathscr{K}\mathscr{C}.$$

Assuming it is true for i,

$$\begin{aligned}(\mathscr{G} \cup I_{\mathrm{c}})^{i+1}\mathscr{K} &= (\mathscr{G} \cup I_{\mathrm{c}})[(\mathscr{G} \cup I_{\mathrm{c}})^i\mathscr{K}] \subseteq (\mathscr{G} \cup I_{\mathrm{c}})(\mathscr{K}\mathscr{C})\\ &= [(\mathscr{G} \cup I_{\mathrm{c}})\mathscr{K}]\mathscr{C} = (\mathscr{G}\mathscr{K} \cup I_{\mathrm{c}} \circ \mathscr{K})\mathscr{C} = (\mathscr{G}\mathscr{K})\mathscr{C} \cup (I_{\mathrm{c}} \circ \mathscr{K})\mathscr{C} = (\mathscr{G}\mathscr{K})\mathscr{C} \cup \mathscr{K}\mathscr{C}.\end{aligned}$$

By Claim 1, $\mathscr{G}\mathscr{K} \subseteq \mathscr{K}\mathscr{G}$, and therefore, still using the Associativity Lemma,

$$(\mathscr{G}\mathscr{K})\mathscr{C} \cup \mathscr{K}\mathscr{C} \subseteq (\mathscr{K}\mathscr{G})\mathscr{C} \cup \mathscr{K}\mathscr{C} \subseteq \mathscr{K}(\mathscr{G}\mathscr{C}) \cup \mathscr{K}\mathscr{C} = \mathscr{K}\mathscr{C},$$

yielding $(\mathscr{G} \cup I_{\mathrm{c}})^{i+1}\mathscr{K} \subseteq \mathscr{K}\mathscr{C}$ and completing the proof of Claim 2.

Using Claim 2 and associativity, we have

$$(\mathscr{K}\mathscr{C})(\mathscr{K}\mathscr{C}) \subseteq \mathscr{K}(\mathscr{C}\mathscr{K})\mathscr{C} \subseteq \mathscr{K}(\mathscr{K}\mathscr{C})\mathscr{C} = (\mathscr{K}\mathscr{K})(\mathscr{C}\mathscr{C}) = \mathscr{K}\mathscr{C},$$

establishing the result that $\mathscr{K}\mathscr{C}$ is a clone. □

Whenever we apply Theorem 1, we just mention the generating functions $g_1, \ldots, g_m$ and $h_1, \ldots, h_n$ of the clones $\mathscr{C}$ and $\mathscr{K}$, respectively, and we let $\mathscr{G} = [g_1, \ldots, g_m, x_1]$, $\mathscr{H} = [h_1, \ldots, h_n, x_1]$.

Let f be an n-ary function, and denote by T_f the *set of true points* of f, i.e., $T_f = \{\mathbf{a} : f(\mathbf{a}) = 1\}$. Let $T_f^{\mathrm{M}} = \{\mathbf{b} \in \mathbb{B}^n : \mathbf{a} \preccurlyeq \mathbf{b}$ for some $\mathbf{a} \in T_f\}$. The *monotone closure* of f, denoted by f^{M}, is defined as the n-ary function whose true points are the members of T_f^{M}.

By definition, $f^{\mathrm{M}} \in M$. We observe that if $f \in T_0$, then $f^{\mathrm{M}} \in M_0$; if $f \in T_{\mathrm{c}}$, then $f^{\mathrm{M}} \in M_{\mathrm{c}}$; and for $m=2, \dots, \infty$, if $f \in T_{\mathrm{c}}U_m$, then $f^{\mathrm{M}} \in M_{\mathrm{c}}U_m$. Also, if $f(\mathbf{a}) = 1$ for some $\mathbf{a}$, then $f^{\mathrm{M}}(\mathbf{a}) = 1$; and if $f^{\mathrm{M}}(\mathbf{a}) = 1$ for some $\mathbf{a}$, then there exists $\mathbf{b} \preccurlyeq \mathbf{a}$ such that $f(\mathbf{b}) = 1$.

3. Compositions of clones

It is a well-known fact that every Boolean function can be represented by DNF and CNF expressions. This fact can be restated as $\Omega = V_{\mathrm{c}} \circ \Lambda_{\mathrm{c}} \circ I^* = \Lambda_{\mathrm{c}} \circ V_{\mathrm{c}} \circ I^*$. It is also known that $M_{\mathrm{c}} = V_{\mathrm{c}} \circ \Lambda_{\mathrm{c}} = \Lambda_{\mathrm{c}} \circ V_{\mathrm{c}}$, so the previous equalities can be written as $\Omega = M_{\mathrm{c}} \circ I^*$.

We also know that every Boolean function is represented by a unique multilinear polynomial over GF(2), called the Zhegalkin or Reed–Muller polynomial or Boolean ring representation (see [3,12,15,21]). This fact can be restated as $\Omega = L_{\mathrm{c}} \circ \Lambda$. Allowing only constant-preserving linear functions is not really a restriction, because $\mathbf{0}$ can be substituted for a variable if necessary.

These facts will be used in what follows. We will present various propositions on whether the composition of two clones is a clone or not. The cases when the composition is a clone and the cases when the composition is not a clone are grouped in Sections 3.1 and 3.2, respectively. For general background, see, e.g., [3,6,11,13].

3.1. Cases when the composition of clones is a clone

We now establish a number of equalities of the form $\mathscr{C}_1\mathscr{C}_2 = \mathscr{C}_3$. The inclusions $\mathscr{C}_1\mathscr{C}_2 \subseteq \mathscr{C}_3$ are obvious in each case from the inclusions in the Post Lattice. It only remains to prove the converse inclusions. We only provide a proof for one of $\mathscr{C}_1\mathscr{C}_2 = \mathscr{C}_3$ and $\mathscr{C}_1^{\mathrm{d}}\mathscr{C}_2^{\mathrm{d}} = \mathscr{C}_3^{\mathrm{d}}$; the other equality follows by duality.

Proposition 1. $I_0 \circ I_1 = I_1 \circ I_0 = I$, $I \circ I^* = \Omega(1)$, $I^* \circ I_0 = I^* \circ I_1 = \Omega(1)$, $L_{\mathrm{c}} \circ I_0 = L_0$, $L_{\mathrm{c}} \circ I_1 = L_1$, $L_{\mathrm{c}} \circ I = L$, $I^* \circ L_{\mathrm{c}} = L_{\mathrm{c}} \circ I^* = LS$.

Proof. Straightforward verification. □

Proposition 2. $U_\infty \circ I_1 = \Omega$, $T_{\mathrm{c}}U_\infty \circ I_1 = T_1$, $MU_\infty \circ I_1 = M$, $M_{\mathrm{c}}U_\infty \circ I_1 = M_1$, $U_\infty \circ V_{\mathrm{c}} = T_0$, $T_{\mathrm{c}}U_\infty \circ V_{\mathrm{c}} = T_{\mathrm{c}}$. *Dually*, $W_\infty \circ I_0 = \Omega$, $T_{\mathrm{c}}W_\infty \circ I_0 = T_0$, $MW_\infty \circ I_0 = M$, $M_{\mathrm{c}}W_\infty \circ I_0 = M_0$, $W_\infty \circ \Lambda_{\mathrm{c}} = T_1$, $T_{\mathrm{c}}W_\infty \circ \Lambda_{\mathrm{c}} = T_{\mathrm{c}}$.

Proof. Let $f \in \Omega$ be n-ary. Define the $(n+1)$-ary function f' as $f' = f(x_1, \dots, x_n) \wedge x_{n+1}$. We observe that $f' \in U_\infty$ and $f = f'(x_1, \dots, x_n, \mathbf{1})$. Since $f' \in U_\infty$and $\mathbf{1} \in I_1$, we have that $U_\infty \circ I_1 = \Omega$.

If $f \in T_1$, then $f' \in T_{\mathrm{c}}U_\infty$. If $f \in M$, then $f' \in MU_\infty$. If $f \in M_1$, then $f' \in M_{\mathrm{c}}U_\infty$. Hence, we have that $T_{\mathrm{c}}U_\infty \circ I_1 = T_1$, $MU_\infty \circ I_1 = M$, $M_{\mathrm{c}}U_\infty \circ I_1 = M_1$.

If $f \in T_0$, then $f' \in U_\infty$. Since in this case $f = f'(x_1, \dots, x_n, x_1 \vee \dots \vee x_n)$ and all disjunctions belong to V_{c}, we have that $U_\infty \circ V_{\mathrm{c}} = T_0$. If $f \in T_{\mathrm{c}}$, then $f' \in T_{\mathrm{c}}U_\infty$, and we have that $T_{\mathrm{c}}U_\infty \circ V_{\mathrm{c}} = T_{\mathrm{c}}$. □

Proposition 3. *For* $m = 2, \dots, \infty$, $T_{\mathrm{c}}U_\infty \circ M_{\mathrm{c}}U_m = T_{\mathrm{c}}U_m$ *and, dually*, $T_{\mathrm{c}}W_\infty \circ M_{\mathrm{c}}W_m = T_{\mathrm{c}}W_m$.

Proof. Let $f \in T_{\mathrm{c}}U_m$ be n-ary. Define the $(n+1)$-ary function f' as $f' = f(x_1, \dots, x_n) \wedge x_{n+1}$. We observe that $f = f'(x_1, \dots, x_n, f^{\mathrm{M}})$, where f^{M} is the monotone closure of f. Because $f' \in T_{\mathrm{c}}U_\infty$, and $f^{\mathrm{M}} \in M_{\mathrm{c}}U_m$, it follows that $T_{\mathrm{c}}U_\infty \circ M_{\mathrm{c}}U_m = T_{\mathrm{c}}U_m$. □

Proposition 4. $V_{\mathrm{c}} \circ U_\infty = T_0$, $V_{\mathrm{c}} \circ T_{\mathrm{c}}U_\infty = T_{\mathrm{c}}$. *Dually*, $\Lambda_{\mathrm{c}} \circ W_\infty = T_1$, $\Lambda_{\mathrm{c}} \circ T_{\mathrm{c}}W_\infty = T_{\mathrm{c}}$.

Proof. Let $f \in T_0$ be n-ary. For $i = 1, \dots, n$, define the n-ary function f_i as $f_i = f \wedge x_i$. We observe that $f_i \in U_\infty$ and $f = f_1 \vee \dots \vee f_n$. Hence, $V_{\mathrm{c}} \circ U_\infty = T_0$. If $f \in T_{\mathrm{c}}$, then $f_i \in T_{\mathrm{c}}U_\infty$, so we have that $V_{\mathrm{c}} \circ T_{\mathrm{c}}U_\infty = T_{\mathrm{c}}$. □

Proposition 5. *For* $m = 2, \dots, \infty$, $M_{\mathrm{c}}U_m \circ T_{\mathrm{c}}U_\infty = T_{\mathrm{c}}U_m$ *and, dually*, $M_{\mathrm{c}}W_m \circ T_{\mathrm{c}}W_\infty = T_{\mathrm{c}}W_m$.

Proof. Let $f \in T_{\mathrm{c}}U_m$ be n-ary. For $i = 1, \dots, n$, define the function f_i as $f_i = f \wedge x_i$. Let $\mathbf{a} = (a_1, \dots, a_n)$. We observe that if $a_i = 0$ then $f_i(\mathbf{a}) = 0$ and if $a_i = 1$ then $f_i(\mathbf{a}) = f(\mathbf{a})$. Therefore, the mapping $\mathbf{a} \mapsto (f_1(\mathbf{a}), \dots, f_n(\mathbf{a}))$ keeps the true points of f fixed and maps the false points to $\mathbf{0}$.

Because $f \leqslant f^{\mathrm{M}}$ and $f^{\mathrm{M}}(\mathbf{0}) = 0$, we have that $f = f^{\mathrm{M}}(f_1, \ldots, f_n)$. Because $f^{\mathrm{M}} \in M_{\mathrm{c}}U_m$ and $f_1, \ldots, f_n \in T_{\mathrm{c}}U_\infty$, we have that $M_{\mathrm{c}}U_m \circ T_{\mathrm{c}}U_\infty = T_{\mathrm{c}}U_m$. □

Proposition 6. *For* $m = 2, \ldots, \infty$, $T_{\mathrm{c}}U_m \circ I_0 = U_m$ *and, dually*, $T_{\mathrm{c}}W_m \circ I_1 = W_m$.

Proof. Let $f \in U_m$ be n-ary. Define the $(n+1)$-ary function f' as

$$f'(a_1, \ldots, a_n, a_{n+1}) = \begin{cases} f(a_1, \ldots, a_n) & \text{if } a_{n+1} = 0, \\ 1 & \text{if } a_{n+1} = 1. \end{cases}$$

We observe that $f = f'(x_1, \ldots, x_n, \mathbf{0})$. Since $f' \in T_{\mathrm{c}}U_m$ and $\mathbf{0} \in I_0$, we have that $T_{\mathrm{c}}U_m \circ I_0 = U_m$. □

Proposition 7. $S \circ I_0 = \Omega$. *Dually*, $S \circ I_1 = \Omega$.

Proof. Let $f \in \Omega$ be n-ary. Define the $(n+1)$-ary function f' as

$$f' = f(x_1 + x_{n+1}, \ldots, x_n + x_{n+1}) + x_{n+1}.$$

We observe that $f = f'(x_1, \ldots, x_n, \mathbf{0})$. Since $f' \in S$ and $\mathbf{0} \in I_0$, we have that $S \circ I_0 = \Omega$. □

Proposition 8. $S_{\mathrm{c}} \circ I = \Omega$.

Proof. Let $f \in \Omega$ be n-ary. Define the $(n+2)$-ary function f' as

$$f' = (x_{n+1} + x_{n+2}) \wedge (f(x_1 + x_{n+2}, \ldots, x_n + x_{n+2}) + x_{n+1} + x_{n+2}) + x_{n+1}.$$

We observe that $f = f'(x_1, \ldots, x_n, \mathbf{1}, \mathbf{0})$. Since $f' \in S_{\mathrm{c}}$ and $\mathbf{0}, \mathbf{1} \in I$, we have that $S_{\mathrm{c}} \circ I = \Omega$. □

Proposition 9. $M_{\mathrm{c}} \circ I^* = \Omega$.

