Multicriteria Decision Analysis
MCDA
Lecture 4 – Kriging

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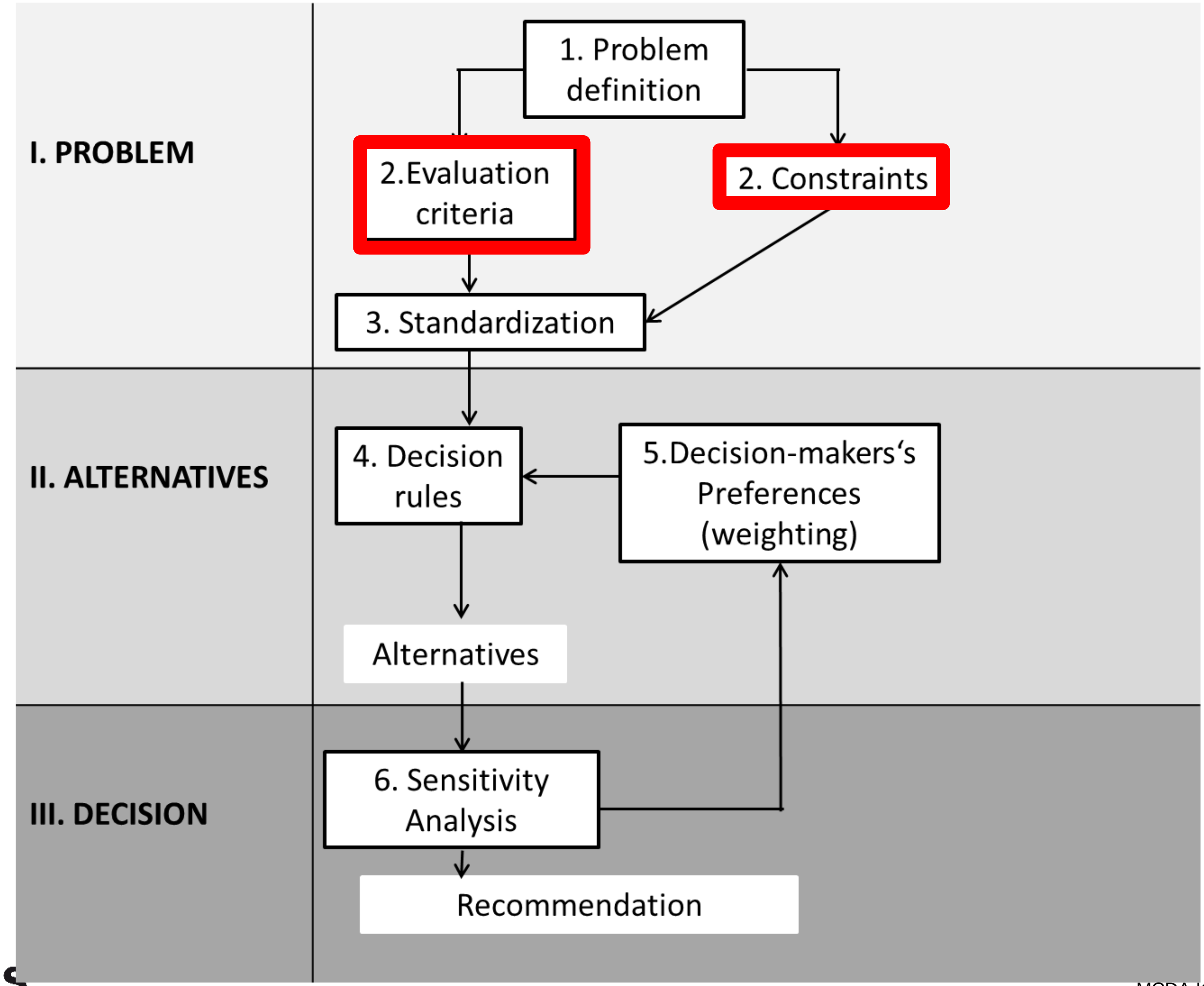
Process in Lecture

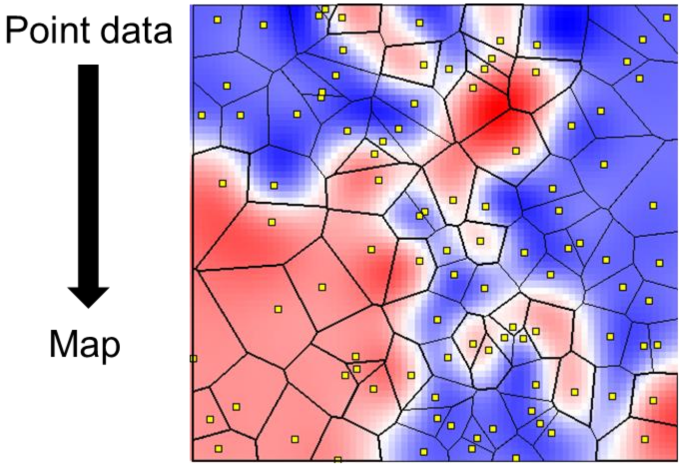


R, ArcGIS

Own Example

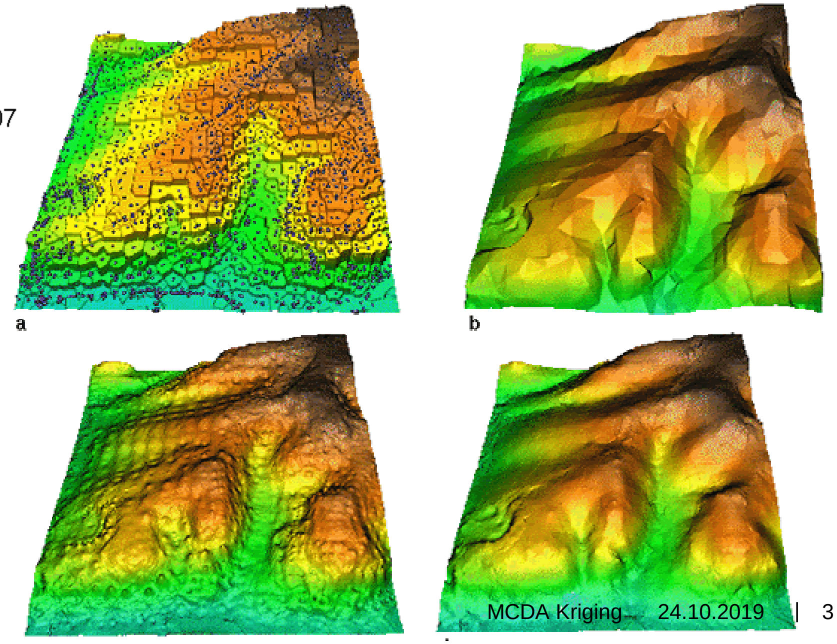
Problem

Lead question: how can we interpolate data points?



UC Davis (soil ressources), 2007

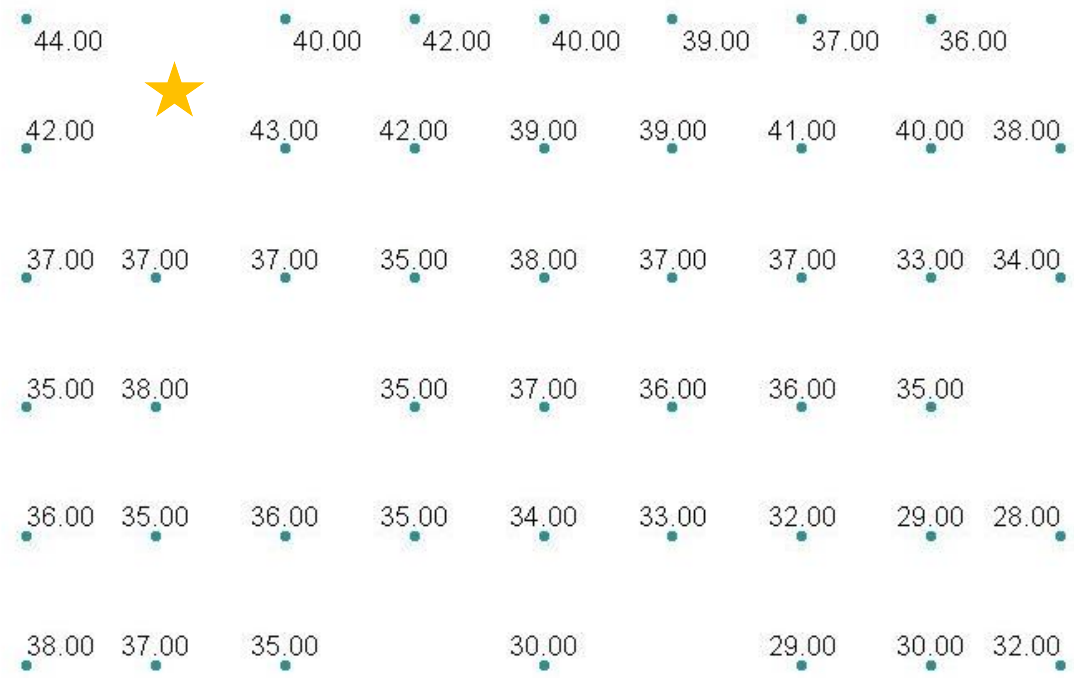
- Variogramm
- Interpolation methods
  - Moving Window Average
  - Inverse Distance Weighting
  - Kriging



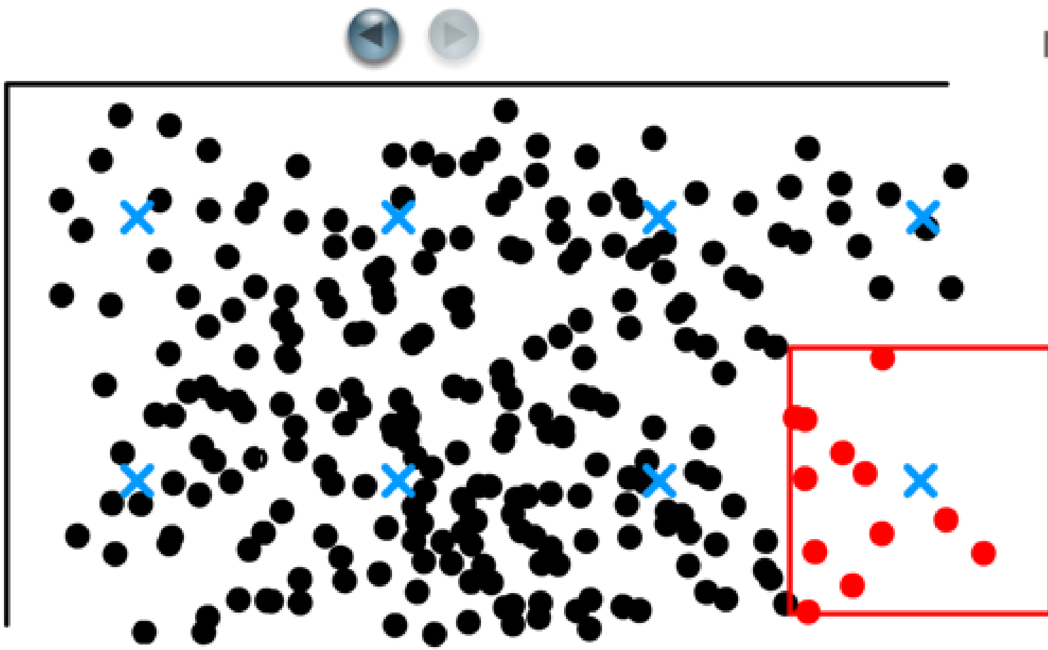
Interpolation

Data points (e.g. energy consumption per household, density of households, etc.….)

