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The algebra of majority consensus

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Abstract. In this note we show that a median algebra can be defined in terms of a single n-ary operation for any $n \geq 5$, so that besides symmetry and a majority condition only one additional identity is required. This provides a short axiomatic characterization of majority consensus for taxonomic structures.

For a number of taxonomic models the majority rule constitutes a feasible method in order to obtain a consensus for partially conflicting models on a given set of taxonomic units. This includes taxonomic models such as "n-trees" (labelled rooted trees), "phylogenetic trees" (labelled unrooted trees), weak orders, and the like. In all these situations a taxonomic model can be regarded as a system of certain subsets, partitions, or pairs of taxonomic units, respectively. If $X_1, \ldots, X_n$ is a profile of such models, then the "majority term"

$$\bigvee_{\substack{I \subseteq \{1, \ldots, n\} \\ \frac{n}{2} < |I| \leq \frac{n}{2}+1}} \bigwedge_{i \in I} X_i$$

is the model proposed by majority consensus – provided that the set M of all models under consideration can be organized as a partial lattice in which all "majority terms" exist. Existence is guaranteed, for instance, when M is a median semilattice – as is the case for the above mentioned types of taxonomic models; cf. Barthélemy, Leclerc and Monjardet (1986). The majority rule is closely related to the median procedure in median semilattices, e.g., majority consensus and median consensus coincide for every profile of odd size. Barthélemy and Janowitz (1991) have axiomatized the median procedure in median semilattices (see Barthélemy and McMorris (1986) for the particular case of n-trees).

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In this paper we wish to axiomatize the majority consensus for profiles of any fixed size $n \geq 5$ by purely algebraic identities, so that we are in the realm of universal algebra. In particular, we are interested in the question of when the system of taxonomic models under consideration receives a median semilattice structure. One should keep in mind that every median semilattice gives rise to a ternary algebra, called a median algebra, where the fundamental operation applied to elements x, y, z can be interpreted as the median consensus or majority consensus for the profile x, y, z. A *median algebra* can be defined (without reference to median semilattices) as a ternary algebra satisfying the following three identities (see Bandelt and Hedlíková (1983) for a survey on median algebras):

$$(uvw) = (vuw) = (wvu),$$

$$(uuv) = u,$$

$$(vw(xyz)) = (x(vwy)(vwz)),$$

where the ternary operation is simply written as $x, y, z \mapsto (xyz)$. This bracket notation will also be used for the n-ary operations f studied in the sequel. So, $f(x_1, \ldots, x_n)$ is expressed as $(x_1 \cdots x_n)$, and if some of the entries are identical, say $x_1 = \cdots = x_k = x$ $(k \geq 0)$, then we write $(x^k x_{k+1} \cdots x_n)$ instead of $(x_1 \cdots x_n)$.

Here is our main result.

THEOREM 1. *Let $x_1, \ldots, x_n \mapsto (x_1 \cdots x_n)$ be a symmetric n-ary operation $(n \geq 5)$ on a set M such that the following two identities hold for some integer s with $\frac{1}{2}n < s \leq \frac{2}{3}n$:*

$$(w_1 \cdots w_{n-s} x^s) = x, \tag{1}$$

$$(w_1 \cdots w_{n-1}(x_1 \cdots x_n)) = ((w_1 \cdots w_{n-1} x_1) \cdots (w_1 \cdots w_{n-1} x_s) x_{s+1} \cdots x_n) \tag{2}$$

for all $w_i, x_i, x \in M$. Then M is a median algebra with respect to the ternary operation $x, y, z \mapsto (xyz)$ defined by

$$(xyz) := (x^{n-s} y^{n-s} z^{2s-n}).$$

Proof. To establish symmetry of the ternary operation $x, y, z \mapsto (xyz)$ requires a little effort. As a prerequisite we need an auxiliary identity:

$$(w^h x^m y^{n-m-h}) = (w^{h-1} x^{m+1} y^{n-m-h}) \qquad \text{for } w, x, y \in M, \tag{3}$$

where $0 < h \le n - s$, $0 \le m < n - s$ and $n - s \le h + m \le s$. We prove this assertion by induction on

$$i = s - n + m + h.$$

Note that $i < h$ and $i \le m$. For $i = 0$ we get $n - m - h = s$, so that either side of (3) equals y by axiom (1). So let $i \ge 1$, and assume that (3) holds whenever y occurs exactly $s - i + 1$ times. Then any feasible number of w's and x's yields the same expression, that is:

$$(w^{n-k-s+i-1}x^k y^{s-i+1}) =: v \qquad \text{for all } k \text{ with } i - 1 \le k \le n - s. \tag{4}$$

In what follows we shall use symmetry of the n-ary operation without explicit mention.