Proof. Let $f \in \Omega$ be n-ary, and let T be the set of true points of f. Let g be the $2n$-ary function whose set of true points is $\{(\mathbf{a}, \overline{\mathbf{a}}) \in \mathbb{B}^{2n} : \mathbf{a} \in T\} \cup \{\mathbf{1}\}$. For all $\mathbf{a} \in \mathbb{B}^n$, $f(\mathbf{a}) = g(\mathbf{a}, \overline{\mathbf{a}}) = g^{\mathrm{M}}(\mathbf{a}, \overline{\mathbf{a}})$, because the values of g remain unchanged in the antichain $\{(\mathbf{a}, \overline{\mathbf{a}}) \in \mathbb{B}^{2n} : \mathbf{a} \in \mathbb{B}^n\}$ when forming the monotone closure. Therefore, $f = g^{\mathrm{M}}(x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n})$. Since $g^{\mathrm{M}} \in M_{\mathrm{c}}$ and $x_i, \overline{x_i} \in I^*$, we conclude that $M_{\mathrm{c}} \circ I^* = \Omega$. □

Proposition 10. $I^* \circ T_0 = \Omega$, $I^* \circ S_{\mathrm{c}} = S$. *Dually*, $I^* \circ T_1 = \Omega$.

Proof. Let $f \in \Omega$. If $f \in T_0$, then clearly $f \in I^* \circ T_0$. If $f \notin T_0$, then $\overline{f} \in T_0$, and so $f = \overline{\overline{f}}$. Because $\overline{x_1} \in I^*$, we have that $I^* \circ T_0 = \Omega$.

Let then $f \in S$. If $f \in S_{\mathrm{c}}$, then clearly $f \in I^* \circ S_{\mathrm{c}}$. If $f \notin S_{\mathrm{c}}$, then $\overline{f} \in S_{\mathrm{c}}$, and we conclude that $f \in I^* \circ S_{\mathrm{c}}$ also in this case. Thus, $I^* \circ S_{\mathrm{c}} = S$. □

Proposition 11. $I^* \circ L_0 = L_0 \circ I^* = L$, $L_0 \circ I_1 = L$, $LS \circ I_0 = L$. *Dually*, $I^* \circ L_1 = L_1 \circ I^* = L$, $L_1 \circ I_0 = L$, $LS \circ I_1 = L$.

Proof. Let $f = a_0\mathbf{1} + a_1x_1 + \cdots + a_nx_n \in L$. If $a_0 = 0$, then $f \in L_0$, and so $f \in I^* \circ L_0$, $f \in L_0 \circ I^*$, and $f \in L_0 \circ I_1$. If $a_0 = 1$, then $\overline{f} \in L_0$, and so $f = \overline{\overline{f}} \in I^* \circ L_0$. Furthermore, the $(n+2)$-ary function $f_1 = a_1x_1 + \cdots + a_nx_n + x_{n+1} + x_{n+2}$ is in L_0, and $f = f_1(x_1, \ldots, x_n, x_1, \overline{x_1}) \in L_0 \circ I^*$. Also the $(n+1)$-ary function $f_2 = a_1x_1 + \cdots + a_nx_n + x_{n+1}$ is in L_0, and $f = f_2(x_1, \ldots, x_n, \mathbf{1}) \in L_0 \circ I_1$. Hence, $I^* \circ L_0 = L$, $L_0 \circ I^* = L$, and $L_0 \circ I_1 = L$.

If an odd number of the coefficients a_i, $i \geqslant 1$, are equal to 1, then $f \in LS$. Otherwise, the $(n+1)$-ary function $f_3 = a_0\mathbf{1} + a_1x_1 + \cdots + a_nx_n + x_{n+1}$ is in LS and $f = f_3(x_1, \ldots, x_n, \mathbf{0}) \in LS \circ I_0$. Thus, $LS \circ I_0 = L$. □

Proposition 12. $I_0 \circ M_{\mathrm{c}} = M_0$, $I \circ M_{\mathrm{c}} = M$, $I_0 \circ M_1 = M$, $I_0 \circ \Lambda_{\mathrm{c}} = \Lambda_0$, $I_1 \circ \Lambda_{\mathrm{c}} = \Lambda_1$, $I \circ \Lambda_{\mathrm{c}} = \Lambda$, $I_0 \circ \Lambda_1 = \Lambda$, $I_1 \circ \Lambda_0 = \Lambda$. *For* $m = 2, \ldots, \infty$, $I_0 \circ M_{\mathrm{c}}U_m = MU_m$. *Dually*, $I_1 \circ M_{\mathrm{c}} = M_1$, $I_1 \circ M_0 = M$, $I_1 \circ V_{\mathrm{c}} = V_1$, $I_0 \circ V_{\mathrm{c}} = V_0$, $I \circ V_{\mathrm{c}} = V$, $I_1 \circ V_0 = V$, $I_0 \circ V_1 = V$. *For* $m = 2, \ldots, \infty$, $I_1 \circ M_{\mathrm{c}}W_m = MW_m$.

Proof. The first eight equalities follow by the definition of the clones M_c, M_0, M, Λ_c, Λ_0, Λ_1, Λ and the fact that for any class $\mathscr{C}$ of functions, $I_0 \circ \mathscr{C} = \mathscr{C} \cup [\mathbf{0}]$, $I_1 \circ \mathscr{C} = \mathscr{C} \cup [\mathbf{1}]$, and $I \circ \mathscr{C} = \mathscr{C} \cup [\mathbf{0}, \mathbf{1}]$. We also observe that for $m = 2, \ldots, \infty$,

$$I_0 \circ M_c U_m = M_c U_m \cup [\mathbf{0}] = (M_c \cap U_m) \cup [\mathbf{0}] = M_0 \cap U_m = M \cap U_m = M U_m,$$

where the penultimate equality holds because $\mathbf{1} \notin U_m$. □

Proposition 13. *$M_c \circ I_0 = M_0$, $M_c \circ I = M$, $M_0 \circ I_1 = M$, $\Lambda_c \circ I_0 = \Lambda_0$, $\Lambda_c \circ I_1 = \Lambda_1$, $\Lambda_c \circ I = \Lambda$, $\Lambda_0 \circ I_1 = \Lambda$, $\Lambda_1 \circ I_0 = \Lambda$. For $m = 2, \ldots, \infty$, $M_c U_m \circ I_0 = M U_m$. Dually, $M_c \circ I_1 = M_1$, $M_1 \circ I_0 = M$, $V_c \circ I_1 = V_1$, $V_c \circ I_0 = V_0$, $V_c \circ I = V$, $V_1 \circ I_0 = V$, $V_0 \circ I_1 = V$. For $m = 2, \ldots, \infty$, $M_c W_m \circ I_1 = M W_m$.*

Proof. By definition, $M_0 \backslash M_c = [\mathbf{0}]$, $M \backslash M_c = [\mathbf{0}, \mathbf{1}]$, $M \backslash M_0 = [\mathbf{1}]$, $\Lambda_0 \backslash \Lambda_c = [\mathbf{0}]$, $\Lambda_1 \backslash \Lambda_c = [\mathbf{1}]$, $\Lambda \backslash \Lambda_c = [\mathbf{0}, \mathbf{1}]$, $\Lambda \backslash \Lambda_0 = [\mathbf{1}]$, $\Lambda \backslash \Lambda_1 = [\mathbf{0}]$. Also, by the observation in the proof of Proposition 12, $M U_m \backslash M_c U_m = [\mathbf{0}]$. The constant functions can be obtained by composing x_1 with $\mathbf{0} \in I_0, I$, $\mathbf{1} \in I_1, I$. □

Proposition 14. *$SM \circ V_c = V_c \circ SM = M_c W_2$. Dually, $SM \circ \Lambda_c = \Lambda_c \circ SM = M_c U_2$.*

Proof. Since the functions μ and $x_1 \vee x_2$ generate *SM* and V_c, respectively, and

$$\mu(x \vee y, a, b) = \mu(x, a, b) \vee \mu(y, a, b),$$
$$\mu(x, y, z) \vee a = \mu(x \vee a, y \vee a, z \vee a),$$

it follows from Theorem 1 that $V_c \circ SM$ and $SM \circ V_c$ are clones, and hence they are equal to $M_c W_2$. □

Proposition 15. *$L_c \circ SM = SM \circ L_c = S_c$.*

Proof. The functions μ and τ are generators of *SM* and L_c, respectively, and

$$\mu(\tau(x, y, z), a, b) = \tau(\mu(x, a, b), \mu(y, a, b), \mu(z, a, b)),$$
$$\tau(\mu(x, y, z), a, b) = \mu(\tau(x, a, b), \tau(y, a, b), \tau(z, a, b)).$$

Theorem 1 implies that $L_c \circ SM$ and $SM \circ L_c$ are clones. Hence, $L_c \circ SM = SM \circ L_c = S_c$. □

Proposition 16. *$SM \circ I^* = S$.*

Proof. The functions $\overline{x_1}$ and μ are generators of I^* and *SM*, respectively, and $\overline{\mu(x, y, z)} = \mu(\overline{x}, \overline{y}, \overline{z})$. Theorem 1 implies that $SM \circ I^*$ is a clone. Hence, $SM \circ I^* = S$. □

Proposition 17. *$SM \circ T_c U_\infty = T_c U_\infty \circ SM = T_c U_2$. Dually, $SM \circ T_c W_\infty = T_c W_\infty \circ SM = T_c W_2$.*

Proof. The functions μ and $x_1(x_2 \vee \overline{x_3})$ are generators of *SM* and $T_c U_\infty$, respectively. We have that

$$\mu(x, y, z)(a \vee \overline{b}) = \mu(x(a \vee \overline{b}), y(a \vee \overline{b}), z(a \vee \overline{b})),$$
$$a(\mu(x, y, z) \vee \overline{b}) = \mu(a(x \vee \overline{b}), a(y \vee \overline{b}), a(z \vee \overline{b})),$$
$$a(b \vee \overline{\mu(x, y, z)}) = \mu(a(b \vee \overline{x}), a(b \vee \overline{y}), a(b \vee \overline{z})).$$

Theorem 1 implies that $SM \circ T_c U_\infty$ is a clone.

We observe that for any function g, $\mu(xg(\mathbf{a}), y, z) = \mu(x, y, z)\mu(g(\mathbf{a}), y, z)$ and $\mu(g, x_i, x_j) \in T_1$ for any variables x_i, x_j. In particular, $\mu(x(y \vee \overline{z}), a, b) = \mu(x, a, b)\mu(y \vee \overline{z}, a, b)$. If $f \in T_1$, then $x_i \wedge f \in T_c U_\infty$ for any variable x_i, and so we conclude that $[\mu, x_1] \circ [x_1(x_2 \vee \overline{x_3}), x_1] \subseteq T_c U_\infty \circ [\mu, x_1]$. Theorem 1 implies that $T_c U_\infty \circ SM$ is a clone. □

Proposition 18. *$M_c \circ L_c = T_c$.*

Proof. The clone L_c is generated by τ, and the functions $x_1 \vee x_2$ and $x_1 \wedge x_2$ generate M_c. We have that

$$\tau(a \vee b, c, d) = (a \wedge \tau(a, b, c)) \vee (b \wedge \tau(b, c, d)) \vee (\tau(a, c, d) \wedge \tau(b, c, d)),$$
$$\tau(a \wedge b, c, d) = (a \vee \tau(a, c, d)) \wedge (b \vee \tau(b, c, d)) \wedge (\tau(a, c, d) \vee \tau(b, c, d)).$$

Theorem 1 implies that $M_c \circ L_c$ is a clone, and it must be T_c. □

Proposition 19. *$M_c U_\infty \circ L_1 = T_1$. Dually, $M_c W_\infty \circ L_0 = T_0$.*

Proof. Let $l_1, \ldots, l_m$ be the $m = 2^n$ n-ary functions of L_1 in any fixed order. Define the function $\lambda : \mathbb{B}^n \to \mathbb{B}^m$ as $\lambda(\mathbf{v}) = (l_1(\mathbf{v}), \ldots, l_m(\mathbf{v}))$. If $\mathbf{a}, \mathbf{b} \in \mathbb{B}^n$ are incomparable, then $\lambda(\mathbf{a}), \lambda(\mathbf{b}) \in \mathbb{B}^m$ are incomparable as well, because all projections are in L_1. If $\mathbf{a} \prec \mathbf{b} \prec \mathbf{1}$, then there are integers $i, j \in \{1, \ldots, n\}$ such that $a_i = b_i = 0$, $a_j = 0$, $b_j = 1$. Consider the n-ary functions $\phi_1 = x_j$, $\phi_2 = x_i + x_j + \mathbf{1}$. We have that $\phi_1, \phi_2 \in L_1$, and $\phi_1(\mathbf{a}) = 0$, $\phi_1(\mathbf{b}) = 1$, $\phi_2(\mathbf{a}) = 1$, $\phi_2(\mathbf{b}) = 0$, so $\lambda(\mathbf{a})$ and $\lambda(\mathbf{b})$ are incomparable also in this case. We conclude that the range of λ consists of an antichain A and $\lambda(\mathbf{1}) = \mathbf{1}$.

Let $f \in T_1$ be n-ary. Let h be the $(m+1)$-ary function for which $h(\mathbf{w}) = 1$ if and only if there is an n-vector $\mathbf{v}$ with $f(\mathbf{v}) = 1$ and $\lambda(\mathbf{v}) = \mathbf{w}$. We see that $f(\mathbf{v}) = h(\lambda(\mathbf{v})) = h^{\mathrm{M}}(\lambda(\mathbf{v}))$. In any true vector of h, the coordinate corresponding to the constant function $\mathbf{1}$ equals 1 by definition, and the same holds for h^{M}, so we have that $h^{\mathrm{M}} \in U_\infty$. Because $f \in T_1$, we also have that $h^{\mathrm{M}} \in M_c$.