Goal: Calculate unknown value based on values of neighboors.



Moving-Window Average



Prinzip der 'Moving Windows Statistik'

No.	Count	Mean	StdDev
1	18	45	2.8
2	35	43	1.7
3	34	46	3.1
4	10	44	4.8
5	26	45	2.7
6	57	44	1.6
7	46	46	2.2
8	12	47	3.6

The distances between points is not included sufficiently in estimation

Gitta, 2009

Inverse Distance Weighting (IDW)

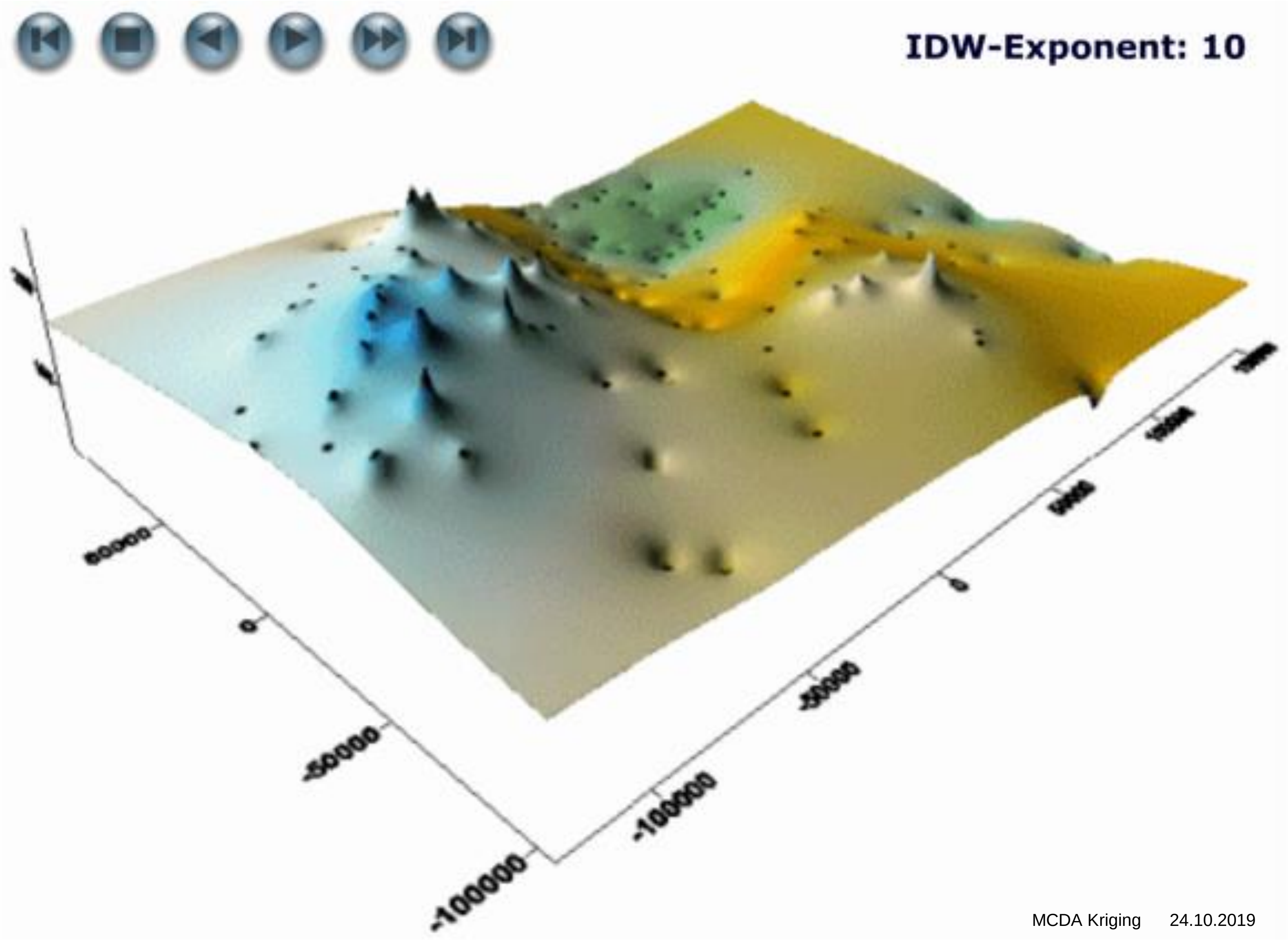
$$\hat{Z}(s_0)_{idw} = \frac{\sum_{i=1}^{p} \frac{Z(s_i)}{d_i^{\alpha}}}{\sum_{i=1}^{p} \frac{1}{d_i^{\alpha}}}$$

d_i = distance between s_i and s_0,

p = number of points

α = distance-decrease factor

IDW



IDW

Advantages of IDW-interpolation:

- Fast algorithm
- Distances between points are included in estimation
- The intensity of the influence of the distance on the value can be driven by the exponent

Disadvantages of IDW-interpolation:

- distance-decreasing factor is constant for entire area
- It is not possible to consider directions in the weighting (relationships in space are not considered, e.g. Mountain ridge)
- Artefacts such as „bull-eyes“ (same values around one point (correction via Shepard IDW).

IDW

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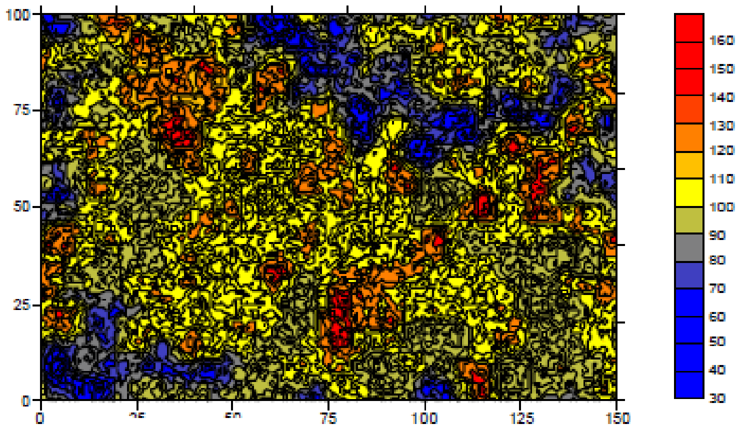
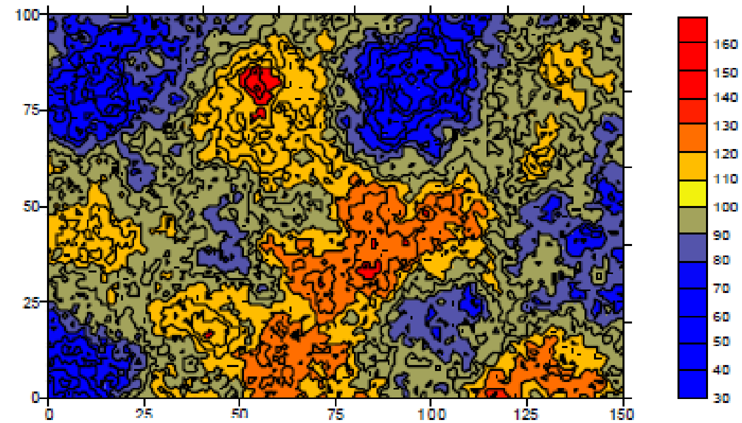
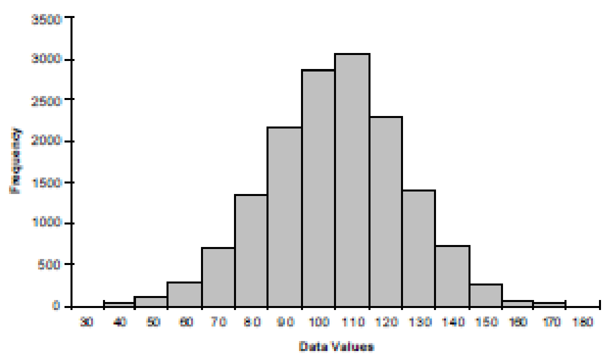
Spatial structures