Since $s \le \frac{2}{3}n$, we immediately get from axiom (1) that

$$(x^{2n-2s}y^{2s-n}) = x.$$

Using this equality and axiom (2) we compute

$$\begin{aligned}(w^h x^m y^{s-i}) &= (w^h x^{m-1}(x^{n-s}x^{n-s}y^{2s-n})y^{s-i})\\ &= (x^{n-s}(w^h x^m y^{s-i})^{n-s}(w^h x^{m-1}y^{s-i+1})^{2s-n}).\end{aligned}$$

Then applying (4) for $k = m - 1$ yields

$$(w^h x^m y^{s-i}) = (x^{n-s}(w^h x^m y^{s-i})^{n-s}v^{2s-n}). \tag{5}$$

In a similar fashion we compute, for $h \ge 2$,

$$\begin{aligned}(w^{h-1}x^{m+1}y^{s-i}) &= (w^{h-2}(w^{n-s}w^{n-s}y^{2s-n})x^{m+1}y^{s-i})\\ &= (w^{n-s}(w^{h-1}x^{m+1}y^{s-i})^{n-s}(w^{h-2}x^{m+1}y^{s-i+1})^{2s-n}).\end{aligned}$$

Now using (4) with $k = m + 1$ gives

$$(w^{h-1}x^{m+1}y^{s-i}) = (w^{n-s}(w^{h-1}x^{m+1}y^{s-i})^{n-s}v^{2s-n}). \tag{6}$$

Now, apply first (5) and (4) for $k = m$, then axiom (2) in two different ways, again (4) for $k = m$, and finally (6), viz.:

$$\begin{aligned}(w^h x^m y^{s-i}) &= (x^{n-s}(w^h x^m y^{s-i})^{n-s} v^{2s-n})\\ &= (x^{n-s}(w^h x^m y^{s-i})^{n-s}(w^{h-1}x^m y^{s-i+1})^{2s-n})\\ &= (w^{h-1}(w^{n-s}x^{n-s}y^{2s-n})x^m y^{s-i})\\ &= (w^{n-s}(w^{h-1}x^{m+1}y^{s-i})^{n-s}(w^{h-1}x^m y^{s-i+1})^{2s-n})\\ &= (w^{n-s}(w^{h-1}x^{m+1}y^{s-i})^{n-s}v^{2s-n})\\ &= (w^{h-1}x^{m+1}y^{s-i}),\end{aligned}$$

as desired. This proves (3).

Letting $h = 2s - n$ and $m = n - s$, we derive from (3) that

$$(w^{2s-n}x^{n-s}y^{n-s}) = (w^{n-s}x^{2s-n}y^{n-s}).$$

Therefore the ternary operation is in fact symmetric. Further, the identity $(xyx) = x$ is an instance of axiom (1). Then we can conclude, according to a theorem of Kolibiar and Marcisová (1974), that M is a median algebra with respect to $x, y, z \mapsto (xyz)$ whenever

$$(wy(xyz)) = (xy(wyz)) \tag{7}$$

holds throughout. In order to show this, we claim that

$$(w^{n-s}x^m y^{s-m-j}(x^{2s-n}y^{n-s}z^{n-s})^j) = (w^m x^{n-s}y^{s-m-j}(w^{2s-n}y^{n-s}z^{n-s})^j)$$

$$\text{for all } j, m \text{ such that } 0 \le j \le 2s-n \text{ and } 0 \le m \le 2s-n-j. \tag{8}$$