Therefore $f = h^{\mathrm{M}}(l_1, \ldots, l_m)$, where $h^{\mathrm{M}} \in M_c U_\infty$ and $l_1, \ldots, l_m \in L_1$, so we conclude that $M_c U_\infty \circ L_1 = T_1$. □

Proposition 20. *$\Lambda_0 \circ SM = MU_2$, $SM \circ \Lambda_1 = M_1$. Dually, $V_1 \circ SM = MW_2$, $SM \circ V_0 = M_0$.*

Proof. It follows from Propositions 2, 12–14 that

$$\Lambda_0 \circ SM = I_0 \circ \Lambda_c \circ SM = I_0 \circ M_c U_2 = MU_2,$$
$$SM \circ \Lambda_1 = SM \circ \Lambda_c \circ I_1 = M_c U_2 \circ I_1 = M_1. \qquad \square$$

Proposition 21. *$SM \circ I_0 = MU_2$, $SM \circ I = M$. Dually, $SM \circ I_1 = MW_2$.*

Proof. By setting $n-1$ variables to 0 in the majority function of arity $2n-1$, we get the n-ary conjunction; and by setting all variables to 0, we get the constant function $\mathbf{0}$. Therefore, $\Lambda_0 \subseteq SM \circ I_0$, so $SM \circ \Lambda_0 \subseteq SM \circ SM \circ I_0 = SM \circ I_0$. Since $I_0 \subseteq \Lambda_0$, we also have that $SM \circ I_0 \subseteq SM \circ \Lambda_0$. Thus, $SM \circ I_0 = SM \circ \Lambda_0 = MU_2$, by Proposition 20. By a similar argument, we can also show that $SM \circ I = M$. □

Proposition 22. *For $m = 2, \ldots, \infty$, $T_c U_\infty \circ MU_m = U_m$, $U_\infty \circ M_c U_m = M_c U_m \circ U_\infty = U_m$ and, dually, $T_c W_\infty \circ MW_m = W_m$, $W_\infty \circ M_c W_m = M_c W_m \circ W_\infty = W_m$.*

Proof. By Propositions 3, 5, 6, 12 and 13, we have that

$$T_c U_\infty \circ MU_m = T_c U_\infty \circ M_c U_m \circ I_0 = T_c U_m \circ I_0 = U_m,$$
$$U_\infty \circ M_c U_m = T_c U_\infty \circ I_0 \circ M_c U_m = T_c U_\infty \circ MU_m = U_m,$$
$$M_c U_m \circ U_\infty = M_c U_m \circ T_c U_\infty \circ I_0 = T_c U_m \circ I_0 = U_m. \qquad \square$$

Proposition 23. *$L \circ \Lambda_c = L_0 \circ \Lambda_1 = L_1 \circ \Lambda_0 = \Omega$. Dually, $L \circ V_c = L_1 \circ V_0 = L_0 \circ V_1 = \Omega$.*

Proof. From the Zhegalkin polynomial representation and Propositions 1, 12 and 13, it follows that

$$\begin{aligned}\Omega &= L_c \circ \Lambda = L_c \circ I \circ \Lambda_c = L \circ \Lambda_c \\ &= L_c \circ I_0 \circ I_1 \circ \Lambda_c = L_0 \circ \Lambda_1 \\ &= L_c \circ I_1 \circ I_0 \circ \Lambda_c = L_1 \circ \Lambda_0. \qquad \square\end{aligned}$$

Proposition 24. *$L_c \circ \Lambda_c = T_c$, $L_c \circ \Lambda_0 = T_0$, $L_c \circ \Lambda_1 = T_1$, $L_0 \circ \Lambda_c = T_0$, $L_1 \circ \Lambda_c = T_1$, $LS \circ \Lambda_0 = \Omega$, $LS \circ \Lambda_1 = \Omega$. Dually, $L_c \circ V_c = T_c$, $L_c \circ V_1 = T_1$, $L_c \circ V_0 = T_0$, $L_1 \circ V_c = T_1$, $L_0 \circ V_c = T_0$, $LS \circ V_1 = \Omega$, $LS \circ V_0 = \Omega$.*

Proof. If $f \in T_c$, then there is no constant term in the Zhegalkin polynomial of f, so in fact $f = g(h_1, \dots, h_n)$, where $g \in L_c$ and $h_i \in \Lambda_c$ $(i = 1, \dots, n)$, so $L_c \circ \Lambda_c = T_c$.

By Propositions 2 and 13 and the equality established above, we also have that

$$L_c \circ \Lambda_0 = L_c \circ \Lambda_c \circ I_0 = T_c \circ I_0 = T_0,$$

$$L_c \circ \Lambda_1 = L_c \circ \Lambda_c \circ I_1 = T_c \circ I_1 = T_1.$$

Furthermore, by applying these equalities and Propositions 1, 10 and 12, we have

$$L_0 \circ \Lambda_c = L_c \circ I_0 \circ \Lambda_c = L_c \circ \Lambda_0 = T_0,$$

$$L_1 \circ \Lambda_c = L_c \circ I_1 \circ \Lambda_c = L_c \circ \Lambda_1 = T_1,$$

$$LS \circ \Lambda_0 = I^* \circ L_c \circ \Lambda_0 = I^* \circ T_0 = \Omega,$$

$$LS \circ \Lambda_1 = I^* \circ L_c \circ \Lambda_1 = I^* \circ T_1 = \Omega. \qquad \square$$

Proposition 25. *$SM \circ \Omega(1) = \Omega$, $SM \circ U_\infty = U_\infty \circ SM = U_2$, $LS \circ SM = S$, $S_c \circ I_0 = T_0$, $SM \circ L_0 = T_0$. Dually, $SM \circ W_\infty = W_\infty \circ SM = W_2$, $S_c \circ I_1 = T_1$, $SM \circ L_1 = T_1$.*

Proof. It follows from Propositions 1, 6, 10, 15, 17, 20–22 and 24 that

$$SM \circ \Omega(1) = SM \circ I \circ I^* = M \circ I^* = \Omega,$$

$$SM \circ U_\infty = SM \circ T_cU_\infty \circ I_0 = T_cU_2 \circ I_0 = U_2,$$

$$U_\infty \circ SM = T_cU_\infty \circ \Lambda_0 \circ SM = T_cU_\infty \circ MU_2 = U_2,$$

$$LS \circ SM = I^* \circ L_c \circ SM = I^* \circ S_c = S,$$

$$S_c \circ I_0 = L_c \circ SM \circ I_0 = L_c \circ MU_2 = T_0,$$

$$SM \circ L_0 = SM \circ L_c \circ I_0 = S_c \circ I_0 = T_0.$$

For the very last equality, we applied one of the previously established equalities. $\square$

Proposition 26. *$V_0 \circ \Lambda_1 = M$, $V_1 \circ \Lambda_0 = M$, $V_c \circ \Lambda_0 = M_0$, $V_c \circ \Lambda_1 = M_1$, $V_0 \circ \Lambda_c = M_0$, $V_1 \circ \Lambda_c = M_1$, $V_c \circ \Lambda = M$, $V \circ \Lambda_c = M$. Dually, $\Lambda_1 \circ V_0 = M$, $\Lambda_0 \circ V_1 = M$, $\Lambda_c \circ V_1 = M_1$, $\Lambda_c \circ V_0 = M_0$, $\Lambda_1 \circ V_c = M_1$, $\Lambda_0 \circ V_c = M_0$, $\Lambda_c \circ V = M$, $\Lambda \circ V_c = M$.*

Proof. From the fact that $V_c \circ \Lambda_c = M_c$ and Propositions 12 and 13, it follows that

$$V_0 \circ \Lambda_1 = I_0 \circ V_c \circ \Lambda_c \circ I_1 = I_0 \circ M_c \circ I_1 = M_0 \circ I_1 = M,$$

$$V_1 \circ \Lambda_0 = I_1 \circ V_c \circ \Lambda_c \circ I_0 = I_1 \circ M_c \circ I_0 = M_1 \circ I_0 = M,$$

$$V_c \circ \Lambda_0 = V_c \circ \Lambda_c \circ I_0 = M_c \circ I_0 = M_0,$$

$$V_c \circ \Lambda_1 = V_c \circ \Lambda_c \circ I_1 = M_c \circ I_1 = M_1,$$

$$V_0 \circ \Lambda_c = I_0 \circ V_c \circ \Lambda_c = I_0 \circ M_c = M_0,$$

$$V_1 \circ \Lambda_c = I_1 \circ V_c \circ \Lambda_c = I_1 \circ M_c = M_1,$$

$$V_c \circ \Lambda = V_c \circ \Lambda_c \circ I = M_c \circ I = M,$$

$$V \circ \Lambda_c = I \circ V_c \circ \Lambda_c = I \circ M_c = M. \qquad \square$$

Proposition 27. *$T_cU_\infty \circ I = \Omega$, $M_cU_\infty \circ I = M$, $T_cU_\infty \circ V_0 = T_0$, $M_cU_\infty \circ \Omega(1) = \Omega$. Dually, $T_cW_\infty \circ I = \Omega$, $M_cW_\infty \circ I = M$, $T_cW_\infty \circ \Lambda_1 = T_1$, $M_cW_\infty \circ \Omega(1) = \Omega$.*

Proof. It follows from Propositions 1, 2, 9 and 13 that

$$\begin{aligned}
&T_cU_\infty \circ I = T_cU_\infty \circ I_1 \circ I_0 = T_1 \circ I_0 = \Omega,\\
&M_cU_\infty \circ I = M_cU_\infty \circ I_1 \circ I_0 = M_1 \circ I_0 = M,\\
&T_cU_\infty \circ V_0 = T_cU_\infty \circ V_c \circ I_0 = T_c \circ I_0 = T_0,\\
&M_cU_\infty \circ \Omega(1) = M_cU_\infty \circ I_1 \circ I_0 \circ I^* = M_1 \circ I_0 \circ I^* = M \circ I^* = \Omega. \qquad \square
\end{aligned}$$

Some of these composition results were presented without proof in [5].

3.2. Cases when the composition of clones is not a clone

Proposition 28. *$I_0 \circ I^* = I^* \cup [\mathbf{0}]$, $I_1 \circ I^* = I^* \cup [\mathbf{1}]$, $I \circ L_0 = L_0 \cup [\mathbf{1}]$, $I \circ L_1 = L_1 \cup [\mathbf{0}]$, $\Omega(1) \circ LS = LS \cup [\mathbf{0}, \mathbf{1}]$; these are not clones.*

Proof. Straightforward verification. □

Proposition 29. *$\Omega(1) \circ M \neq \Omega$, $\Omega(1) \circ T_c \neq \Omega$, $I_0 \circ T_c \neq T_0$, $I_0 \circ L_c \neq L_0$, $I^* \circ SM \neq S$. For $m = 2, \ldots, \infty$, $I_0 \circ T_cU_m \neq U_m$. (But $\Omega(1) \vee M = \Omega$, $\Omega(1) \vee T_c = \Omega$, $I_0 \vee T_c = T_0$, $I_0 \vee L_c = L_0$, $I^* \vee SM = S$, $I_0 \vee T_cU_m = U_m$.) Dually, $I_1 \circ T_c \neq T_1$, $I_1 \circ L_c \neq L_1$,* For $m = 2, \ldots, \infty$, *$I_1 \circ T_cW_m \neq W_m$.*

Proof. The function $x_1 + x_2 \in L_0$ is not constant, monotone, the negation of a monotone function, constant-preserving, nor the negation of a constant-preserving function. Therefore, $\Omega(1) \circ M \neq \Omega$, $\Omega(1) \circ T_c \neq \Omega$,$I_0 \circ T_c \neq T_0$, $I_0 \circ L_c \neq L_0$.

The function $x_1 + x_2 + x_3 \in S$ is not monotone nor the negation of a monotone self-dual function. Therefore, $I^* \circ SM \neq S$.

The function $(x_1 + x_2)x_3 \in U_m$, for any $m = 2, \ldots, \infty$, is not constant-preserving nor $\mathbf{0}$. Therefore, $I_0 \circ T_cU_m \neq U_m$. □

Proposition 30. *$M \circ T_0 \neq \Omega$ (but $M \vee T_0 = \Omega$). Dually, $M \circ T_1 \neq \Omega$.*

Proof. The only unary functions in T_0 are $\mathbf{0}$ and x_1. These are monotone, so any function in M composed with unary functions in T_0 will be in M. Thus, $\overline{x_1} \notin M \circ T_0$, and so $M \circ T_0 \neq \Omega$. □

Proposition 31. *$L \circ S \neq \Omega$ (but $L \vee S = \Omega$).*

Proof. The binary self-dual functions are variables and negations of variables. These are linear, so $L \circ S \neq \Omega$. □

Proposition 32. *$I_0 \circ SM \neq MU_2$, $I_0 \circ S_c \neq T_0$ (but $I_0 \vee SM = MU_2$, $I_0 \vee S_c = T_0$). Dually, $I_1 \circ SM \neq MW_2$, $I_1 \circ S_c \neq T_1$.*

Proof. Since in the chains $SM \subset M_cU_2 \subset MU_2$, $S_c \subset T_c \subset T_0$ all subset inclusions are proper, we must have that $I_0 \circ SM = SM \cup [\mathbf{0}] \neq MU_2$, $I_0 \circ S_c = S_c \cup [\mathbf{0}] \neq T_0$. □

Proposition 33. *$S \circ T_c \neq \Omega$ (but $S \vee T_c = \Omega$).*

Proof. Let $f \in S$ be n-ary, and let $g_1, \ldots, g_n \in T_c$ be m-ary. Now $f(\mathbf{0}) \neq f(\mathbf{1})$ and for all $i = 1, \ldots, n$, $g_i(\mathbf{0}) = 0$, $g_i(\mathbf{1}) = 1$, so we have that

$$f(g_1(\mathbf{0}), \ldots, g_n(\mathbf{0})) = f(\mathbf{0}) \neq f(\mathbf{1}) = f(g_1(\mathbf{1}), \ldots, g_n(\mathbf{1})).$$

Therefore, there are functions that are not in $S \circ T_c$, e.g., all constant functions. □

Proposition 34. *$U_2 \circ I^* \neq \Omega$ (but $U_2 \vee I^* = \Omega$). Dually, $W_2 \circ I^* \neq \Omega$.*

Proof. We show that $\mathbf{1} \notin U_2 \circ I^*$. Let $f \in U_2$ be m-ary and let $g_1, \ldots, g_m \in I^*$ be n-ary variables or negations of variables. Then

$$f(g_1(\mathbf{0}), \ldots, g_n(\mathbf{0})) = f(a_1, \ldots, a_n),$$
$$f(g_1(\mathbf{1}), \ldots, g_n(\mathbf{1})) = f(\overline{a_1}, \ldots, \overline{a_n})$$

cannot both be equal to 1, because $(a_1, \ldots, a_n) \wedge (\overline{a_1}, \ldots, \overline{a_n}) = \mathbf{0}$. Therefore, $f(g_1, \ldots, g_n) \neq \mathbf{1}$, and so $U_2 \circ I^* \neq \Omega$. □

Proposition 35. *$U_2 \circ S \neq \Omega$ (but $U_2 \vee S = \Omega$). Dually, $W_2 \circ S \neq \Omega$.*

Proof. The unary constant function $\mathbf{1}$ is not in $U_2 \circ S$: there cannot exist unary $s_1, \ldots, s_n \in S$ and $f \in U_2$ such that $f(s_1, \ldots, s_n) = \mathbf{1}$, because the unary functions in S are exactly those in I^*, and for every function $f \in U_2$ it holds that if $f(\mathbf{a}) = 1$, then $f(\overline{\mathbf{a}}) = 0$. Hence, $U_2 \circ S \neq \Omega$. □

Proposition 36. *$\Lambda \circ L \neq \Omega$, $\Lambda_0 \circ L_0 \neq T_0$, $\Lambda_1 \circ L_1 \neq T_1$, $\Lambda \circ S \neq \Omega$* (but *$\Lambda \vee L = \Omega$, $\Lambda_0 \vee L_0 = T_0$, $\Lambda_1 \vee L_1 = T_1$, $\Lambda \vee S = \Omega$). Dually, $V \circ L \neq \Omega$, $V_1 \circ L_1 \neq T_1$, $V_0 \circ L_0 \neq T_0$, $V \circ S \neq \Omega$.*

Proof. Consider the 4-ary function $\phi = x_1x_2 + x_3x_4$. We show that $\phi \notin \Lambda \circ L$. Assume, on the contrary, that there exists an n-ary function $f \in \Lambda$ and quaternary functions $g_1, \ldots, g_n \in L$ such that $\phi = f(g_1, \ldots, g_n)$. Since ϕ is not a constant function, f is not a constant function but rather a conjunction. We can assume, without loss of generality, that $f = x_1 \wedge \cdots \wedge x_n$.