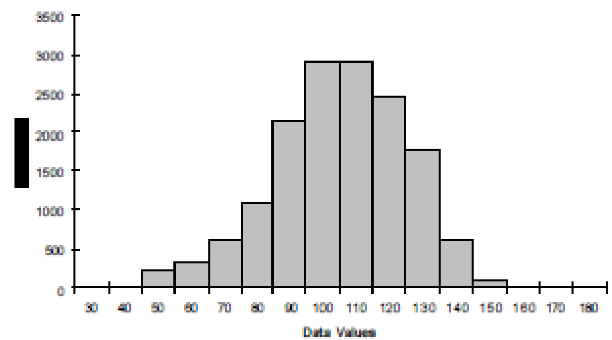
Data Set **A** Contour Plot



Data Set **B** Contour Plot



Data Set **A** Histogram



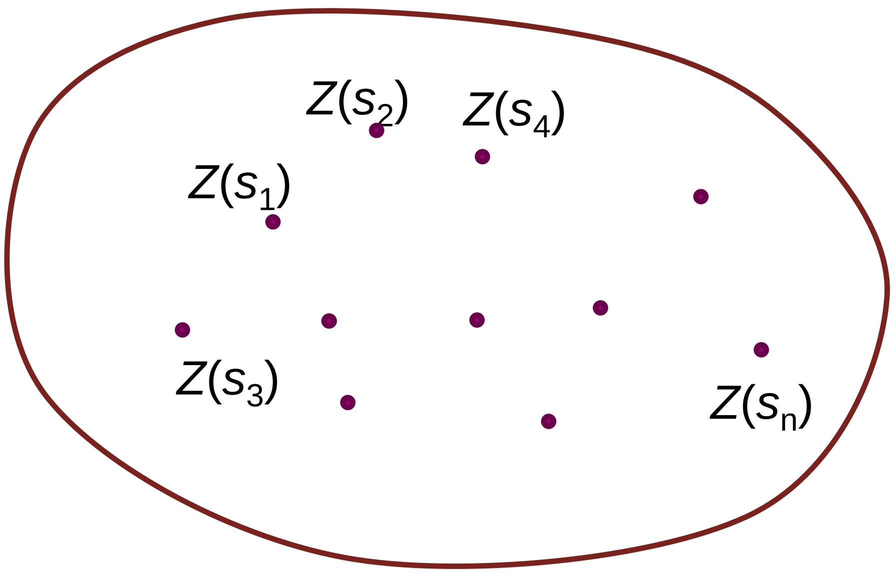
Data Set **B** Histogram

Can a single value describe space? Where do I find high and low values?

Random field

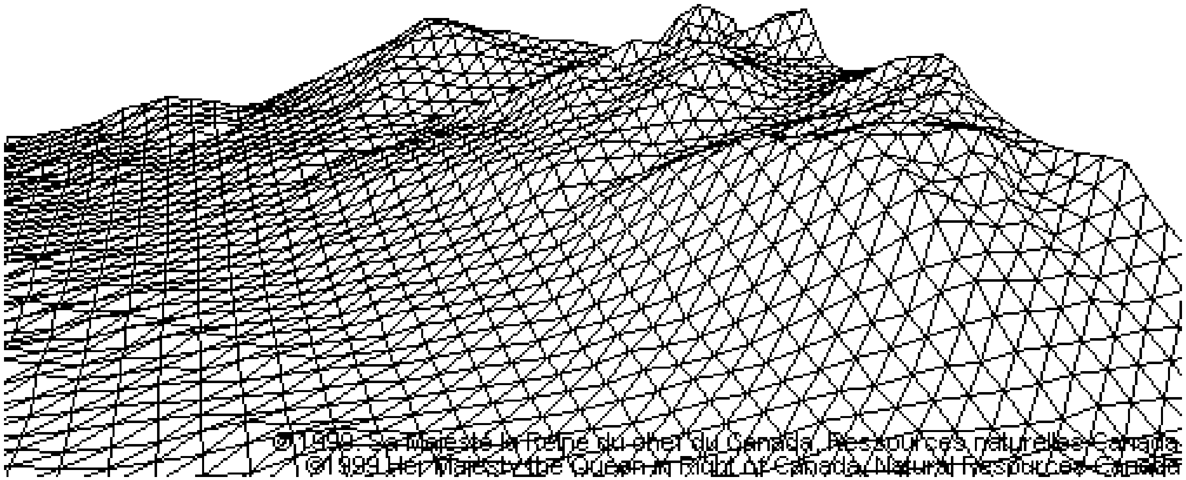
Spatial pattern = synthesis of a specific process in space and time

$$Z(s) = [Z(s_1), \ldots\ldots, Z(s_n)]' = \mu(s) + e(s)$$



Where μ (s) is the mean of the random field and *e*(s) is a spatially correlated stochastic term

Variogram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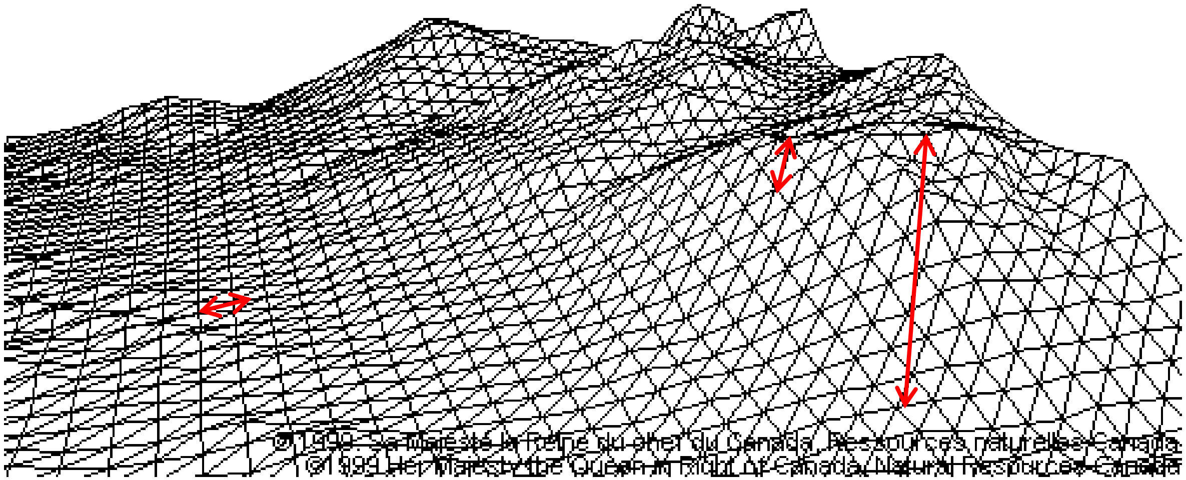
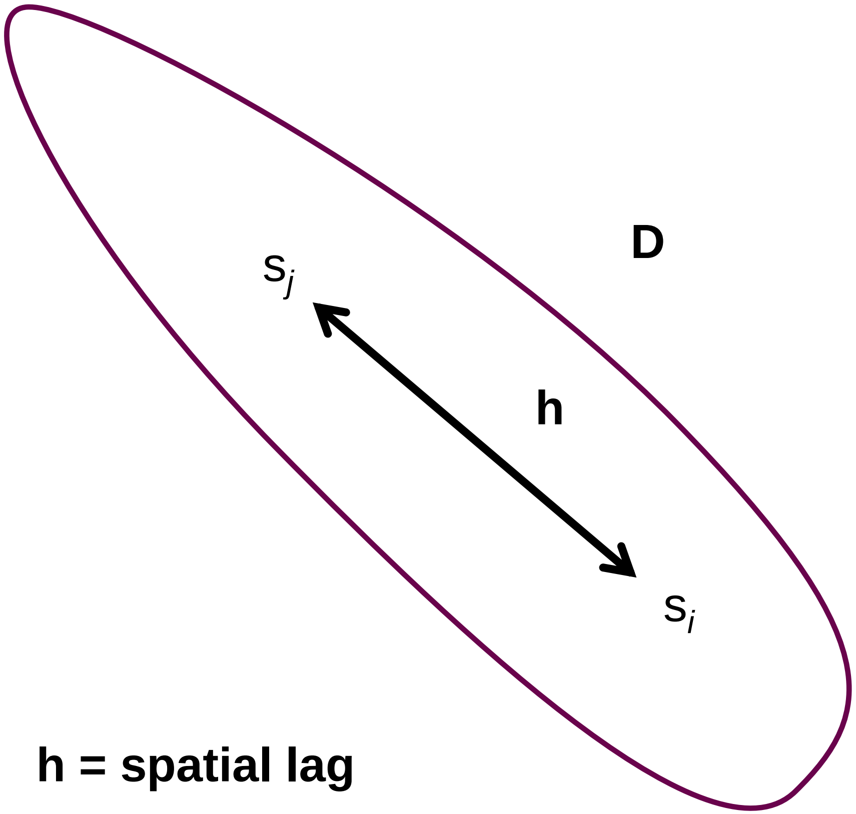
Variogram