We proceed by induction on j. First let $j = 0$. Then $n - s \le s - m$, so that (8) is simply a consequence of (3) for $h = n - s$. Now let $j \ge 1$. Then from axiom (2) and the induction hypothesis we deduce

$$\begin{aligned}&(w^{n-s}x^m y^{s-m-j}(x^{2s-n}y^{n-s}z^{n-s})^j)\\ &= (w^{n-s}x^m y^{s-m-j}(x^{2s-n}y^{n-s}z^{n-s})^{j-1}(x^{2s-n}y^{n-s}z^{n-s}))\\ &= ((w^{n-s}x^{m+1}y^{s-m-j}(x^{2s-n}y^{n-s}z^{n-s})^{j-1})^{2s-n}\\ &\quad (w^{n-s}x^m y^{s-m-j+1}(x^{2s-n}y^{n-s}z^{n-s})^{j-1})^{n-s}z^{n-s})\end{aligned}$$

$$= ((w^{m+1}x^{n-s}y^{s-m-j}(w^{2s-n}y^{n-s}z^{n-s})^{j-1})^{2s-n}$$

$$(w^m x^{n-s}y^{s-m-j+1}(w^{2s-n}y^{n-s}z^{n-s})^{j-1})^{n-s}z^{n-s})$$

$$= (w^m x^{n-s}y^{s-m-j}(w^{2s-n}y^{n-s}z^{n-s})^{j-1}(w^{2s-n}y^{n-s}z^{n-s}))$$

$$= (w^m x^{n-s}y^{s-m-j}(w^{2s-n}y^{n-s}z^{n-s})^{j}).$$

This concludes the induction step. Then, in particular, (8) holds for $j = 2s - n$ and $m = 0$, that is:

$$(w^{n-s}y^{n-s}(x^{2s-n}y^{n-s}z^{n-s})^{2s-n}) = (x^{n-s}y^{n-s}(w^{2s-n}y^{n-s}z^{n-s})^{2s-n}).$$

This is equivalent to the asserted equality (8) in view of symmetry. This finishes the proof of the theorem. □

In connection with Theorem 1 some questions come up naturally; e.g., what about subdirect representation? The next proposition settles this.

PROPOSITION. *Let M be an n-ary algebra satisfying the conditions of Theorem 1 for some s with $\frac{1}{2}n < s \le \frac{2}{3}n$. Then M is a subdirect product of copies of the two-element algebras $\underline{2} = \{0, 1\}$ with*

$$(0^{n-j}1^j) = \begin{cases} 0 & \text{if } j < r \\ 1 & \text{otherwise,} \end{cases}$$

for $\frac{1}{2}n < r \le s$.

Proof. From the definition of the ternary median and identity (2) in Theorem 1 we infer that

$$(w_1 \cdots w_{n-1}(xyz)) = (x(w_1 \cdots w_{n-1}y)(w_1 \cdots w_{n-1}z))$$

for all $w_i, x, y, z \in M$. Now, every median algebra is a subdirect power of the two-element median algebra $\underline{2}$. This is expressed by the fact that the median homomorphisms from M to $\underline{2}$ separate the points, that is, whenever $x \neq y$ in M there is a homomorphism $f: M \to \underline{2}$ with $fx \neq fy$. The kernel $\ker f$ of any such mapping f is also a congruence of the n-ary algebra because $fx = fy$ implies

$$\begin{aligned} f(x_1 \cdots x_{n-1}x) &= f(x_1 \cdots x_{n-1}(xxy)) \\ &= f(x(x_1 \cdots x_{n-1}x)(x_1 \cdots x_{n-1}y)) \\ &= (fx\, f(x_1 \cdots x_{n-1}x) f(x_1 \cdots x_{n-1}y)) \\ &= (fy\, f(x_1 \cdots x_{n-1}x) f(x_1 \cdots x_{n-1}y)) \\ &\;\vdots \\ &= f(x_1 \cdots x_{n-1}y) \end{aligned}$$

for all $x_i, x, y \in M$. Therefore the two-element set $\underline{2} = M/\ker f$ becomes an n-ary algebra so that f is also a homomorphism between the n-ary algebras. Now consider a specific n-ary operation on $\underline{2}$ satisfying our axioms. Without loss of generality we may assume that

$$(0^{n-k}1^k) = 0, \qquad \text{where } \frac{n-1}{2} \le k \le \frac{n}{2}.$$

Let r be the largest integer such that

$$(0^{n-r+1}1^{r-1}) = 0.$$

Certainly r exceeds $n/2$. Now, if $(0^{n-j}1^j) = 0$ for $j \ge 2$, then

$$\begin{aligned} (0^{n-j+1}1^{j-1}) &= (0^{n-j}1^{j-1}(010)) \\ &= (0(0^{n-j}1^j)(0^{n-j+1}1^{j-1})) \\ &= (00(0^{n-j+1}1^{j-1})) \\ &= 0. \end{aligned}$$