Therefore, if for some $\mathbf{a} = (a_1, a_2, a_3, a_4)$, $\phi(\mathbf{a}) = 1$, then for all $1 \leqslant i \leqslant n$, $g_i(\mathbf{a}) = 1$. Also, if $\phi(\mathbf{a}) = 0$, then for some $1 \leqslant i \leqslant n$, $g_i(\mathbf{a}) = 0$. It follows that there exists $i \in \{1, \ldots, n\}$ such that $g_i(1, 1, 0, 0) = g_i(1, 1, 0, 1) = g_i(1, 1, 1, 0) = 1$, $g_i(1, 1, 1, 1) = 0$. But we see that it is not possible that g_i be linear, a contradiction. Hence, $\Lambda \circ L \neq \Omega$.

We note also that $\phi \in T_0$ and $\overline{\phi} \in T_1$. A similar argument shows that $\phi \notin \Lambda_0 \circ L_0$ and $\overline{\phi} \notin \Lambda_1 \circ L_1$. Hence, $\Lambda_0 \circ L_0 \neq T_0$ and $\Lambda_1 \circ L_1 \neq T_1$.

A similar argument shows also that $\phi \notin \Lambda \circ S$. Assume, on the contrary, that there exists an n-ary function $f \in \Lambda$ and quaternary functions $g_1, \ldots, g_n \in S$ such that $\phi = f(g_1, \ldots, g_n)$. We conclude that there exists a self-dual function g_i such that $g_i(0, 0, 0, 0) = g_i(1, 1, 1, 1) = 0$, a contradiction. Hence, $\Lambda \circ S \neq \Omega$. □

Proposition 37. *$U_3 \circ L_0 \neq T_0$ (but $U_3 \vee L_0 = T_0$). Dually, $W_3 \circ L_1 \neq T_1$.*

Proof. Consider the binary function $\phi = x_1 \vee x_2$. We see that $\phi \in T_0$ but $\phi \notin U_3, L_0$. Suppose, on the contrary, that $U_3 \circ L_0 = T_0$. Then $\phi = f(g_1, \ldots, g_m)$, where $f \in U_3$ is m-ary for some m and $g_1, \ldots, g_m \in L_0$ are binary. The binary functions in L_0 are $\mathbf{0}, x_1, x_2, x_1 + x_2$. With some identification of variables, permutation of variables, and addition of inessential variables, we can assume that f is 4-ary and $g_1 = \mathbf{0}$, $g_2 = x_1$, $g_3 = x_2$, $g_4 = x_1 + x_2$. Then we have that

$$0 = \phi(0, 0) = f(g_1, g_2, g_3, g_4)(0, 0) = f(0, 0, 0, 0),$$
$$1 = \phi(0, 1) = f(g_1, g_2, g_3, g_4)(0, 1) = f(0, 0, 1, 1),$$
$$1 = \phi(1, 0) = f(g_1, g_2, g_3, g_4)(1, 0) = f(0, 1, 0, 1),$$
$$1 = \phi(1, 1) = f(g_1, g_2, g_3, g_4)(1, 1) = f(0, 1, 1, 0).$$

But now $(0, 0, 1, 1) \wedge (0, 1, 0, 1) \wedge (0, 1, 1, 0) = (0, 0, 0, 0)$, a contradiction with the fact that $f \in U_3$ and so the set $\{(0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0)\}$ should be 1-separating. □

Proposition 38. *$T_cU_2 \circ S_c \neq T_c$ (but $T_cU_2 \vee S_c = T_c$). Dually, $T_cW_2 \circ S_c \neq T_c$.*

Proof. Consider the binary function $\phi = x_1 \vee x_2$. We see that $\phi \in T_c$ but $\phi \notin T_cU_2, S_c$. The binary functions in S_c are just variables, but these are also in T_cU_2, so a composition of any function in T_cU_2 with binary functions in S_c belongs to T_cU_2. Therefore, we conclude that $\phi \notin T_cU_2 \circ S_c$. Thus $T_cU_2 \circ S_c \neq T_c$. □

Proposition 39. *If $\mathscr{C}$ is a proper subclone of T_0 or T_1, then $\Omega(1) \circ \mathscr{C} \neq \Omega$.*

Proof. Since for every function f either $f \in T_0$ or $\overline{f} \in T_0$, $\{T_0, \overline{T_0}\}$ is a partition of Ω (we denote $\overline{\mathscr{C}} = \{f : \overline{f} \in \mathscr{C}\}$), and so for any proper subset $\mathscr{C}$ of T_0, we have that $\mathscr{C} \cup \overline{\mathscr{C}} \neq \Omega$. If $\mathscr{C}$ is in addition a clone, then there are several non-constant functions that are in T_0 but not in $\mathscr{C}$, so it is easily seen that $\Omega(1) \circ \mathscr{C} = \mathscr{C} \cup \overline{\mathscr{C}} \cup [\mathbf{0}, \mathbf{1}] \neq \Omega$.

Similarly, we can prove that if $\mathscr{C}$ is a proper subclone of T_1, then $\Omega(1) \circ \mathscr{C} \neq \Omega$. □

Proposition 39 implies in particular that $\Omega(1) \circ T_c$, $\Omega(1) \circ M_0$, $\Omega(1) \circ M_1$, $\Omega(1) \circ U_2$, $\Omega(1) \circ W_2$ are not clones.

Proposition 40. *If $\mathscr{C}$ is a proper subclone of M and $\mathscr{C} \neq M_0, M_1, M_c$, then $I \circ \mathscr{C} \neq M$. If $\mathscr{C}$ is a proper subclone of M_0 and $\mathscr{C} \neq M_c$, then $I_0 \circ \mathscr{C} \neq M_0$.*

Proof. The equality $I \circ \mathscr{C} = \mathscr{C} \cup [\mathbf{0}, \mathbf{1}] = M$ holds only if $\mathscr{C} \in \{M, M_0, M_1, M_c\}$. The equality $I_0 \circ \mathscr{C} = \mathscr{C} \cup [\mathbf{0}] = M_0$ holds only if $\mathscr{C} \in \{M_0, M_c\}$. □

Proposition 40 implies in particular that $I \circ MU_2$, $I_1 \circ M_cU_2$ are not clones. Dually, $I \circ MW_2$, $I_0 \circ M_cW_2$ are not clones.

Proposition 41. *$\Lambda_1 \circ M_cU_2 \neq M_1$, $\Lambda \circ MU_2 \neq M$, $M_0 \circ T_c \neq T_0$. For $n = 2, \ldots, \infty$, $MU_n \circ T_cU_n \neq U_n$. (But $\Lambda_1 \vee M_cU_2 = M_1$, $\Lambda \vee MU_2 = M$, $M_0 \vee T_c = T_0$, $MU_n \vee T_cU_n = U_n$.) Dually, $V_0 \circ M_cW_2 \neq M_0$, $V \circ MW_2 \neq M$, $M_1 \circ T_c \neq T_1$. For $n = 2, \ldots, \infty$, $MW_n \circ T_cW_n \neq W_n$.*

Proof. By Proposition 12, we have that

$$\Lambda_1 \circ M_cU_2 = I_1 \circ \Lambda_c \circ M_cU_2 = I_1 \circ M_cU_2 \neq M_1,$$

$$\Lambda \circ MU_2 = I \circ \Lambda_c \circ MU_2 = I \circ MU_2 \neq M,$$

$$M_0 \circ T_c = I_0 \circ M_c \circ T_c = I_0 \circ T_c \neq T_0,$$

$$MU_n \circ T_cU_n = I_0 \circ M_cU_n \circ T_cU_n = I_0 \circ T_cU_n \neq U_n.$$

The inequalities follow from Propositions 29 and 40. □

Proposition 42. *$L_0 \circ S_c \neq T_0$, $U_2 \circ S_c \neq T_0$, $\Lambda_c \circ S_c \neq T_c$, $T_cU_3 \circ L_c \neq T_c$ (but $L_0 \vee S_c = T_0$, $U_2 \vee S_c = T_0$, $\Lambda_c \vee S_c = T_c$, $T_cU_3 \vee L_c = T_c$). Dually, $L_1 \circ S_c \neq T_1$, $W_2 \circ S_c \neq T_1$, $V_c \circ S_c \neq T_c$, $T_cW_3 \circ L_c \neq T_c$.*

Proof. Suppose, on the contrary, that $L_0 \circ S_c = T_0$, $U_2 \circ S_c = T_0$, $\Lambda_c \circ S_c = T_c$, or $T_cU_3 \circ L_c = T_c$. It follows from Propositions 1, 2, 9–11, 16, 31, 35–37 that

$$\Omega \neq L \circ S_c = I^* \circ L_0 \circ S_c = I^* \circ T_0 = \Omega,$$

$$\Omega \neq U_2 \circ S = U_2 \circ S_c \circ I^* = T_0 \circ I^* = \Omega,$$

$$\Omega \neq \Lambda_c \circ S = \Lambda_c \circ S_c \circ I^* = T_c \circ I^* = \Omega,$$

$$T_0 \neq T_cU_3 \circ L_0 = T_cU_3 \circ L_c \circ I_0 = T_c \circ I_0 = T_0.$$

We have reached a contradiction in each case. □

3.3. Clone composition theorems

Based on the previous two sections, one can construct a clone composition table which indicates for all clones $\mathscr{C}_1$, $\mathscr{C}_2$ whether the composition $\mathscr{C}_1\mathscr{C}_2$ is a clone or not, see Table 1. The correctness of the table can be verified by drawing obvious consequences of each of the Propositions using the general rules of Section 2.

Theorem 2 summarizes the composition table. For each clone $\mathscr{C}$ that is the composition of two proper subclones, Theorem 3 then gives the possible decompositions $\mathscr{C} = \mathscr{C}_1\mathscr{C}_2$. The correctness of Theorems 2 and 3 can be verified making use of the clone composition table and the Post Lattice, and cross-checking. (Use of a coloured pencil and a photocopier recommended.)