➡ Dissimilarity between data pairs



$$\gamma \text{ is a function of } (\{Z(s_i) - Z(s_j)\}^2)$$

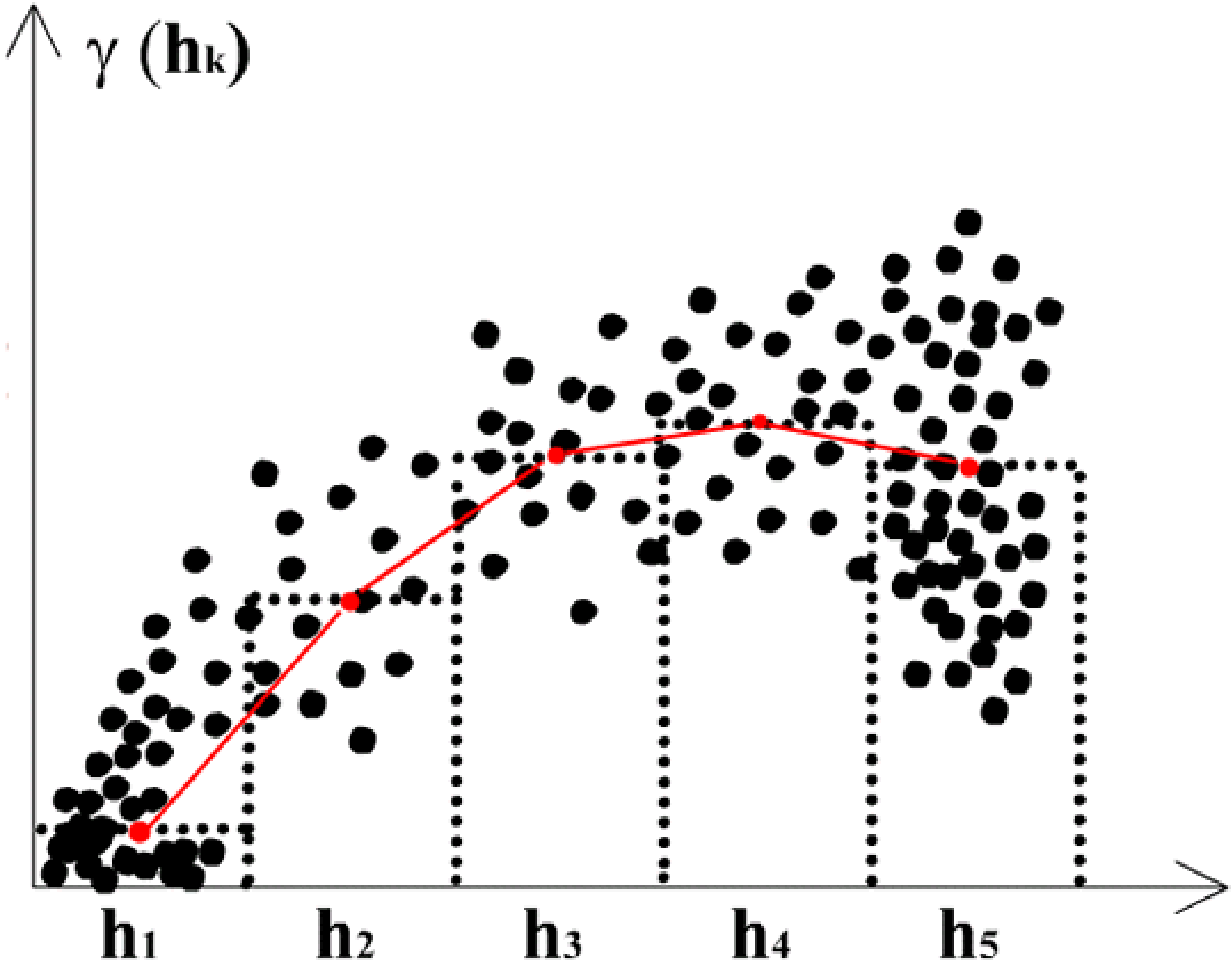
Semi-Variogram



h = spatial lag

$$\gamma(h) = \frac{1}{2N(h)} \sum_{i=1}^{R} \{Z(s_i) - Z(s_i + h)\}^2$$

Semi-Variogram (Experimental)



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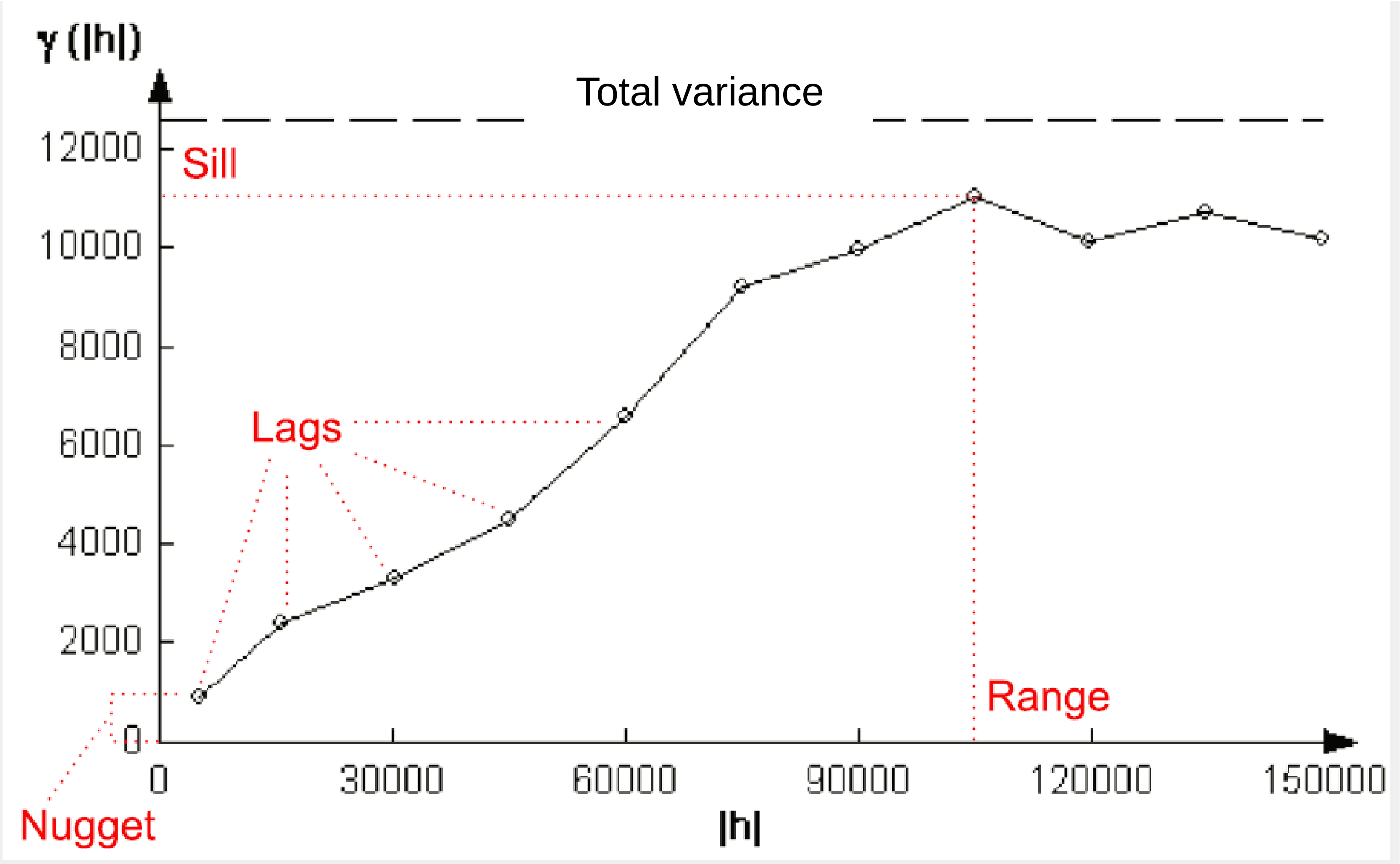
Semi-Variogram



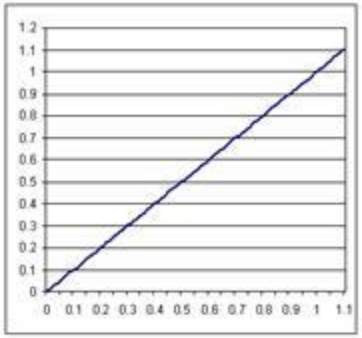
Semi-Variogram

Short questions

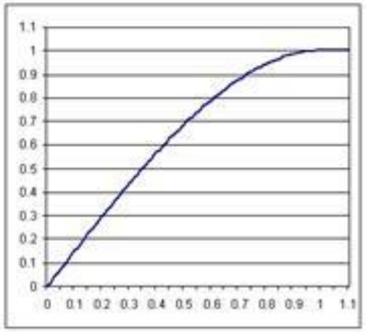
- What happens when all the points have the same value?
- Does it matter in which order the numbers are inputed into the equation?
- Why does the curve not start at point (0;0)

Semi-Variogram (theoretics)