So, a trivial induction shows that $(0^{n-j}1^j)$ equals 0 if and only if $j < r$. Necessarily, $r \le s$. □

Using Birkhoff's (1979) terminology, the Proposition says that our n-ary algebras are dyadic. Further, they form – what Fried and Pixley (1979) called – a dual discriminator variety.

Theorem 1 states that the ternary median operation is a term function of the n-ary algebra, or in other words, it belongs to the clone of the latter. Conversely, does the given n-ary operation appear to be a term function of the median algebra? If so then both clones of operations would coincide (and hence the n-ary algebra and the median algebra were cryptoautomorphic sensu Birkhoff (1979)). The concluding theorem shows that this is exactly the case when n is odd and s equals $(n+1)/2$. Note that there can be at most one such majority function (for each odd number n) on a median algebra M (or any dyadic algebra) such that every homomorphism of the algebra into $\underline{2}$ is also one for this majority function (since this is trivially true for the algebra $\underline{2}$). In particular, if M is ordered as a median semilattice, then (according to Lemma 3.4 of Bandelt and Hedlíková (1983)) we obtain for $n = 2s - 1$,

$$(x_1 \cdots x_{2s-1}) = \bigvee_{\substack{I \subseteq \{1,\ldots,2s-1\} \\ |I| = s}} \bigwedge_{i \in I} x_i$$

for all $x_i \in M$. We will refer to this function as the *n-ary median operation*. The fact that this operation is a term function of the median algebra M follows already from results of Monjardet (1972, 1975).

THEOREM 2. *Let M be an n-ary algebra satisfying the conditions of Theorem 1. Assume that $r > \frac{1}{2}n$ is the smallest integer such that*

$$(x^r y^{n-r}) = x \qquad \textit{for all } x, y \in M.$$

Then the clones of M and the associated median algebra coincide if and only if $n = 2r - 1$. In this case the n-ary median is obtained from the $(n-2)$-ary and ternary medians via the recurrence

$$(x_1 \cdots x_{2r-1}) = (x_1 \cdots x_{r-2}(x_{r-1}x_r x_{r+1})(x_{r-1}x_r x_{r+1}x_{r+2}^2) \cdots (x_{r-1} \cdots x_{2r-2}x_{2r-1}^{r-1})), \qquad x_i \in M,$$

where

$$(w_1 \cdots w_{i+1} x^i) = ((\cdots((w_1 w_2 x) w_3 x) \cdots) w_{i+1} x), \qquad w_i, \quad x \in M.$$

Proof. First we assert that the algebra M satisfies the axioms (1) and (2) for $s = r$. By the Proposition, M is a subdirect product of two-element algebras. For each factor $\underline{2}$ we have

$$(0^{n-j}1^j) = 1 \quad \text{and} \quad (0^j 1^{n-j}) = 0 \quad \text{if } j \geq r.$$

Then, of course, (1) is true for any $s \geq r$. In order to establish (2), consider the mapping $x \mapsto (w_1 \cdots w_{n-1} x)$ on $\underline{2}$. Then, by virtue of the preceding Proposition and its proof, this mapping is either the identity map or one of the two constant maps. In the former case (2) holds trivially. In the latter case either side of (2) equals this constant because (1) is true for $s \geq r$.

Now, assume that $n < 2r - 1$. In order to show that the n-ary operation is not a term function of the median algebra, we will compare the binary term functions available in either algebra. Since every two-element subset is a median subalgebra, the two projections $x, y \mapsto x$ and $x, y \mapsto y$ are the only binary term functions of the median algebra. We claim that the n-ary algebra does admit binary term functions which are not just projections. Indeed, let

$$x + y := (x^{n-r+1} y^{r-1}) \qquad \text{for } x, y \in M.$$

By assumption, $n - r + 1 \leq r - 1$. Since r was chosen to be minimal, $x + y$ equals neither x nor y for all pairs x, y, that is, the operation $+$ is not a projection. Therefore the clone of the n-ary algebra properly contains the clone of the median algebra.