Table 1
Clone composition table

$\circ$	Ω	T_0	T_1	T_c	M	M_0	M_1	M_c	S	S_c	SM	L	L_0	L_1	LS	L_c	Λ	Λ_0	Λ_1	Λ_c	V	V_0	V_1	V_c	U_n	MU_n	T_cU_n	M_cU_n	W_n	MW_n	T_cW_n	M_cW_n	$\Omega(1)$	I	I_0	I_1	I^*	I_c
Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω
T_0	Ω	T_0	Ω	T_0	Ω	T_0	Ω	T_0	Ω	T_0	T_0	Ω	T_0	Ω	Ω	T_0	Ω	T_0	Ω	T_0	Ω	T_0	Ω	T_0	T_0	T_0	T_0	T_0	Ω	Ω	T_0	T_0	Ω	Ω	T_0	Ω	Ω	T_0
T_1	Ω	Ω	T_1	T_1	Ω	Ω	T_1	T_1	Ω	T_1	T_1	Ω	Ω	T_1	Ω	T_1	Ω	Ω	T_1	T_1	Ω	Ω	T_1	T_1	Ω	Ω	T_1	T_1	T_1	T_1	T_1	T_1	Ω	Ω	Ω	T_1	Ω	T_1
T_c	Ω	T_0	T_1	T_c	Ω	T_0	T_1	T_c	Ω	T_c	T_c	Ω	T_0	T_1	Ω	T_c	Ω	T_0	T_1	T_c	Ω	T_0	T_1	T_c	T_0	T_0	T_c	T_c	T_1	T_1	T_c	T_c	Ω	Ω	T_0	T_1	Ω	T_c
M	Ω	$[\Omega]$	$[\Omega]$	$[\Omega]$	M	M	M	M	Ω	$[\Omega]$	M	Ω	$[\Omega]$	$[\Omega]$	Ω	$[\Omega]$	M	M	M	M	M	M	M	M	$[\Omega]$	M	$[\Omega]$	M	$[\Omega]$	M	$[\Omega]$	M	Ω	M	M	M	Ω	M
M_0	Ω	T_0	$[\Omega]$	$[T_0]$	M	M_0	M	M_0	Ω	$[T_0]$	M_0	Ω	T_0	$[\Omega]$	Ω	$[T_0]$	M	M_0	M	M_0	M	M_0	M	M_0	T_0	M_0	$[T_0]$	M_0	$[\Omega]$	M	$[T_0]$	M_0	Ω	M	M_0	M	Ω	M_0
M_1	Ω	$[\Omega]$	T_1	$[T_1]$	M	M	M_1	M_1	Ω	$[T_1]$	M_1	Ω	$[\Omega]$	T_1	Ω	$[T_1]$	M	M	M_1	M_1	M	M	M_1	M_1	$[\Omega]$	M	$[T_1]$	M_1	T_1	M_1	$[T_1]$	M_1	Ω	M	M	M_1	Ω	M_1
M_c	Ω	T_0	T_1	T_c	M	M_0	M_1	M_c	Ω	T_c	M_c	Ω	T_0	T_1	Ω	T_c	M	M_0	M_1	M_c	M	M_0	M_1	M_c	T_0	M_0	T_c	M_c	T_1	M_1	T_c	M_c	Ω	M	M_0	M_1	Ω	M_c
S	Ω	Ω	Ω	$[\Omega]$	Ω	Ω	Ω	$[\Omega]$	S	S	S	Ω	Ω	Ω	S	S	Ω	Ω	Ω	$[\Omega]$	Ω	Ω	Ω	$[\Omega]$	Ω	Ω	$[\Omega]$	$[\Omega]$	Ω	Ω	$[\Omega]$	$[\Omega]$	Ω	Ω	Ω	Ω	S	S
S_c	Ω	T_0	T_1	T_c	Ω	T_0	T_1	T_c	S	S_c	S_c	Ω	T_0	T_1	S	S_c	Ω	T_0	T_1	T_c	Ω	T_0	T_1	T_c	T_0	T_0	T_c	T_c	T_1	T_1	T_c	T_c	Ω	Ω	T_0	T_1	S	S_c
SM	Ω	T_0	T_1	T_c	M	M_0	M_1	M_c	S	S_c	SM	Ω	T_0	T_1	S	S_c	M	MU_2	M_1	M_cU_2	M	M_0	MW_2	M_cW_2	U_2	MU_2	T_cU_2	M_cU_2	W_2	MW_2	T_cW_2	M_cW_2	Ω	M	MU_2	MW_2	S	SM
L	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	$[\Omega]$	$[\Omega]$	$[\Omega]$	L	L	L	L	L	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω	L	L	L	L	L	L
L_0	Ω	T_0	Ω	T_0	Ω	T_0	Ω	T_0	$[\Omega]$	$[T_0]$	$[T_0]$	L	L_0	L	L	L_0	Ω	T_0	Ω	T_0	Ω	T_0	Ω	T_0	T_0	T_0	T_0	T_0	Ω	Ω	T_0	T_0	L	L	L_0	L	L	L_0
L_1	Ω	Ω	T_1	T_1	Ω	Ω	T_1	T_1	$[\Omega]$	$[T_1]$	$[T_1]$	L	L	L_1	L	L_1	Ω	Ω	T_1	T_1	Ω	Ω	T_1	T_1	Ω	Ω	T_1	T_1	T_1	T_1	T_1	T_1	L	L	L	L_1	L	L_1
LS	Ω	Ω	Ω	$[\Omega]$	Ω	Ω	Ω	$[\Omega]$	S	S	S	L	L	L	LS	LS	Ω	Ω	Ω	$[\Omega]$	Ω	Ω	Ω	$[\Omega]$	Ω	Ω	$[\Omega]$	$[\Omega]$	Ω	Ω	$[\Omega]$	$[\Omega]$	L	L	L	L	LS	LS
L_c	Ω	T_0	T_1	T_c	Ω	T_0	T_1	T_c	S	S_c	S_c	L	L_0	L_1	LS	L_c	Ω	T_0	T_1	T_c	Ω	T_0	T_1	T_c	T_0	T_0	T_c	T_c	T_1	T_1	T_c	T_c	L	L	L_0	L_1	LS	L_c
Λ	Ω	$[\Omega]$	$[\Omega]$	$[\Omega]$	M	M	M	M	$[\Omega]$	$[\Omega]$	$[M]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	Λ	Λ	Λ	Λ	M	M	M	M	$[\Omega]$	$[M]$	$[\Omega]$	$[M]$	$[\Omega]$	M	$[\Omega]$	M	$[\Omega]$	Λ	Λ	Λ	$[\Omega]$	Λ
Λ_0	Ω	T_0	$[\Omega]$	$[T_0]$	M	M_0	M	M_0	$[\Omega]$	$[T_0]$	MU_2	$[\Omega]$	$[T_0]$	$[\Omega]$	$[\Omega]$	$[T_0]$	Λ	Λ_0	Λ	Λ_0	M	M_0	M	M_0	U_n	MU_n	$[U_n]$	MU_n	$[\Omega]$	M	$[T_0]$	M_0	$[\Omega]$	Λ	Λ_0	Λ	$[\Omega]$	Λ_0
Λ_1	Ω	$[\Omega]$	T_1	$[T_1]$	M	M	M_1	M_1	$[\Omega]$	$[T_1]$	$[M_1]$	$[\Omega]$	$[\Omega]$	$[T_1]$	$[\Omega]$	$[T_1]$	Λ	Λ	Λ_1	Λ_1	M	M	M_1	M_1	$[\Omega]$	$[M]$	$[T_1]$	$[M_1]$	T_1	M_1	$[T_1]$	M_1	$[\Omega]$	Λ	Λ	Λ_1	$[\Omega]$	Λ_1
Λ_c	Ω	T_0	T_1	T_c	M	M_0	M_1	M_c	$[\Omega]$	$[T_c]$	M_cU_2	$[\Omega]$	$[T_0]$	$[T_1]$	$[\Omega]$	$[T_c]$	Λ	Λ_0	Λ_1	Λ_c	M	M_0	M_1	M_c	U_n	MU_n	T_cU_n	M_cU_n	T_1	M_1	T_c	M_c	$[\Omega]$	Λ	Λ_0	Λ_1	$[\Omega]$	Λ_c
V	Ω	$[\Omega]$	$[\Omega]$	$[\Omega]$	M	M	M	M	$[\Omega]$	$[\Omega]$	$[M]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	M	M	M	M	V	V	V	V	$[\Omega]$	M	$[\Omega]$	M	$[\Omega]$	$[M]$	$[\Omega]$	$[M]$	$[\Omega]$	V	V	V	$[\Omega]$	V
V_0	Ω	T_0	$[\Omega]$	$[T_0]$	M	M_0	M	M_0	$[\Omega]$	$[T_0]$	$[M_0]$	$[\Omega]$	$[T_0]$	$[\Omega]$	$[\Omega]$	$[T_0]$	M	M_0	M	M_0	V	V_0	V	V_0	T_0	M_0	$[T_0]$	M_0	$[\Omega]$	$[M]$	$[T_0]$	$[M_0]$	$[\Omega]$	V	V_0	V	$[\Omega]$	V_0
V_1	Ω	$[\Omega]$	T_1	$[T_1]$	M	M	M_1	M_1	$[\Omega]$	$[T_1]$	MW_2	$[\Omega]$	$[\Omega]$	$[T_1]$	$[\Omega]$	$[T_1]$	M	M	M_1	M_1	V	V	V_1	V_1	$[\Omega]$	M	$[T_1]$	M_1	W_n	MW_n	$[W_n]$	MW_n	$[\Omega]$	V	V	V_1	$[\Omega]$	V_1
V_c	Ω	T_0	T_1	T_c	M	M_0	M_1	M_c	$[\Omega]$	$[T_c]$	M_cW_2	$[\Omega]$	$[T_0]$	$[T_1]$	$[\Omega]$	$[T_c]$	M	M_0	M_1	M_c	V	V_0	V_1	V_c	T_0	M_0	T_c	M_c	W_n	MW_n	T_cW_n	M_cW_n	$[\Omega]$	V	V_0	V_1	$[\Omega]$	V_c
U_m	Ω	T_0	Ω	T_0	Ω	T_0	Ω	T_0	$[\Omega]$	$[T_0]$	U_2	Ω	$T_0\dagger$	Ω	$[\Omega]$	$[T_0]$	Ω	U_m	Ω	U_m	Ω	T_0	Ω	T_0	U_p	U_p	U_p	U_p	Ω	Ω	T_0	T_0	Ω	Ω	U_m	Ω	$[\Omega]$	U_m
MU_m	Ω	T_0	$[\Omega]$	$[T_0]$	M	M_0	M	M_0	$[\Omega]$	$[T_0]$	MU_2	Ω	$T_0\dagger$	$[\Omega]$	$[\Omega]$	$[T_0]$	M	MU_m	M	MU_m	M	M_0	M	M_0	U_p	MU_p	$[U_p]$	MU_p	$[\Omega]$	M	$[T_0]$	M_0	Ω	M	MU_m	M	$[\Omega]$	MU_m
T_cU_m	Ω	T_0	T_1	T_c	Ω	T_0	T_1	T_c	$[\Omega]$	$[T_c]$	T_cU_2	Ω	$T_0\dagger$	T_1	$[\Omega]$	$[T_c]$	Ω	U_m	T_1	T_cU_m	Ω	T_0	T_1	T_c	U_p	U_p	T_cU_p	T_cU_p	T_1	T_1	T_c	T_c	Ω	Ω	U_m	T_1	$[\Omega]$	T_cU_m
M_cU_m	Ω	T_0	T_1	T_c	M	M_0	M_1	M_c	$[\Omega]$	$[T_c]$	M_cU_2	Ω	$T_0\dagger$	T_1	$[\Omega]$	$[T_c]$	M	MU_m	M_1	M_cU_m	M	M_0	M_1	M_c	U_p	MU_p	T_cU_p	M_cU_p	T_1	M_1	T_c	M_c	Ω	M	MU_m	M_1	$[\Omega]$	M_cU_m
W_m	Ω	Ω	T_1	T_1	Ω	Ω	T_1	T_1	$[\Omega]$	$[T_1]$	W_2	Ω	Ω	$T_1\dagger$	$[\Omega]$	$[T_1]$	Ω	Ω	T_1	T_1	Ω	Ω	W_m	W_m	Ω	Ω	T_1	T_1	W_p	W_p	W_p	W_p	Ω	Ω	Ω	W_m	$[\Omega]$	W_m
MW_m	Ω	$[\Omega]$	T_1	$[T_1]$	M	M	M_1	M_1	$[\Omega]$	$[T_1]$	MW_2	Ω	$[\Omega]$	$T_1\dagger$	$[\Omega]$	$[T_1]$	M	M	M_1	M_1	M	M	MW_m	MW_m	$[\Omega]$	M	$[T_1]$	M_1	W_p	MW_p	$[W_p]$	MW_p	Ω	M	M	MW_m	$[\Omega]$	MW_m
T_cW_m	Ω	T_0	T_1	T_c	Ω	T_0	T_1	T_c	$[\Omega]$	$[T_c]$	T_cW_2	Ω	T_0	$T_1\dagger$	$[\Omega]$	$[T_c]$	Ω	T_0	T_1	T_c	Ω	T_0	W_m	T_cW_m	T_0	T_0	T_c	T_c	W_p	W_p	T_cW_p	T_cW_p	Ω	Ω	T_0	W_m	$[\Omega]$	T_cW_m
M_cW_m	Ω	T_0	T_1	T_c	M	M_0	M_1	M_c	$[\Omega]$	$[T_c]$	M_cW_2	Ω	T_0	$T_1\dagger$	$[\Omega]$	$[T_c]$	M	M_0	M_1	M_c	M	M_0	MW_m	M_cW_m	T_0	M_0	T_c	M_c	W_p	MW_p	T_cW_p	M_cW_p	Ω	M	M_0	MW_m	$[\Omega]$	M_cW_m
$\Omega(1)$	Ω	Ω	Ω	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	L	L	L	$[L]$	$[L]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$\Omega(1)$	$\Omega(1)$	$\Omega(1)$	$\Omega(1)$	$\Omega(1)$	$\Omega(1)$
I	Ω	$[\Omega]$	$[\Omega]$	$[\Omega]$	M	M	M	M	$[\Omega]$	$[\Omega]$	$[M]$	L	$[L]$	$[L]$	$[L]$	$[L]$	Λ	Λ	Λ	Λ	V	V	V	V	$[\Omega]$	$[M]$	$[\Omega]$	$[M]$	$[\Omega]$	$[M]$	$[\Omega]$	$[M]$	$\Omega(1)$	I	I	I	$\Omega(1)$	I
I_0	Ω	T_0	$[\Omega]$	$[T_0]$	M	M_0	M	M_0	$[\Omega]$	$[T_0]$	$[MU_2]$	L	L_0	$[L]$	$[L]$	$[L_0]$	Λ	Λ_0	Λ	Λ_0	V	V_0	V	V_0	U_n	MU_n	$[U_n]$	MU_n	$[\Omega]$	$[M]$	$[T_0]$	$[M_0]$	$\Omega(1)$	I	I_0	I	$[\Omega(1)]$	I_0
I_1	Ω	$[\Omega]$	T_1	$[T_1]$	M	M	M_1	M_1	$[\Omega]$	$[T_1]$	$[MW_2]$	L	$[L]$	L_1	$[L]$	$[L_1]$	Λ	Λ	Λ_1	Λ_1	V	V	V_1	V_1	$[\Omega]$	$[M]$	$[T_1]$	$[M_1]$	W_n	MW_n	$[W_n]$	MW_n	$\Omega(1)$	I	I	I_1	$[\Omega(1)]$	I_1
I^*	Ω	Ω	Ω	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	S	S	$[S]$	L	L	L	LS	LS	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$[\Omega]$	$\Omega(1)$	$\Omega(1)$	$\Omega(1)$	$\Omega(1)$	I^*	I^*
I_c	Ω	T_0	T_1	T_c	M	M_0	M_1	M_c	S	S_c	SM	L	L_0	L_1	LS	L_c	Λ	Λ_0	Λ_1	Λ_c	V	V_0	V_1	V_c	U_n	MU_n	T_cU_n	M_cU_n	W_n	MW_n	T_cW_n	M_cW_n	$\Omega(1)$	I	I_0	I_1	I^*	I_c

In the table we have enumerated $\mathscr{C}_1\mathscr{C}_2$. Rows correspond to $\mathscr{C}_1$ and columns correspond to $\mathscr{C}_2$. Whenever $\mathscr{C}_1\mathscr{C}_2$ is not a clone, we have written $\mathscr{C}_1 \vee \mathscr{C}_2$ in square brackets. The clone appearing as an entry marked with † is $\mathscr{C}_1\mathscr{C}_2$ for $m = 2$ only, and $\mathscr{C}_1 \vee \mathscr{C}_2$ otherwise. We denote $p = \min(m, n)$.

Let $\mathscr{C}_1, \dots, \mathscr{C}_n, \mathscr{D}_1, \dots, \mathscr{D}_m$ be Post classes. As usual, $[\mathscr{C}_i, \mathscr{D}_j]$ denotes the set of Post classes $\mathscr{C}$ with $\mathscr{C}_i \subseteq \mathscr{C} \subseteq \mathscr{D}_j$. The union $\bigcup_{i,j} [\mathscr{C}_i, \mathscr{D}_j]$ is denoted by $[\{\mathscr{C}_1, \dots, \mathscr{C}_n\}, \{\mathscr{D}_1, \dots, \mathscr{D}_m\}]$, where the set braces are omitted when $n=1$ or $m=1$.