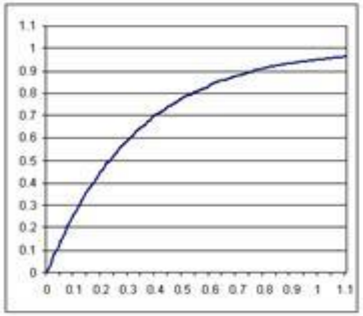
Linear

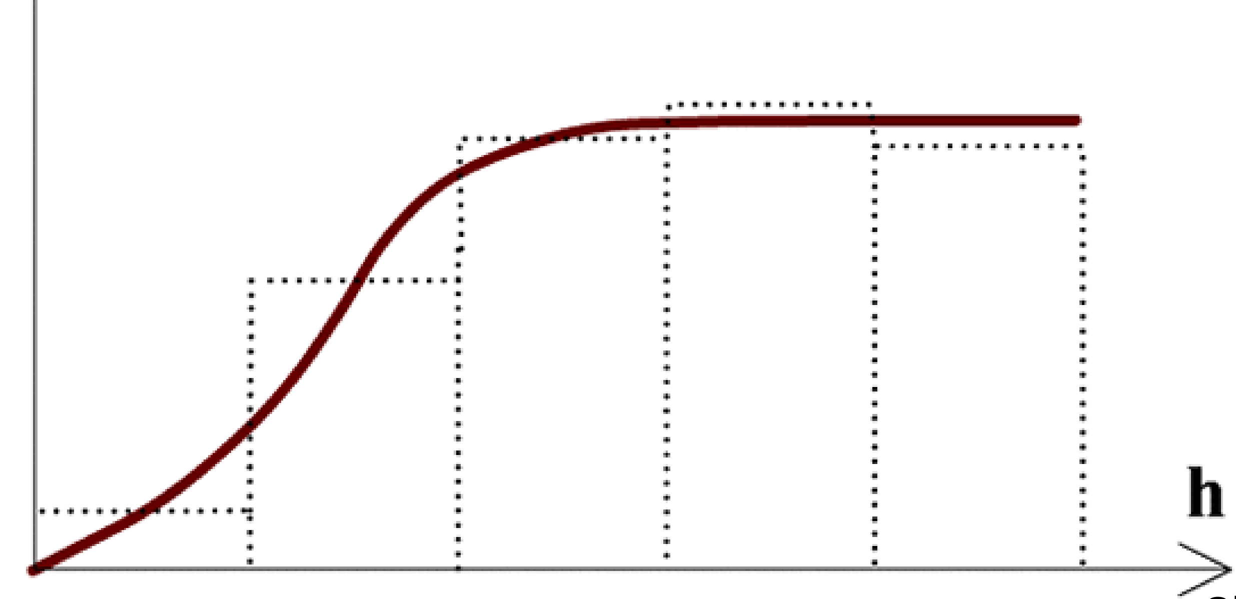


Spherical



Exponential





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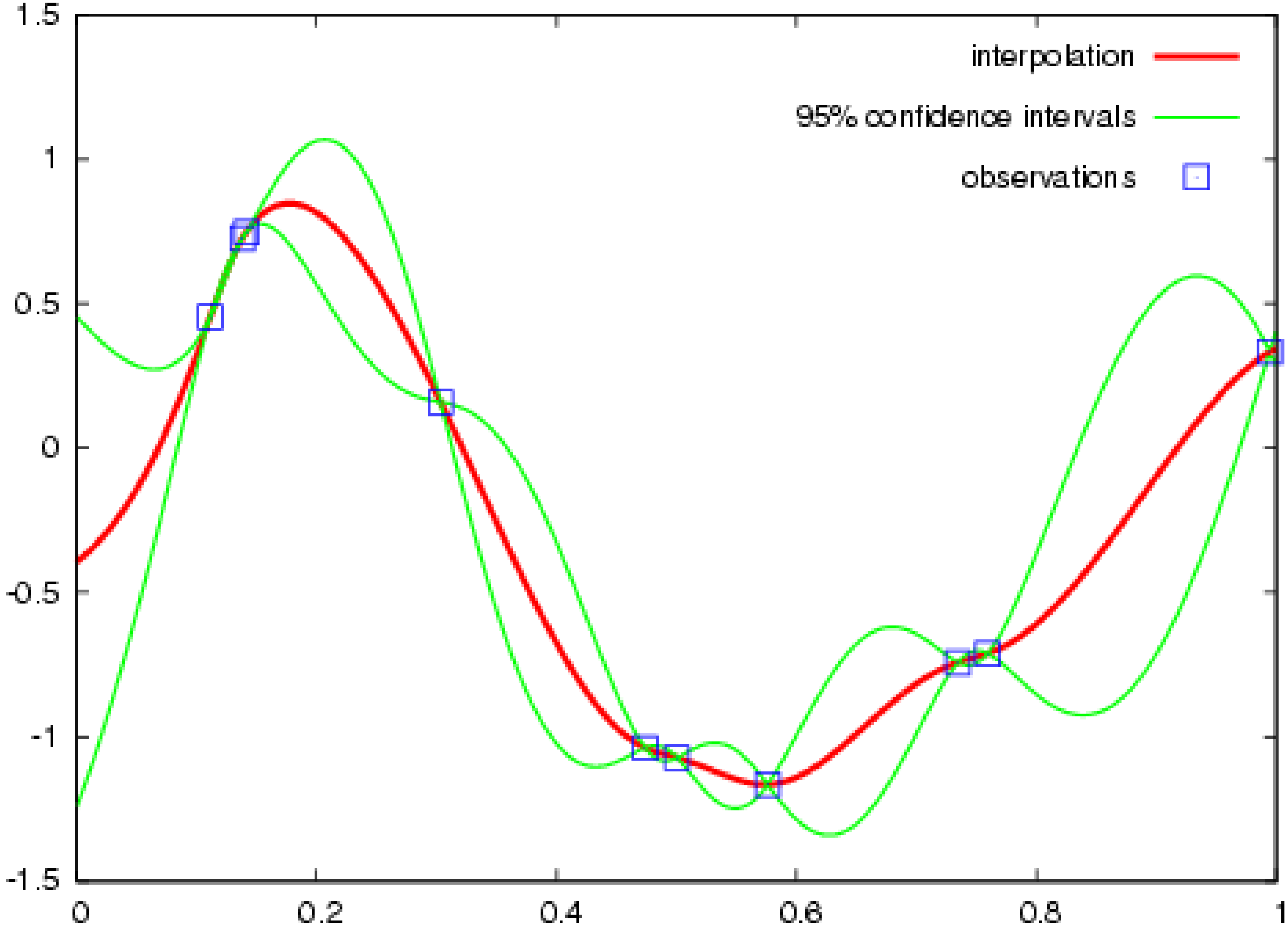
ETH zürich

Kriging



Gitta, 2009

PLUS

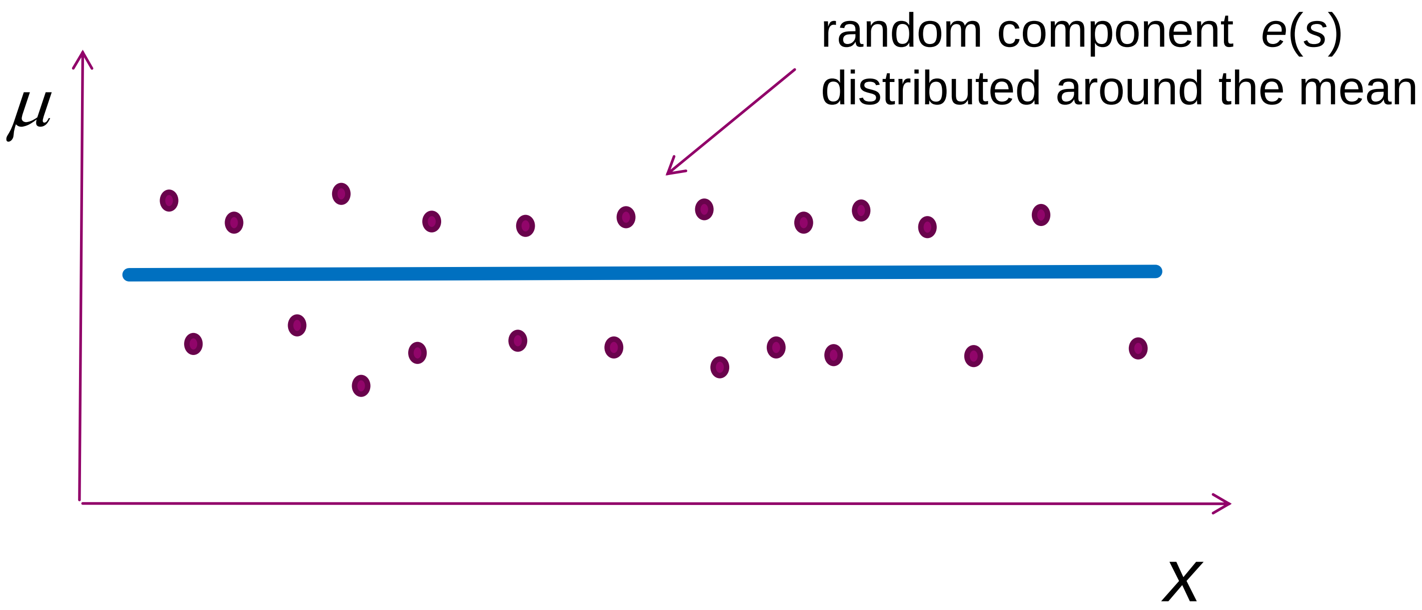
Kriging

Various kriging methods exist with different level of sophistication

- Ordinary kriging
- Simple kriging
- Universal kriging

Stationarity

When a process is stationary, the mean value $\mu(s)$ is constant over space, in such a way that the process can be described by a constant stochastic value over the area, e(s)



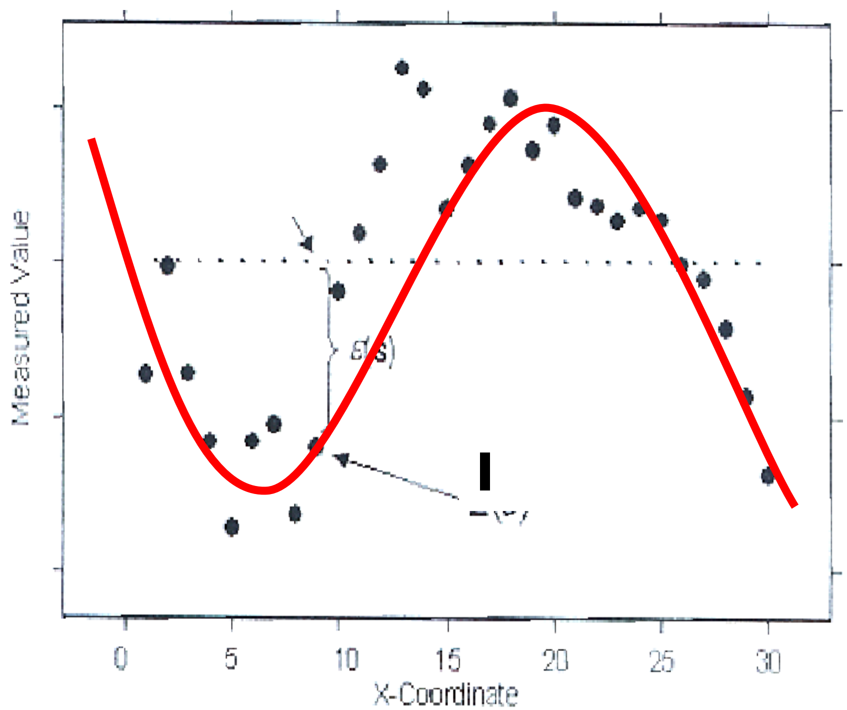
Ordinary Kriging

When the mean value μ (s) of the process is unknown and constant over the area, one can use ordinary kriging:

$$\hat{Z}(s_0) = \sum_{i=1}^{n} e_i Z(\mathrm{s}_i)$$

Based on semi-variogram

e_i = weights of point i
$Z(\mathbf{s}_i)$ = values of point i



Ordinary kriging

A b

$$\begin{pmatrix} \gamma(h_{11}) & \gamma(h_{12}) & \gamma(h_{13}) & 1 \\ \gamma(h_{21}) & \gamma(h_{22}) & \gamma(h_{23}) & 1 \\ \gamma(h_{31}) & \gamma(h_{23}) & \gamma(h_{33}) & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e'_3 \\ \lambda \end{pmatrix} = \begin{pmatrix} \gamma(h_{1P}) \\ \gamma(h_{2P}) \\ \gamma(h_{3P}) \\ 1 \end{pmatrix}$$