Finally assume that $n = 2r - 1$. Since we are in a dyadic algebra, we may assume that $M = \underline{2}$ in order to verify the recurrence. First note that $(w_1 \cdots w_{i+1} x^i)$ equals x (in $M = \underline{2}$) if and only if at least one of the w_i's equals x. The same is true for the dexter of the asserted equality in the Proposition. By induction we can assume that the $(n-2)$-ary majority term is available in the median algebra $\underline{2}$. So it remains to check the equality for $(x_1 \cdots x_{2r-1})$. Say, $(x_1 \cdots x_{2r-1}) = 0$. Then $x_i = 0$ for a majority of indices i. We proceed by induction on r. For $r = 2$ there is nothing to show. So let $r \geq 3$. We distinguish three cases.

Case 1. $x_1 = x_2 = \cdots = x_{r-2} = 0 = x_{2r-1}$.

Then $x_j = 0$ for at least one index j with $r - 1 \leq j \leq 2r - 2$. Hence

$$(x_{r-1} \cdots x_{2r-2} x_{2r-1}^{r-1}) = 0,$$

and consequently, the $(n-2)$-median must be 0.

Case 2. $x_1 = x_2 = \cdots = x_{r-2} = 1 = x_{2r-1}$.

Then the remaining entries $x_{r-1}, \ldots, x_{2r-2}$ equal 0, whence all $r - 1$ brackets within the $(n-2)$-median receive 0, thus yielding 0 altogether.

Case 3. At least one of the x_i's with $i \leq r-2$ is different from x_{2r-1}, say: $x_1 \neq x_{2r-1}$. Then a majority of $x_2, \ldots, x_{2r-2}$ equals 0. Therefore, by the induction hypothesis, also a majority of the terms $x_2, \ldots, x_{r-2}, (x_{r-1}x_rx_{r+1}), \ldots, (x_{r-1} \cdots x_{2r-3}x_{2r-2}^{r-2})$ equals 0. Assume that

$$(x_{r-1} \cdots x_{2r-2}x_{2r-1}^{r-1}) = 1,$$

for otherwise, the $(n-2)$-median would receive 0, as required. It is impossible that all r entries $x_{r-1}, \ldots, x_{2r-2}$ equal 1. Hence $x_{2r-1} = 1$ and $x_1 = 0$. This again gives a majority of zero entries in the $(n-2)$-median. This completes the proof of the theorem. □

Expressing the $(2r-1)$-ary median by the $(2r-3)$-ary median via the recurrence of Theorem 2 incorporates $\binom{r}{2}$ ternary medians. So, when intermediate values are stored, one can obtain any $(2r-1)$-ary median after $\binom{r+1}{3}$ computations of ternary medians. The actual length of the corresponding term in $2r-1$ variables, however, seems to increase exponentially. Iterating the recurrence, we get

$$(x_1 \cdots x_5) = (x_1(x_2x_3x_4)((x_2x_3x_5)x_4x_5)),$$

$$(x_1 \cdots x_7) = ((((x_3x_4x_7)x_5x_7)x_6x_7)(x_1x_2(x_3x_4x_5))$$
$$((x_1x_2((x_3x_4x_6)x_5x_6))(x_3x_4x_5)((x_3x_4x_6)x_5x_6))),$$

i.e., expressions of length 9 and 27, respectively. For $n = 9$ we have formulae of length 77. The formulae suggested by Birkhoff (1979) for $n = 5, 7, 9, 11$ are much longer (after going through the necessary iterations), viz.: for $n = 5, 7$ the corresponding lengths are 27 and $(27)^3 = 19.683$. Anyway, recursion formulae for the odd medians exist in abundance. The following one is just another sample (which the reader will find easy to verify):

$$(x_1 \cdots x_{2r-1}) = (((x_1x_{2r-2}x_{2r-3})x_2x_3 \cdots x_{2r-3})(x_1(x_2x_{2r-2}x_{2r-3})x_3 \cdots x_{2r-3})$$
$$\cdots (x_1 \cdots x_{2r-4}(x_{2r-3}x_{2r-2}x_{2r-1}))).$$

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