Theorem 2. *Let $\mathscr{C}_1$, $\mathscr{C}_2$ be Post classes. If for some $i \in \{1, \dots, 10\}$, $\mathscr{C}_1 \in A_i$ and $\mathscr{C}_2 \in B_i$ or $\mathscr{C}_1^{\mathrm{d}} \in A_i$ and $\mathscr{C}_2^{\mathrm{d}} \in B_i$, where the sets A_i, B_i are given below, then $\mathscr{C}_1\mathscr{C}_2$ and $\mathscr{C}_1^{\mathrm{d}}\mathscr{C}_2^{\mathrm{d}}$ are not clones. Otherwise, $\mathscr{C}_1\mathscr{C}_2 = \mathscr{C}_1 \vee \mathscr{C}_2$ and $\mathscr{C}_1^{\mathrm{d}}\mathscr{C}_2^{\mathrm{d}} = \mathscr{C}_1^{\mathrm{d}} \vee \mathscr{C}_2^{\mathrm{d}}$ are clones.*

- $A_1 = [\{I_0, \Lambda_{\mathrm{c}}\}, \{\Omega(1), \Lambda, U_2\}]$, $B_1 = [L_{\mathrm{c}}, S]$;
- $A_2 = [I^*, \Omega(1)]$, $B_2 = [\Lambda_{\mathrm{c}}, \{U_2, M, T_{\mathrm{c}}\}]$;
- $A_3 = [LS, S]$, $B_3 = [\Lambda_{\mathrm{c}}, T_{\mathrm{c}}]$;
- $A_4 = [\Lambda_{\mathrm{c}}, \Lambda]$, $B_4 = [\{I^*, L_{\mathrm{c}}\}, \{L, S\}]$;
- $A_5 = [I_0, \{L, V\}]$, $B_5 = [SM, S]$;
- $A_6 = [I_0, M]$, $B_6 = [\{L_{\mathrm{c}}, T_{\mathrm{c}}U_\infty, T_{\mathrm{c}}W_\infty\}, T_1]$;
- $A_7 = [I_1, \Lambda]$, $B_7 = [M_{\mathrm{c}}U_\infty, MU_2]$;
- $A_8 = [M_{\mathrm{c}}U_\infty, U_3]$, $B_8 = \{L_0\}$;
- $A_9 = [\{I_0, \Lambda_{\mathrm{c}}\}, U_2]$, $B_9 = \{I^*\}$;
- $A_{10} = \{I^*\}$, $B_{10} = \{SM\}$.

For a given clone $\mathscr{C}$, there are generally several factorizations of the form $\mathscr{C} = \mathscr{C}_1\mathscr{C}_2$, where $\mathscr{C}_1$ and $\mathscr{C}_2$ are also clones. We say that $\mathscr{C}_1\mathscr{C}_2$ is a *minimal factorization of $\mathscr{C}$ into two clones* if $\mathscr{C}_1\mathscr{C}_2 = \mathscr{C}$ and for all subclones $\mathscr{C}_1' \subseteq \mathscr{C}_1$ and $\mathscr{C}_2' \subseteq \mathscr{C}_2$, whenever at least one of the subset inclusions is proper, we have that $\mathscr{C}_1'\mathscr{C}_2' \neq \mathscr{C}$.

It is customary to call the clones covering I_{c} *minimal*. The minimal clones are I_0, I_1, I^*, Λ_{c}, V_{c}, L_{c}, SM. We say that a clone is *prime* if it is not a composition of two proper subclones. The prime clones are the seven minimal ones and $T_{\mathrm{c}}U_\infty$, $T_{\mathrm{c}}W_\infty$, $M_{\mathrm{c}}U_n$, $M_{\mathrm{c}}W_n$ for $n = 3, \dots, \infty$.

Theorem 3. *The following list of minimal factorizations of each non-prime clone into two clones is complete up to duality. That is, whenever we have listed a minimal factorization $\mathscr{C} = \mathscr{C}_1\mathscr{C}_2$, we also have the minimal factorization $\mathscr{C}^{\mathrm{d}} = \mathscr{C}_1^{\mathrm{d}}\mathscr{C}_2^{\mathrm{d}}$. Furthermore, for each clone $\mathscr{C}$ and subclones $\mathscr{D}_1$ and $\mathscr{D}_2$ of $\mathscr{C}$, $\mathscr{C} = \mathscr{D}_1\mathscr{D}_2$ if and only if there is a minimal factorization $\mathscr{C} = \mathscr{C}_1\mathscr{C}_2$ such that $\mathscr{C}_1 \subseteq \mathscr{D}_1$ and $\mathscr{C}_2 \subseteq \mathscr{D}_2$.*

- $\Omega = M_{\mathrm{c}} \circ I^* = S \circ I_0 = S_{\mathrm{c}} \circ I = SM \circ \Omega(1) = L \circ \Lambda_{\mathrm{c}} = L_0 \circ \Lambda_1 = L_1 \circ \Lambda_0 = LS \circ \Lambda_0 = LS \circ \Lambda_1 = L_{\mathrm{c}} \circ \Lambda = U_\infty \circ I_1 = T_{\mathrm{c}}U_\infty \circ I = M_{\mathrm{c}}U_\infty \circ \Omega(1) = I^* \circ T_0$;
- $T_0 = S_{\mathrm{c}} \circ I_0 = SM \circ L_0 = L_0 \circ \Lambda_{\mathrm{c}} = L_0 \circ V_{\mathrm{c}} = L_{\mathrm{c}} \circ \Lambda_0 = L_{\mathrm{c}} \circ V_0 = V_{\mathrm{c}} \circ U_\infty = U_\infty \circ V_{\mathrm{c}} = T_{\mathrm{c}}U_\infty \circ V_0 = T_{\mathrm{c}}W_\infty \circ I_0 = M_{\mathrm{c}}W_\infty \circ L_0$;
- $T_{\mathrm{c}} = M_{\mathrm{c}} \circ L_{\mathrm{c}} = L_{\mathrm{c}} \circ \Lambda_{\mathrm{c}} = V_{\mathrm{c}} \circ T_{\mathrm{c}}U_\infty = T_{\mathrm{c}}U_\infty \circ V_{\mathrm{c}}$;
- $M = M_{\mathrm{c}} \circ I = SM \circ I = \Lambda \circ V_{\mathrm{c}} = \Lambda_0 \circ V_1 = \Lambda_1 \circ V_0 = \Lambda_{\mathrm{c}} \circ V = MU_\infty \circ I_1 = M_{\mathrm{c}}U_\infty \circ I = I \circ M_{\mathrm{c}} = I_0 \circ M_1$;
- $M_0 = SM \circ V_0 = \Lambda_0 \circ V_{\mathrm{c}} = \Lambda_{\mathrm{c}} \circ V_0 = V_0 \circ \Lambda_{\mathrm{c}} = V_{\mathrm{c}} \circ \Lambda_0 = M_{\mathrm{c}}W_\infty \circ I_0 = I_0 \circ M_{\mathrm{c}}$;
- $M_{\mathrm{c}} = \Lambda_{\mathrm{c}} \circ V_{\mathrm{c}}$;
- $S = SM \circ I^* = LS \circ SM = I^* \circ S_{\mathrm{c}}$;
- $S_{\mathrm{c}} = SM \circ L_{\mathrm{c}} = L_{\mathrm{c}} \circ SM$;
- $L = L_0 \circ I_1 = L_0 \circ I^* = LS \circ I_0 = L_{\mathrm{c}} \circ I = I^* \circ L_0$;
- $L_0 = L_{\mathrm{c}} \circ I_0$;
- $LS = L_{\mathrm{c}} \circ I^* = I^* \circ L_{\mathrm{c}}$;
- $U_2 = SM \circ U_\infty = U_\infty \circ SM$;
- $U_m = U_\infty \circ M_{\mathrm{c}}U_m = T_{\mathrm{c}}U_m \circ I_0 = T_{\mathrm{c}}U_\infty \circ MU_m = M_{\mathrm{c}}U_m \circ U_\infty$;
- $MU_2 = SM \circ I_0 = \Lambda_0 \circ SM$;
- $MU_m = I_0 \circ M_{\mathrm{c}}U_m = M_{\mathrm{c}}U_m \circ I_0$;
- $T_{\mathrm{c}}U_2 = SM \circ T_{\mathrm{c}}U_\infty = T_{\mathrm{c}}U_\infty \circ SM$;
- $T_{\mathrm{c}}U_m = T_{\mathrm{c}}U_\infty \circ M_{\mathrm{c}}U_m = M_{\mathrm{c}}U_m \circ T_{\mathrm{c}}U_\infty$;
- $M_{\mathrm{c}}U_2 = SM \circ \Lambda_{\mathrm{c}} = \Lambda_{\mathrm{c}} \circ SM$;
- $\Lambda = \Lambda_0 \circ I_1 = \Lambda_1 \circ I_0 = \Lambda_{\mathrm{c}} \circ I = I \circ \Lambda_{\mathrm{c}} = I_0 \circ \Lambda_1 = I_1 \circ \Lambda_0$;
- $\Lambda_0 = \Lambda_{\mathrm{c}} \circ I_0 = I_0 \circ \Lambda_{\mathrm{c}}$;

- $\Lambda_1 = \Lambda_c \circ I_1 = I_1 \circ \Lambda_c$;
- $\Omega(1) = I \circ I^* = I^* \circ I_0$;
- $I = I_0 \circ I_1$.

4. Normal forms

It follows from Theorem 3 that every clone can be represented as a product of prime clones. In fact, the clone Ω can be represented as a product of minimal clones. (By a product, we mean a composition of 0, 1, 2, 3, or more clones. We adopt the convention that an empty composition of clones equals I_c.)

Note that each of the seven minimal clones is generated by a single function. We refer to the minimum arity of such a generating function as the *arity* of the clone. For each of the minimal clones, there is a unique generating function of minimum arity. The minimal clones, their generating functions of minimum arity, and their arities are the following: SM, μ, 3; L_c, τ, 3; Λ_c, $x_1 \wedge x_2$, 2; V_c, $x_1 \vee x_2$, 2; I^*, $\overline{x_1}$, 1; I_0, $\mathbf{0}$, 1; I_1, $\mathbf{1}$, 1.

In order to develop the concept of well-behaved factorization, we impose two simple conditions on the factorization $\mathscr{C} = \mathscr{C}_1 \cdots \mathscr{C}_n$ of a clone $\mathscr{C}$ into minimal clones.

Condition 1. *The factors occur in descending order of arity with no repetitions of factors.*

Condition 2. *For any factorization* $\mathscr{C} = \mathscr{D}_1 \cdots \mathscr{D}_m$ *of* $\mathscr{C}$ *into minimal clones satisfying* Condition 1, *there are no integers* i, j, k, l *with* $0 \leqslant i \leqslant j \leqslant n$, $0 \leqslant k \leqslant l \leqslant m$, *such that*

$$\mathscr{D}_1 \cdots \mathscr{D}_k \subseteq \mathscr{C}_1 \cdots \mathscr{C}_i,$$

$$\mathscr{D}_{k+1} \cdots \mathscr{D}_l \subset \mathscr{C}_{i+1} \cdots \mathscr{C}_j,$$

$$\mathscr{D}_{l+1} \cdots \mathscr{D}_m \subseteq \mathscr{C}_{j+1} \cdots \mathscr{C}_n.$$

Note that Condition 2 implies in particular that no factor can be dropped off. We say that a factorization satisfying Condition 1 is *redundant*, if it does not satisfy Condition 2.

A *descending irredundant factorization* of the clone $\mathscr{C}$ is a factorization of $\mathscr{C}$ into minimal clones that satisfies Conditions 1 and 2.

Theorem 4. *The descending irredundant factorizations of* Ω *are exactly the following*:

$$SM \circ I^* \circ I_0, \quad SM \circ I^* \circ I_1, \quad L_c \circ \Lambda_c \circ I_0 \circ I_1, \quad L_c \circ \Lambda_c \circ I_1 \circ I_0,$$

$$L_c \circ V_c \circ I_0 \circ I_1, \quad L_c \circ V_c \circ I_1 \circ I_0, \quad \Lambda_c \circ V_c \circ I^*, \quad V_c \circ \Lambda_c \circ I^*.$$

Proof. Consider a descending irredundant factorization $\Omega = \mathscr{C}_1 \cdots \mathscr{C}_n$. It is obvious that $n \geqslant 3$, because Ω is not a minimal clone nor a product of two minimal clones.

$\mathscr{C}_1$ cannot be I^*, I_0 or I_1; otherwise we could only obtain a subclass of $\Omega(1)$.

Consider the case that $\mathscr{C}_1 = \Lambda_c$. It is not possible that $\mathscr{C}_2 = I^*, I_0, I_1$, because then we could only obtain a subclass of $\Lambda_c \circ \Omega(1) \neq \Omega$. If $\mathscr{C}_2 = V_c$ (note that $\Lambda_c \circ V_c = M_c$), then we could have the factorization $\Omega = \Lambda_c \circ V_c \circ I^*$; but if $\mathscr{C}_3$ were I_0 or I_1, then we would still need the factor I^* in order to obtain Ω, and then the factorization would become redundant.

By duality, if $\mathscr{C}_1 = V_c$, then the only possible factorization is $\Omega = V_c \circ \Lambda_c \circ I^*$.

Consider the case that $\mathscr{C}_1 = L_c$. It is not possible that $\mathscr{C}_2 = I^*, I_0, I_1$; otherwise we could only obtain a subclass of $L = L_c \circ \Omega(1)$. If $\mathscr{C}_2 = \Lambda_c$ (note that $L_c \circ \Lambda_c = T_c$), then $\mathscr{C}_3$ cannot be V_c because then the factorization becomes redundant ($L_c \circ \Lambda_c \circ V_c = L_c \circ \Lambda_c = T_c$). If $\mathscr{C}_3 = I^*$, then we would have the factorization $\Omega = L_c \circ \Lambda_c \circ I^*$, which satisfies Condition 1; but it does not satisfy Condition 2, because we also have that $\Omega = V_c \circ \Lambda_c \circ I^*$ and $V_c \circ \Lambda_c = M_c \subset T_c = L_c \circ \Lambda_c$. If $\mathscr{C}_3 = I_0, I_1$, then we have the factorizations $\Omega = L_c \circ \Lambda_c \circ I_0 \circ I_1 = L_c \circ \Lambda_c \circ I_1 \circ I_0$; any I^* occurring after $\mathscr{C}_3$ would give rise to a redundant factorization. By duality, the only possible factorizations with $\mathscr{C}_2 = V_c$ are $\Omega = L_c \circ V_c \circ I_0 \circ I_1 = L_c \circ V_c \circ I_1 \circ I_0$. We will discuss later the case that $\mathscr{C}_1 = L_c$ and $\mathscr{C}_2 = SM$.

Consider the case that $\mathscr{C}_1 = SM$. If $\mathscr{C}_2 = I^*, I_0, I_1$, then we could have the factorizations $\Omega = SM \circ I^* \circ I_0 = SM \circ I^* \circ I_0 = SM \circ I_0 \circ I_1 \circ I^* = SM \circ I_1 \circ I_0 \circ I^*$. But the last two do not satisfy Condition 2, because we also have that $\Omega = \Lambda_c \circ V_c \circ I^*$ and $\Lambda_c \circ V_c = M_c \subset M = SM \circ I_0 \circ I_1 = SM \circ I_1 \circ I_0$.