Solution for $e = A^{-1} \times b$

$P = s_0$, the point to be interpolated

Ordinary Kriging

$$e_1\gamma(h_{11})+e_2\gamma(h_{12})+e3\gamma(h_{13})+\lambda=\gamma(h_{10})$$

$$e_1\gamma(h_{12})+e_2\gamma(h_{22})+e_3\gamma(h_{23})+\lambda=\gamma(h_{20})$$

$$e_1\gamma(h_{13})+e_2\gamma(h_{32})+e_3\gamma(h_{33})+\lambda=\gamma(h_{30})$$

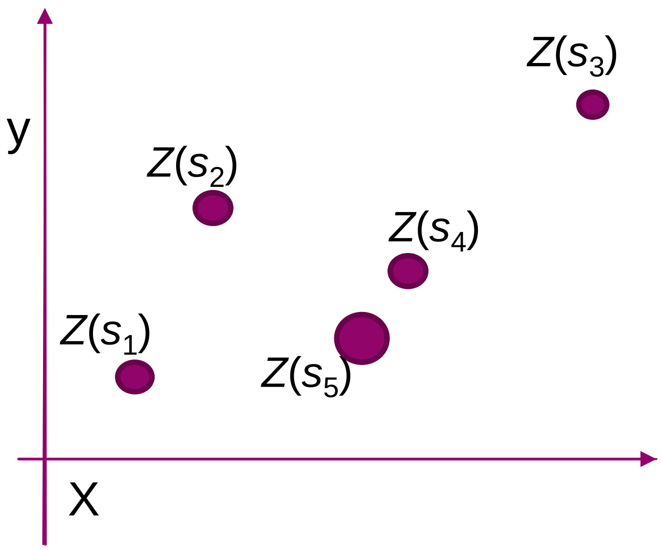
$$e_1+e_2+e_3=1$$

$\gamma(h_{ij})$ = semi-variance at h

$$\boxed{Z(s_0)=e_1Z(s_1)+e_2Z(s_2)+e_3Z(s_3)}$$

EXAMPLE KRIGING

The data:



Point number	1	2	3	4	5	0
X- Coordinate	2	3	9	6	5	5
Y- Coordinate	2	7	9	5	3	5
Value (Z)	3	4	2	4	6	-

The Variogram is assumed to be **spherical** and can be described by the following equation

$$\gamma(h) = c + b \cdot \left\{ \frac{3}{2} \cdot \frac{h}{a} - 0.5 \left(\frac{h}{a} \right)^3 \right\}$$

EXAMPLE KRIGING

The distance between the points can be represented by a distance matrix

Point Nr	1	2	3	4	5		0
1	0.0	5.099	9.899	5.0	3.162		4.243
2	5.099	0.0	6.325	3.606	4.472		2.828
3	9.899	6.325	0.0	5.0	7.211		5.657
4	5.0	3.606	5.0	0.0	2.236		1.0
5	3.162	4.472	7.211	2.236	0.0		2.0

Distance of the known points → to 0

The variogram (nugget=2.5, sill= 7.5 and range= 10) is given by:

$$\gamma(h) = 2.5 + 7.5\left\{\frac{3}{2} \cdot \frac{h}{10} - 0.5\left(\frac{h}{10}\right)^3\right\}$$

And thus for h= 5.0

$$\gamma(5.0) = 7.656$$

EXAMPLE KRIGING

From the distances of the points the γ_{ij} (semivariances) are calculated and inserted into the matrix A

	1	2	3	4	5	6		0
1	2.500	7.739	9.999	7.656	5.939	1.000		7.151
2	7.656	2.500	8.667	6.381	7.196	1.000		5.597
3	9.999	8.667	2.500	7.656	9.206	1.000		8.815
4	7.656	6.381	7.656	2.500	4.936	1.000		3.621
5	5.939	7.196	9.206	4.936	2.500	1.000		4.720
6	1.000	1.000	1.000	1.000	1.000	0.000		1.000

To obtain the matrix A, the distance matrix was additionally modified (column / row). The next step is to invert this matrix, what means, A^{-1} must be calculated

EXAMPLE KRIGING

The inverse matrix is:

$$\mathbf{A}^{-1} = \begin{pmatrix} -0.172 & 0.050 & 0.022 & -0.026 & 0.126 & 0.273 \\ 0.050 & -0.167 & 0.032 & 0.077 & 0.007 & 0.207 \\ 0.022 & 0.032 & -0.111 & 0.066 & -0.010 & 0.357 \\ -0.026 & 0.077 & 0.066 & -0.307 & 0.190 & 0.030 \\ 0.126 & 0.007 & -0.010 & 0.190 & -0.313 & 0.134 \\ 0.273 & 0.207 & 0.357 & 0.003 & 0.134 & -6.873 \end{pmatrix}$$

And the product is:

$$\mathbf{A}^{-1}\mathbf{b} = \mathbf{e} = \begin{pmatrix} 0.0175 \\ 0.2281 \\ -0.0891 \\ 0.6437 \\ 0.1998 \\ 0.1182 \end{pmatrix}$$

EXAMPLE KRIGING

Note that the sum of the first components of the resulting vector is:

$$\sum_{i=1}^{5} \mathrm{e}_i = 1$$

since this is the λ_i, which should sum 1.
The value at the point x_0 can be calculated as:

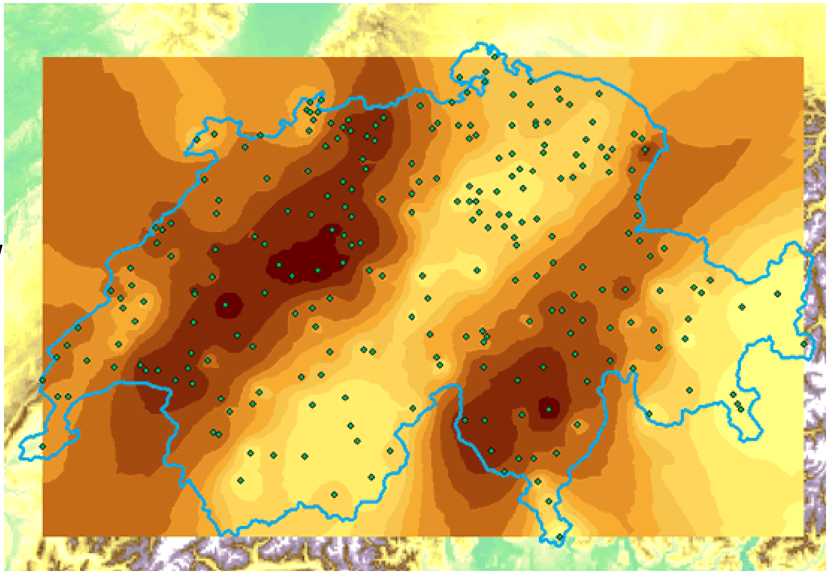
$$\hat{Z}(x_0) = 0.0175 \cdot 3 + 0.2281 \cdot 4 - 0.0891 \cdot 2 + 0.6437 \cdot 4 + 0.1998 \cdot 6 = 4.560$$

And the correspondig variance is:

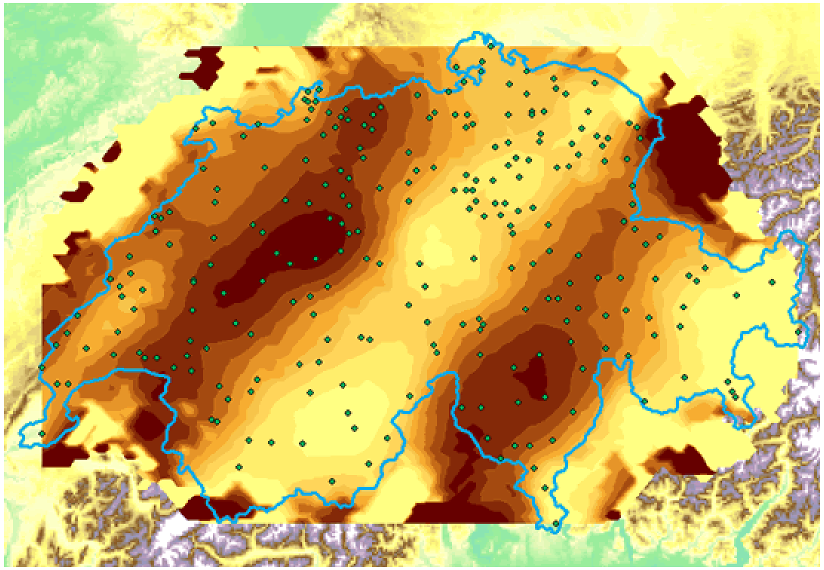
$$\hat{\sigma^2} = 0.0175 \cdot 7.151 + 0.2281 \cdot 5.597 - 0.0891 \cdot 8.815 + 0.6437 \cdot 3.621 + 0.1998 \cdot 4.720 + 0.1182 = 4.008$$