It is not possible that $\mathscr{C}_1 = SM$ and $\mathscr{C}_2 = \Lambda_{\mathrm{c}}$ (note that $SM \circ \Lambda_{\mathrm{c}} = M_{\mathrm{c}}U_2$). If $\mathscr{C}_3 = V_{\mathrm{c}}$, then Condition 2 is not satisfied, because $SM \circ \Lambda_{\mathrm{c}} \circ V_{\mathrm{c}} = M_{\mathrm{c}} = \Lambda_{\mathrm{c}} \circ V_{\mathrm{c}}$. If $\mathscr{C}_3 = I^*$, I_0, I_1, then we could obtain the factorizations:

$$\Omega = SM \circ \Lambda_{\mathrm{c}} \circ I^* \circ I_0 = SM \circ \Lambda_{\mathrm{c}} \circ I^* \circ I_1 = SM \circ \Lambda_{\mathrm{c}} \circ I_0 \circ I_1 \circ I^* = SM \circ \Lambda_{\mathrm{c}} \circ I_1 \circ I_0 \circ I^*,$$

but here we also have redundancy because of the factorizations mentioned in the previous paragraph. By duality, it is not possible that $\mathscr{C}_1 = SM$ and $\mathscr{C}_2 = V_{\mathrm{c}}$.

Consider then the case that $\mathscr{C}_1 = SM$, $\mathscr{C}_2 = L_{\mathrm{c}}$, or that $\mathscr{C}_1 = L_{\mathrm{c}}$, $\mathscr{C}_2 = SM$ (note that $SM \circ L_{\mathrm{c}} = L_{\mathrm{c}} \circ SM = S_{\mathrm{c}}$). Because $S_{\mathrm{c}} \circ I = \Omega$ is a minimal factorization of Ω into two factors, we could have the factorizations:

$$\Omega = SM \circ L_{\mathrm{c}} \circ I_0 \circ I_1 = SM \circ L_{\mathrm{c}} \circ I_1 \circ I_0 = L_{\mathrm{c}} \circ SM \circ I_0 \circ I_1 = L_{\mathrm{c}} \circ SM \circ I_1 \circ I_0.$$

But these do not satisfy Condition 2, because we also have that $\Omega = SM \circ I^* \circ I_0 = L_{\mathrm{c}} \circ \Lambda_{\mathrm{c}} \circ I_0 \circ I_1$, and $I^* \circ I_0 = \Omega(1) \subset L = L_{\mathrm{c}} \circ I_0 \circ I_1 = L_{\mathrm{c}} \circ I_1 \circ I_0$ and $\Lambda_{\mathrm{c}} \circ I_0 \circ I_1 = \Lambda \subset M = SM \circ I_0 \circ I_1 = SM \circ I_1 \circ I_0$. If $\mathscr{C}_3 = \Lambda_{\mathrm{c}}$ or $\mathscr{C}_3 = V_{\mathrm{c}}$, then Condition 2 is not satisfied, because

$$SM \circ L_{\mathrm{c}} \circ \Lambda_{\mathrm{c}} = SM \circ L_{\mathrm{c}} \circ V_{\mathrm{c}} = L_{\mathrm{c}} \circ SM \circ \Lambda_{\mathrm{c}} = L_{\mathrm{c}} \circ SM \circ V_{\mathrm{c}} = T_{\mathrm{c}} = L_{\mathrm{c}} \circ \Lambda_{\mathrm{c}} = L_{\mathrm{c}} \circ V_{\mathrm{c}}.$$

If $\mathscr{C}_3 = I^*$, then Condition 2 is not satisfied, because

$$SM \circ L_{\mathrm{c}} \circ I^* = L_{\mathrm{c}} \circ SM \circ I^* = S = SM \circ I^*.$$

Thus, we only have the eight factorizations:

$$\begin{aligned}\Omega &= \Lambda_{\mathrm{c}} \circ V_{\mathrm{c}} \circ I^* = V_{\mathrm{c}} \circ \Lambda_{\mathrm{c}} \circ I^* = L_{\mathrm{c}} \circ \Lambda_{\mathrm{c}} \circ I_0 \circ I_1 = L_{\mathrm{c}} \circ \Lambda_{\mathrm{c}} \circ I_1 \circ I_0\\ &= L_{\mathrm{c}} \circ V_{\mathrm{c}} \circ I_0 \circ I_1 = L_{\mathrm{c}} \circ V_{\mathrm{c}} \circ I_1 \circ I_0 = SM \circ I^* \circ I_0 = SM \circ I^* \circ I_1\end{aligned}$$

that clearly satisfies Condition 1. It is straightforward to verify that these do not violate Condition 2. □

Replacing the sequence of unary clones by their composition in the eight descending irredundant factorizations of Theorem 4, we get the following five factorizations of Ω:

$$\Omega = V_{\mathrm{c}} \circ \Lambda_{\mathrm{c}} \circ I^*, \qquad (1)$$

$$\Omega = \Lambda_{\mathrm{c}} \circ V_{\mathrm{c}} \circ I^*, \qquad (2)$$

$$\Omega = L_{\mathrm{c}} \circ \Lambda_{\mathrm{c}} \circ I, \qquad (3)$$

$$\Omega = L_{\mathrm{c}} \circ V_{\mathrm{c}} \circ I, \qquad (4)$$

$$\Omega = SM \circ \Omega(1). \qquad (5)$$

Factorizations (1) and (2) express the representability of every Boolean function in DNF and CNF, respectively. Factorization (3) expresses the existence of the Zhegalkin polynomial representation for every function. We shall see that the other two factorizations express representability of functions in other normal forms.

The factorization $\Omega = L_{\mathrm{c}} \circ V_{\mathrm{c}} \circ I$ relates to $\Omega = L_{\mathrm{c}} \circ \Lambda_{\mathrm{c}} \circ I$ in the same way as CNF relates to DNF. Essentially the same as Factorization (4) is expressed by $\Omega = L \circ V_{\mathrm{c}} = L_0 \circ V_1$. These latter factorizations express the fact that every function can be represented as a sum of terms, where each term is a disjunction or **1**. It is not difficult to prove that this representation is unique up to permutation and repetition of terms and permutation and repetition of variables within terms.

The factorization $\Omega = SM \circ \Omega(1)$ expresses the fact that every function can be expressed by repeated applications of the ternary majority function μ to variables, negated variables, and constants. (For early research on the role of ternary majority (median) and a related ternary rejection in Boolean algebra, see [2,8,9,17,20].)

To make a formal comparison between the various expressions of functions corresponding to these factorizations, we need the following definitions.

Let $\Omega = \mathscr{C}_1 \cdots \mathscr{C}_{k-1}\mathscr{C}_k$ be a factorization into clones. Let $\mathscr{C}_k$ contain only variables, negated variables, or constant functions. Let $\mathscr{C}_1, \ldots, \mathscr{C}_{k-1}$ be generated by single functions $\gamma_1, \ldots, \gamma_{k-1}$, where the γ_i's are pairwise distinct and none of them is in $\mathscr{C}_k$. The pair of sequences $\mathscr{C}_1, \ldots, \mathscr{C}_k$ and $\gamma_1, \ldots, \gamma_{k-1}$ is called a *normal form system*. In this case, $\mathscr{C}_1^{\mathrm{d}}, \ldots, \mathscr{C}_k^{\mathrm{d}}$ and $\gamma_1^{\mathrm{d}}, \ldots, \gamma_{k-1}^{\mathrm{d}}$ also constitute a normal form system called the *dual system* and the factorization

$\Omega = \mathscr{C}_1^{\mathrm{d}} \cdots \mathscr{C}_k^{\mathrm{d}}$ is called the *dual factorization*. Note that (1) and (3) are dual to (2) and (4), respectively, and that (5) is self-dual. An *n-ary formula* corresponding to the system $\mathscr{C}_1, \ldots, \mathscr{C}_k$ and $\gamma_1, \ldots, \gamma_{k-1}$ is defined as a string over $\mathscr{C}_k^{(n)} \cup \{\gamma_1, \ldots, \gamma_{k-1}\}$, where $\mathscr{C}_k^{(n)}$ denotes the set of the n-ary functions of $\mathscr{C}_k$, by the following recursion:

(1) The elements in $\mathscr{C}_k^{(n)}$ are n-ary formulas.
(2) If γ_i is m-ary and $a_1, \ldots, a_m$ are n-ary formulas and none of the a_i's starts with γ_j with $i > j$, then $\gamma_i a_1 \cdots a_m$ is an n-ary formula.

By a *formula* corresponding to the system we mean an n-ary formula for some n, and the length of a formula f as a string of symbols is denoted by $|f|$. If the generators $\gamma_1, \ldots, \gamma_{k-1}$ are clear from the context, then we only refer to the factorization $\mathscr{C}_1 \cdots \mathscr{C}_k$ and to formulas corresponding to $\mathscr{C}_1 \cdots \mathscr{C}_k$.

Clearly every n-ary formula represents an n-ary function, and every n-ary function is represented by an n-ary formula.

For illustration, we consider representations of the n-ary function $\overline{x_i}$ by formulas of different lengths corresponding to various factorizations $\Omega = \mathscr{C}_1 \cdots \mathscr{C}_k$. If $\mathscr{C}_k$ contains negated variables, as in Factorizations (1), (2), (5), then the function $\overline{x_i}$ can be represented by the formula $\overline{x_i}$ of length 1. On the other hand, if one of the clones $\mathscr{C}_1, \ldots, \mathscr{C}_{k-1}$ is generated by a negated variable, i.e., it is the clone I^*, then the function $\overline{x_i}$ can be represented by the formula $\neg x_i$ of length 2, where $\neg$ denotes the unary function $0 \mapsto 1$, $1 \mapsto 0$. Finally, $\overline{x_i}$ can be represented by the formula $\tau\mathbf{01}x_i$ of length 4 corresponding to Factorizations (3) and (4) (τ generates L_{c} and $\mathbf{0}, \mathbf{1}, x_i \in I$).

Factorizations (1)–(5) together with the generators $\vee, \wedge, \tau, \mu$ for the clones $V_{\mathrm{c}}, \Lambda_{\mathrm{c}}, L_{\mathrm{c}}, SM$ will be called *disjunctive*, *conjunctive*, *polynomial*, *dual polynomial*, and *median normal form systems*, denoted $\mathbf{D}$, $\mathbf{C}$, $\mathbf{P}$, $\mathbf{P}^{\mathrm{d}}$, $\mathbf{M}$, respectively, and the corresponding formulas will be called *disjunctive*, *conjunctive*, *polynomial*, *dual polynomial*, and *median formulas.* The term median is motivated by the fact that $(\mathbb{B}, \mu)$ is the only possible median algebra on the two-element set $\mathbb{B}$, see, e.g., [1,10,19].

Representation of functions by disjunctive and conjunctive formulas are just variants of the well-known DNF and CNF representations. Polynomial formulas are just variants of Zhegalkin polynomial representations. Representation by dual polynomial formulas relates to polynomial formulas just as CNF relates to DNF. In the remainder of this paper, we will compare the efficiency of these five formula representations, and we will show that median formulas are in some sense more efficient than the others.

For a normal form system A, denote by F_A the set of formulas corresponding to A. For a function $f \in \Omega$, we define the *A-complexity* of f, denoted $C_A(f)$, as

$$\min\{|\Phi| : \Phi \in F_A,\ \ \Phi \text{ represents } f\}.$$

For normal form systems A and B, we say that A is *polynomially as efficient as* B, denoted $A \preccurlyeq B$, if there is a polynomial p with integer coefficients such that $C_A(f) \leqslant p(C_B(f))$ for all $f \in \Omega$. Indeed, the relation "polynomially as efficient as" is a preorder on any set of normal form systems. In fact (Theorem 5, below) it is also anti-symmetric on $\{\mathbf{D}, \mathbf{C}, \mathbf{P}, \mathbf{P}^{\mathrm{d}}, \mathbf{M}\}$, and thus it is a partial order on that set. If neither $A \preccurlyeq B$ nor $B \preccurlyeq A$ holds, we say that A and B are *uncomparable* or, to be more descriptive, that A and B *provide representations of uncomparable complexity.* In the case of $A \preccurlyeq B$ but $B \not\preccurlyeq A$, we say that A is *polynomially more efficient than* B, or that A *provides a representation of lower complexity than* B.

Theorem 5. *The disjunctive*, *conjunctive*, *polynomial*, *and dual polynomial normal form systems provide representations of pairwise uncomparable complexity*. *The median normal form system* $\mathbf{M}$ *provides representations of lower complexity than the other four normal form systems* $\mathbf{D}$, $\mathbf{C}$, $\mathbf{P}$, $\mathbf{P}^{\mathrm{d}}$.

Proof. We shall make use of the basic theory of disjunctive and conjunctive normal forms and implicants and implicata.

To prove that $\mathbf{D} \not\preccurlyeq \mathbf{C}$, let n be an even positive integer and let f_n be the n-ary Boolean function

$$(x_1 \vee x_2) \wedge \cdots \wedge (x_{2i-1} \vee x_{2i}) \wedge \cdots \wedge (x_{n-1} \vee x_n).$$

The $\mathbf{C}$-complexity of f_n is less than $3n$. But f_n has $2^{n/2}$ prime implicants and each has to appear as a separate term in any DNF. Since each implicant has $n/2$ variables, $C_{\mathbf{D}}(f_n) \geqslant (n/2)2^{n/2}$. Therefore, there can be no polynomial p such that $C_{\mathbf{D}}(f_n) \leqslant p(C_{\mathbf{C}}(f_n))$ would hold for all f_n's. The proof of $\mathbf{C} \not\preccurlyeq \mathbf{D}$ is similar but based on the dual family of functions

$$(x_1 \wedge x_2) \vee \cdots \vee (x_{2i-1} \wedge x_{2i}) \vee \cdots \vee (x_{n-1} \wedge x_n).$$

Next we show that $\mathbf{D} \not\preccurlyeq \mathbf{P}, \mathbf{P}^{\mathrm{d}}$ and $\mathbf{C} \not\preccurlyeq \mathbf{P}, \mathbf{P}^{\mathrm{d}}$. For each odd $n \geqslant 1$, consider the n-ary function

$$f_n = x_1 + \cdots + x_n.$$

Both the $\mathbf{P}$- and $\mathbf{P}^{\mathrm{d}}$-complexities of f_n are less than $2n$. However, f_n has 2^{n-1} prime implicants, each of which is a product of n variables or negated variables. Therefore, $C_{\mathbf{D}}(f_n) \geqslant n2^{n-1}$. This shows that $\mathbf{D} \not\preccurlyeq \mathbf{P}, \mathbf{P}^{\mathrm{d}}$. Similarly, $C_{\mathbf{C}}(f_n) \geqslant n2^{n-1}$, which shows that $\mathbf{C} \not\preccurlyeq \mathbf{P}, \mathbf{P}^{\mathrm{d}}$.