IDW-Kriging

WITH IDW



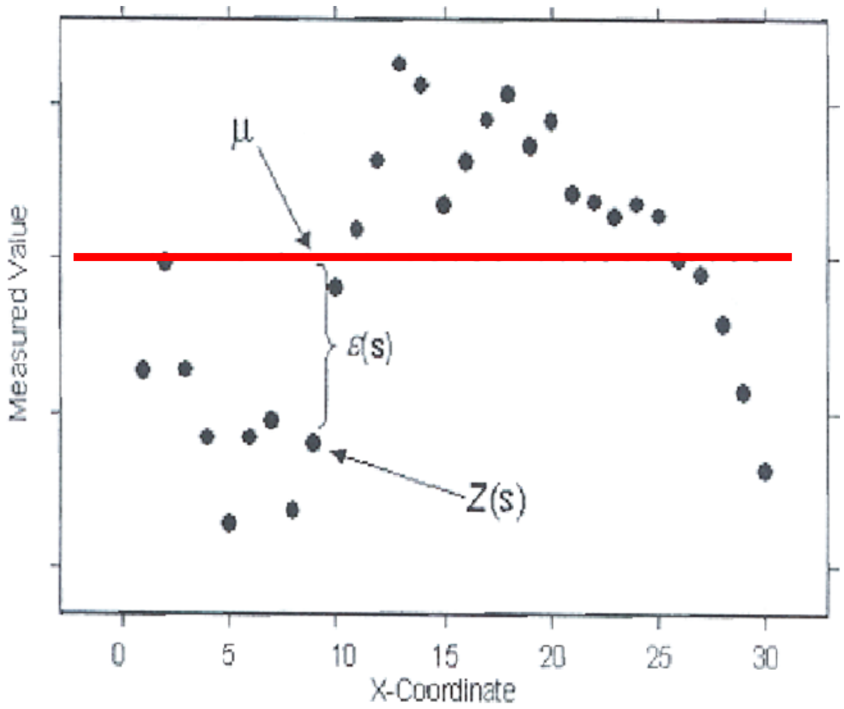
WITH Kriging



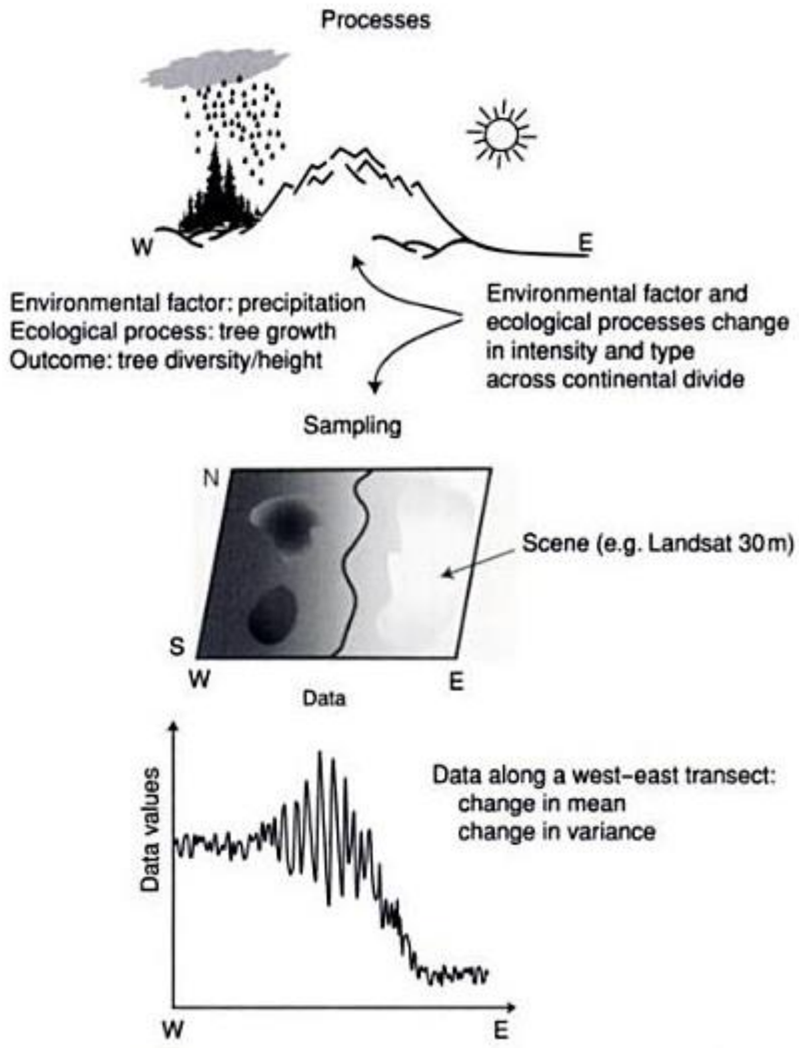
Gitta, 2009

Simple Kriging

If you know the mean value μ (s) and it is constant over the area, one can use simple kriging.

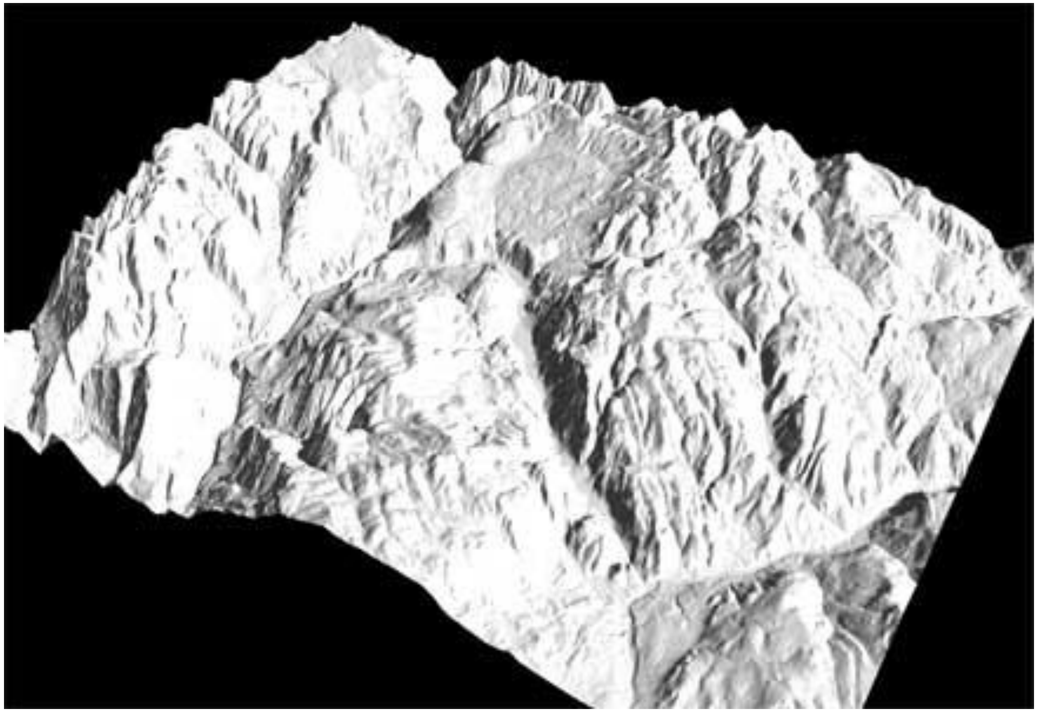


Non-stationarity



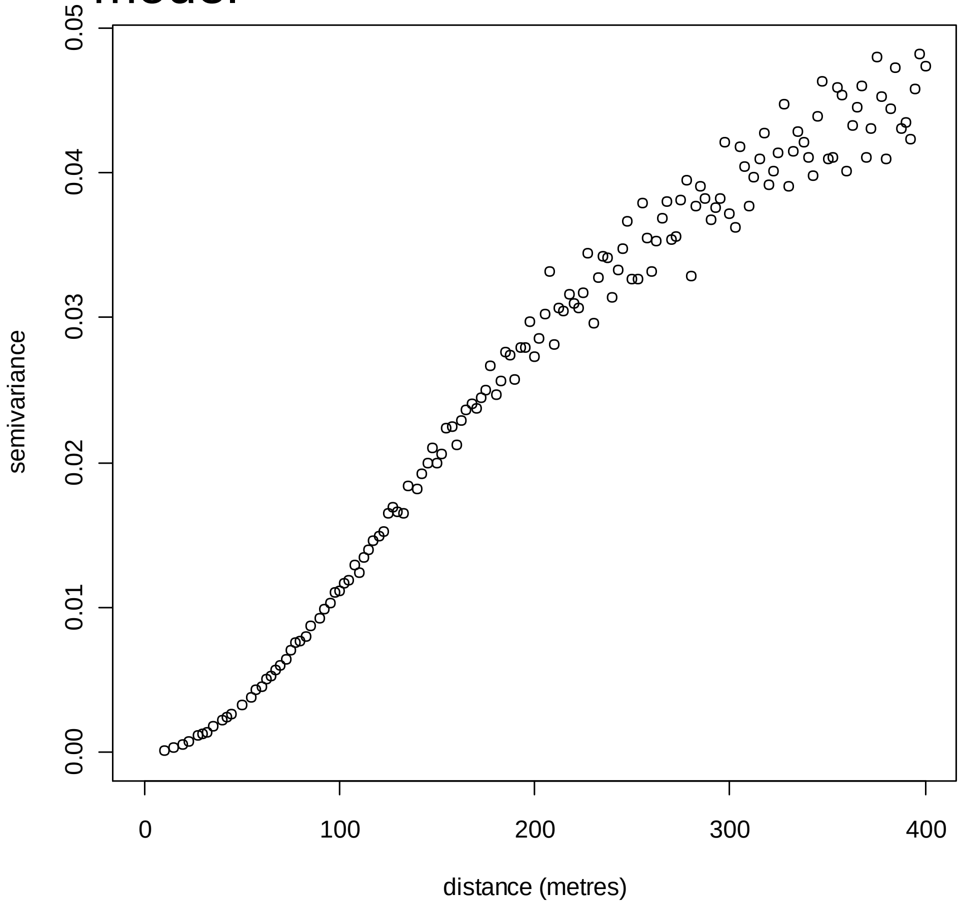
Non-Stationarity

Digital elevation model:



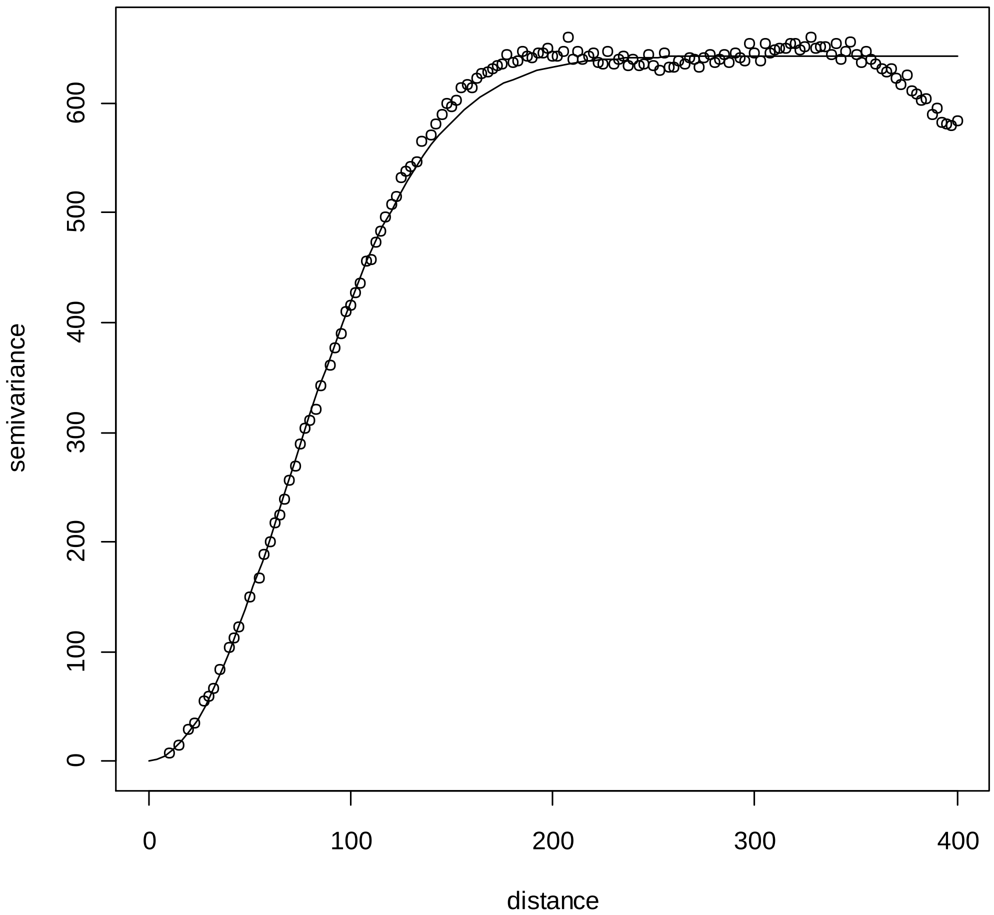
Universal Kriging

Semi-variogram of a digital elevation model



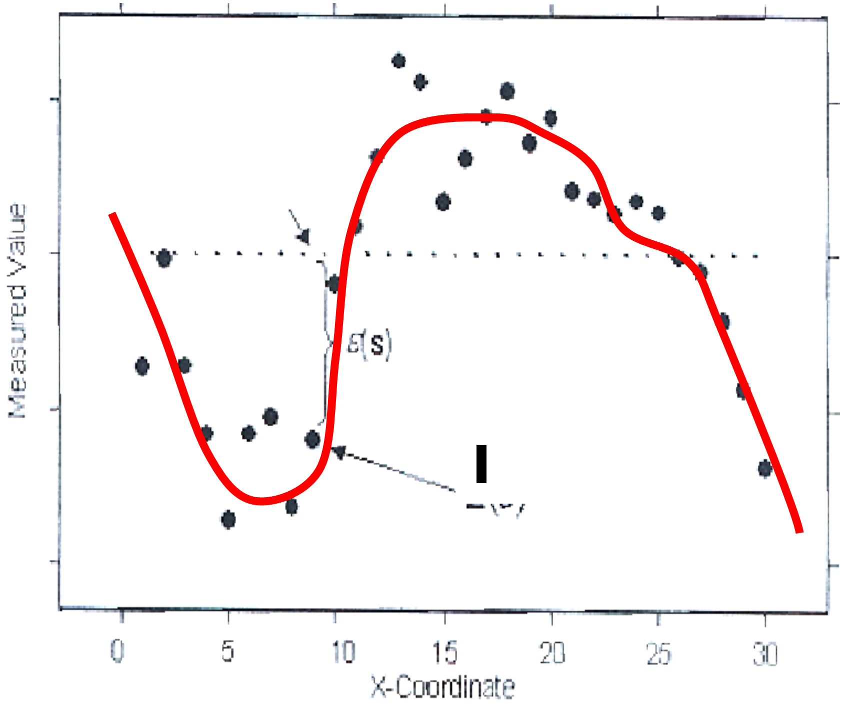
Universal Kriging

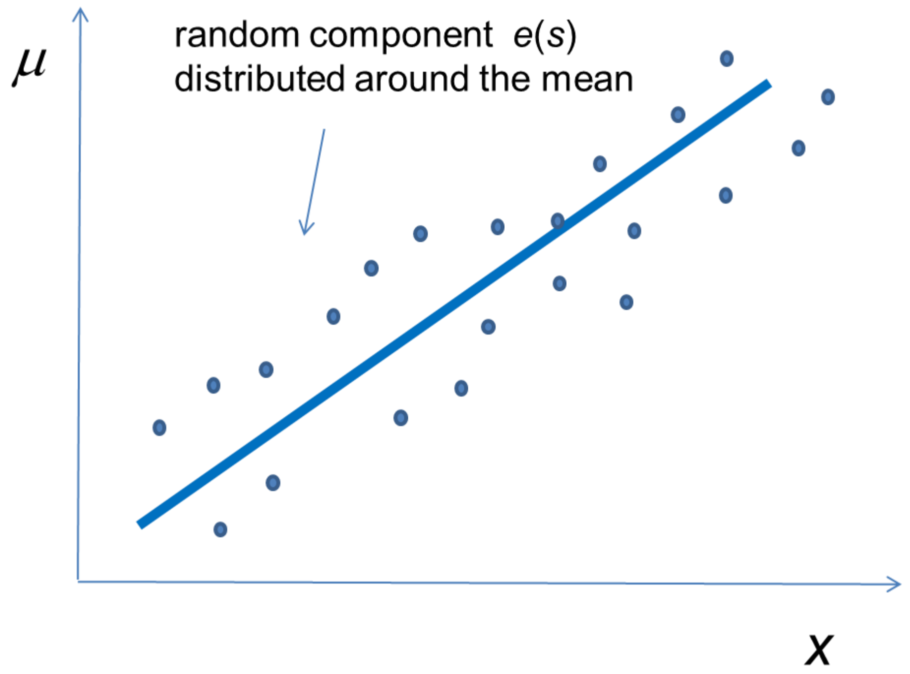
Empirical semi-variogram when trend of digital elevation model has been removed (third-order polynomial)



Universal Kriging for non-stationarity

If the mean value changes over space μ (s), one uses universal kriging

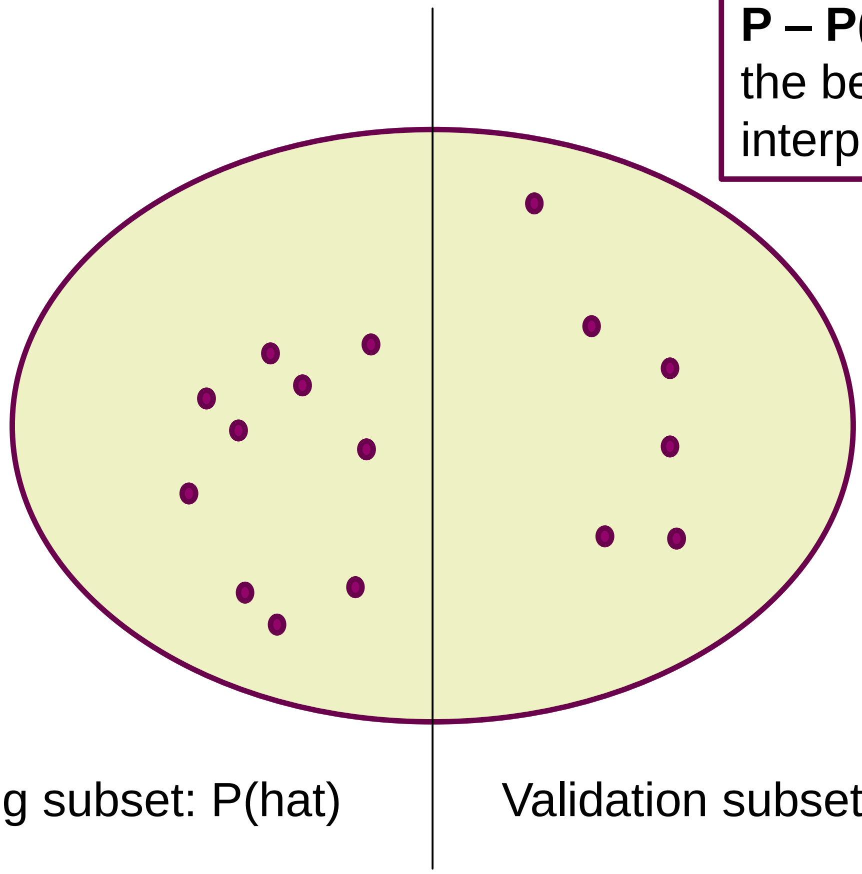




Validation

Interpolated data vs. observed data

The smaller
P – P(hat) = error,
the better the
interpolation



Training subset: P(hat)

Validation subset: (P)

Validation

The performance of interpolation methods can also be evaluated through the root mean squared error (RMSE):

$$\mathrm{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(value_{vs_i} - value_{int_i})^2}$$

where ***n*** is the number of observations in the validation set, $value_{vsi}$ is the value in the validation set at location ***i***, and $value_{nti}$ is the value of the corresponding interpolated variable at location ***i***.

Note that the term ($value_{vsi}$ - $value_{inti}$) corresponds to the error of the interpolation method.

Sample questions

Sample questions after 5th lecture:

1. Which spatial process leads to a flat variogram?

2. What are the differences between the 3 types of interpolation methods (advantages and disadvantages

3. Small exercise....

On 24 April 2013 you read the following article in the NZZ:

NZZ.CH — Neue Zürcher Zeitung — ZÜRICH

Kantonale Schulraumstrategie

Regierung plant zwei bis drei neue Kantonsschulen

Im Kanton Zürich steigt die Zahl der Mittelschüler bis 2027 um 3000 an. Der Bau neuer und die Instandsetzung bestehender Gymnasien erfordert gemäss kantonaler Schulraumstrategie Investitionen von weit über einer Milliarde Franken.

Because of your expertise in GIS and MCDA the Cantonal authorities ask you to propose suitable locations for these two new schools.

You decide to start with the collection of data that is relevant for this problem. It may be important to build the school in an area where there are many potential students (so that students don't have to travel that far), so you want to have a raster map indicating the average grades of primary school pupils in Zurich (i.e. only pupils with a high average grade can go to the gymnasium). To a large number of random homes in Canton Zurich you send out a questionnaire asking the parents to disclose the average primary school grades of their children. After 5000 questionnaires were returned you want to interpolate the average school grades over the whole Canton. For the interpolation you decide to use ordinary kriging (OK). For OK a semivariogram needs to be created. Below is a table with some of the average school grades and coordinates of the pupil's houses.

Pupil	**X-coordinate**	**Y-coordinate**	**Average grade**
A	693579	239785	4.5
B	694079	239785	5.7
C	693829	240218	4.9

1) *For each of the distances (h, rounded to 0 decimals) between pupil's houses, calculate the semivariance ($\hat{\gamma}(h)$) from the average grade. Write down all the calculation steps. Hint: start with calculating the distances (h).*

2) What shape could the empirical semivariogram that results from the dataset of average school grades have?

Plot the empirical semivariogram and describe why you would expect it to have that certain shape.

Is the 'nugget' in the origin (point 0,0) of your semivariogram? Explain why this is or is not the case.

CRITERIA SELECTION (Step 2)

«Homework»

2. Step: Determine the criteria (factors/constraints)

- how much details are needed in the analysis affects the set of criteria to be used
- Criteria should be measurable
- If not determinable, use proxies i.e. slope stability can be represented by slope gradient