To see that $\mathbf{P} \not\preccurlyeq \mathbf{D}, \mathbf{C}, \mathbf{P}^{\mathrm{d}}$, consider for each $n \geqslant 2$ the n-ary function

$$f_n = x_1 \vee \cdots \vee x_n.$$

The $\mathbf{D}$-, $\mathbf{C}$- and $\mathbf{P}^{\mathrm{d}}$-complexities of f_n are less than $2n$. However, the (unique) Zhegalkin polynomial of f_n is the sum of all $2^n - 1$ non-constant monomials in n indeterminates. This implies that the $\mathbf{P}$-complexity of f_n is at least $2^n - 1$. The proof of $\mathbf{P}^{\mathrm{d}} \not\preccurlyeq \mathbf{D}, \mathbf{C}, \mathbf{P}$ is similarly based on the dual family of functions $f_n = x_1 \wedge \cdots \wedge x_n$.

Next we show that $\mathbf{D}, \mathbf{C}, \mathbf{P}, \mathbf{P}^{\mathrm{d}} \not\preccurlyeq \mathbf{M}$. For each $k \geqslant 1$, let $n = 2k+1$, and consider the n-ary self-dual monotone function f_k defined inductively as follows:

$$\begin{aligned} f_1 &= \mu(x_1, x_2, x_3) \\ &= (x_1 \wedge x_2) \vee (x_2 \wedge x_3) \vee (x_3 \wedge x_1) = (x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_1) \\ &= (x_1 \wedge x_2) + (x_2 \wedge x_3) + (x_3 \wedge x_1) = (x_1 \vee x_2) + (x_2 \vee x_3) + (x_3 \vee x_1), \end{aligned}$$

$$f_{k+1} = \mu(f_{k'}, x_{2k+1}, x_{2k+3}),$$

where $f_{k'}$ denotes the $(2k+3)$-ary cylindrification of f_k. The $\mathbf{M}$-complexity of f_k is at most $3k+1$. By induction we see that the number of prime implicants is $2^{k+1} - 1$, which is also the number of prime implicata. Thus, the $\mathbf{D}$- and $\mathbf{C}$-complexities of f_n are at least $2^{k+1} - 1$, from which it follows that $\mathbf{D}, \mathbf{C} \not\preccurlyeq \mathbf{M}$. Also by induction, the Zhegalkin polynomial of f_k has $2^{k+1} - 1$ terms and thus $C_{\mathbf{P}}(f_k) \geqslant 2^{k+1} - 1$, and hence $\mathbf{P} \not\preccurlyeq \mathbf{M}$. Similarly, we can see that $C_{\mathbf{P}^{\mathrm{d}}}(f_k) \geqslant 2^{k+1} - 1$, and hence $\mathbf{P}^{\mathrm{d}} \not\preccurlyeq \mathbf{M}$.

To see that $\mathbf{M} \preccurlyeq \mathbf{D}$, let us define the map $T : F_{\mathbf{D}} \to F_{\mathbf{M}}$ recursively as follows:

(1) $T(a) = a$, whenever a is a variable or a negated variable,
(2) $T(\wedge\Phi\Psi) = \mu T(\Phi) T(\Psi) \mathbf{0}$, for $\Phi, \Psi, \wedge\Phi\Psi \in F_{\mathbf{D}}$, and $T(\vee\Phi\Psi) = \mu T(\Phi) T(\Psi) \mathbf{1}$, for $\Phi, \Psi, \vee\Phi\Psi \in F_{\mathbf{D}}$.

Clearly, for every $\Phi \in F_{\mathbf{D}}$, $T(\Phi) \in F_{\mathbf{M}}$ represents the same function as Φ, and by induction it follows that $|T(\Phi)| < 2|\Phi|$. This shows that $\mathbf{M} \preccurlyeq \mathbf{D}$. Similarly, it follows that $\mathbf{M} \preccurlyeq \mathbf{C}$.

Finally, we show that $\mathbf{M} \preccurlyeq \mathbf{P}, \mathbf{P}^{\mathrm{d}}$. For $n \geqslant 0$, consider the functions

$$f_n = x_1 + \cdots + x_{3^n}.$$

We define the median formulas Φ_n for $n \geqslant 0$ recursively as follows: $\Phi_0 = x_1$, and for $n \geqslant 0$, we obtain Φ_{n+1} by substituting

$$\mu\mu\overline{x_{3i-2}}x_{3i-1}x_{3i}\,\mu x_{3i-2}\overline{x_{3i-1}}x_{3i}\,\mu x_{3i-2}x_{3i-1}\overline{x_{3i}}$$

and

$$\mu\mu x_{3i-2}\overline{x_{3i-1}}\,\overline{x_{3i}}\,\mu\overline{x_{3i-2}}x_{3i-1}\overline{x_{3i}}\,\mu\overline{x_{3i-2}}\,\overline{x_{3i-1}}x_{3i}$$

for each occurrence of x_i and $\overline{x_i}$ in Φ_n, respectively. It is easy to prove by induction that Φ_n represents f_n, because $\overline{\mu(x, y, z)} = \mu(\overline{x}, \overline{y}, \overline{z})$ and

$$\tau(x, y, z) = \mu(\mu(\overline{x}, y, z), \mu(x, \overline{y}, z), \mu(x, y, \overline{z})).$$

Denote by $|\Phi|_a$ the number of occurrences of the symbol a in the formula Φ, and for any variable x_i, denote $|\Phi|_{\widetilde{x_i}} = |\Phi|_{x_i} + |\Phi|_{\overline{x_i}}$. We prove by induction on n that $|\Phi_n|_{\widetilde{x_i}} = 3^n$ for $i = 1, \ldots, 3^n$ and $|\Phi_n|_\mu = (3^{2n} - 1)/2$. It is clear that the claim holds for Φ_0. Assume that it holds for Φ_n, and then consider Φ_{n+1}. By the inductive hypothesis, $|\Phi_n|_{\widetilde{x_i}} = 3^n$ for $i = 1, \dots, 3^n$, and so $|\Phi_{n+1}|_{\widetilde{x_i}} = 3^{n+1}$ for $i = 1, \dots, 3^{n+1}$ and $|\Phi_{n+1}|_\mu = |\Phi_n|_\mu + 4 \cdot 3^{2n} = (3^{2n+2} - 1)/2$. We also conclude that $|\Phi_n| = (3^{2n+1} - 1)/2$.

Let $f \in \Omega$ and let Θ be the shortest polynomial formula representing f. Then f is a sum of $m = 1 + 2|\Theta|_\tau$ monomial terms. For $i = 1, \dots, m$, let Θ_i be the polynomial formula (occurring as a substring of Θ) that represents the ith monomial term. Since the monomial terms are conjunctions of variables, each Θ_i is in fact a disjunctive formula, and by the previous proof that $\mathbf{M} \preccurlyeq \mathbf{D}$, there is a median formula Ξ_i representing the ith monomial term such that $|\Xi_i| < 2|\Theta_i|$.

For any median formula Γ, denote by Γ' the median formula obtained from Γ by substituting $\overline{x_i}$ for x_i, x_i for $\overline{x_i}$, $\mathbf{0}$ for $\mathbf{1}$, and $\mathbf{1}$ for $\mathbf{0}$. It is clear that $|\Gamma| = |\Gamma'|$, and Γ' represents the negation of the function represented by Γ. Hence, the $\mathbf{M}$-complexity of any function is equal to the $\mathbf{M}$-complexity of its negation.

Let n be the smallest integer such that $m \leqslant 3^n$. For $i = m + 1, \dots, 3^n$, let $\Xi_i = \Theta_i = \mathbf{0}$. Now, we can construct a median formula Ξ that represents f by substituting Ξ_i for x_i and Ξ_i' for $\overline{x_i}$ in Φ_n for $i = 1, \dots, 3^n$. We have that

$$\begin{aligned} C_{\mathbf{M}}(f) &\leqslant |\Xi| = |\Phi_n|_\mu + \sum_{i=1}^{3^n} |\Phi_n|_{\widetilde{x_i}} |\Xi_i| \\ &< \frac{3^{2n}-1}{2} + \sum_{i=1}^{3^n} 3^n \cdot 2|\Theta_i| = \frac{3^{2n}-1}{2} + 2 \cdot 3^n \sum_{i=1}^{m} |\Theta_i| + 2 \cdot 3^n \sum_{i=m+1}^{3^n} |\mathbf{0}| \\ &\leqslant \frac{3^{2n}-1}{2} + 2 \cdot 3^n \sum_{i=1}^{m} |\Theta_i| + 2 \cdot 3^n \cdot 3^n = \frac{5 \cdot 3^{2n} - 1}{2} + 2 \cdot 3^n \sum_{i=1}^{m} |\Theta_i|. \end{aligned}$$

If $|\Theta|_\tau \neq 0$, then using

$$C_{\mathbf{P}}(f) = |\Theta| = |\Theta|_\tau + \sum_{i=1}^{m} |\Theta_i|$$

and $|\Theta|_\tau \geqslant 3^{n-1}/2$, we have that

$$\begin{aligned} (C_{\mathbf{P}}(f))^2 &= (|\Theta|_\tau)^2 + 2|\Theta|_\tau \sum_{i=1}^{m} |\Theta_i| + \left(\sum_{i=1}^{m} |\Theta_i| \right)^2 \geqslant (|\Theta|_\tau)^2 + 2|\Theta|_\tau \sum_{i=1}^{m} |\Theta_i| \\ &\geqslant \frac{3^{2n-2}}{4} + 3^{n-1} \sum_{i=1}^{m} |\Theta_i| = \frac{5 \cdot 3^{2n}}{180} + \frac{2 \cdot 3^n}{6} \sum_{i=1}^{m} |\Theta_i|, \end{aligned}$$

and so

$$C_{\mathbf{M}}(f) < \frac{5 \cdot 3^{2n}}{2} + 2 \cdot 3^n \sum_{i=1}^{m} |\Theta_i| \leqslant 180(C_{\mathbf{P}}(f))^2.$$

Also, if $|\Theta|_\tau = 0$, then clearly $C_{\mathbf{M}}(f) \leqslant 180(C_{\mathbf{P}}(f))^2$. We conclude that $\mathbf{M} \preccurlyeq \mathbf{P}$. A similar argument proves that $\mathbf{M} \preccurlyeq \mathbf{P}^{\mathrm{d}}$. □

The above proof of Theorem 5 actually provides algorithms for converting DNF, CNF, and Zhegalkin (Reed–Muller) representations into a generally more efficient median normal form.

Appendix A. Post classes

We make use of notations and terminology appearing in [7] and in [11].

- Ω denotes the clone of all Boolean functions;
- T_0 and T_1 denote the clones of 0- and 1-preserving functions, respectively, i.e., $T_0 = \{f \in \Omega : f(0, \dots, 0) = 0\}$, $T_1 = \{f \in \Omega : f(1, \dots, 1) = 1\}$;

- T_c denotes the clone of constant-preserving functions, i.e., $T_c = T_0 \cap T_1$.
- M denotes the clone of all monotone functions, i.e., $M = \{f \in \Omega : f(\mathbf{a}) \leqslant f(\mathbf{b})$, whenever $\mathbf{a} \preccurlyeq \mathbf{b}\}$;
- $M_0 = M \cap T_0$, $M_1 = M \cap T_1$, $M_c = M \cap T_c$;
- S denotes the clone of all self-dual functions, i.e., $S = \{f \in \Omega : f^d = f\}$;
- $S_c = S \cap T_c$, $SM = S \cap M$;
- L denotes the clone of all linear functions, i.e., $L = \{f \in \Omega : f = c_0\mathbf{1} + c_1x_1 + \cdots + c_nx_n$ for some n and $c_0, \ldots, c_n \in \mathbb{B}\}$;
- $L_0 = L \cap T_0$, $L_1 = L \cap T_1$, $LS = L \cap S$, $L_c = L \cap T_c$.

Let $a \in \{0, 1\}$. A set $A \subseteq \{0, 1\}^n$ is said to be *a-separating* if there is i, $1 \leqslant i \leqslant n$, such that for every $(a_1, \ldots, a_n) \in A$ we have $a_i = a$. A function f is said to be *a-separating* if $f^{-1}(a)$ is a-separating. A function f is said to be *a-separating of rank* $k \geqslant 2$ if every subset $A \subseteq f^{-1}(a)$ of size at most k is a-separating.

- For $m \geqslant 2$, U_m and W_m denote the clones of all 1- and 0-separating functions of rank m, respectively;
- U_∞ and W_∞ denote the clones of all 1- and 0-separating functions, respectively, i.e., $U_\infty = \bigcap_{k \geqslant 2} U_k$ and $W_\infty = \bigcap_{k \geqslant 2} W_k$;
- $T_cU_m = T_c \cap U_m$ and $T_cW_m = T_c \cap W_m$, for $m = 2, \ldots, \infty$;
- $MU_m = M \cap U_m$ and $MW_m = M \cap W_m$, for $m = 2, \ldots, \infty$;
- $M_cU_m = M_c \cap U_m$ and $M_cW_m = M_c \cap W_m$, for $m = 2, \ldots, \infty$;
- Λ denotes the clone of all conjunctions and constants, i.e., $\Lambda = \{f \in \Omega : f = \mathbf{0}, \mathbf{1}, x_{i_1} \wedge \cdots \wedge x_{i_n}$ for some $n \geqslant 1$ and i_j's$\}$;
- $\Lambda_0 = \Lambda \cap T_0$, $\Lambda_1 = \Lambda \cap T_1$, $\Lambda_c = \Lambda \cap T_c$;
- V denotes the clone of all disjunctions and constants, i.e., $V = \{f \in \Omega : f = \mathbf{0}, \mathbf{1}, x_{i_1} \vee \ldots \vee x_{i_n}$ for some $n \geqslant 1$ and i_j's$\}$;
- $V_0 = V \cap T_0$, $V_1 = V \cap T_1$, $V_c = V \cap T_c$;
- $\Omega(1)$ denotes the clone of all variables, negated variables, and constants;
- I^* denotes the clone of all variables and negated variables;
- I denotes the clone of all variables and constants;
- $I_0 = I \cap T_0$, $I_1 = I \cap T_1$;
- I_c denotes the smallest clone containing only variables, i.e., $I_c = I \cap T_c$.

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