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(2016-2017)

Chapter 3

The Equation of Radiative Transfer.

3.1 Absorption only: Beer's Law.

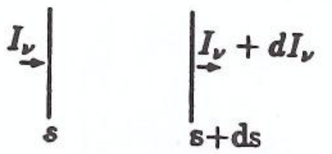


Figure 3.1: Absorption of radiation.

Suppose that we have EM radiationof specific intensity I_ν incident on a thin slab, thickness ds, of material. Experimentally it can be shown that the change in specific intensity dI_ν is proportional to:

1. The amount of absorber in the path $\rho_a ds$ where ρ_a is the absorber density.

2. The intensity of the incident radiation I_ν

We get

$$dI_\nu = -k_\nu I_\nu \rho_a ds$$

Here k_ν is the constant of proportionality, known as the absorption coefficient. It is often a very rapidly varying function of frequency.

Question: Why do we have a negative sign in the equation above?

By definition k_ν is a positive quantity. − sign shows that there is a decrease in specific intensity.

If we integrate the equation from distance $s = 0$ to $s = S$ we have

$$\int_{s=0}^{S} \frac{dI_\nu}{I_\nu} = -\int_{s=0}^{S} k_\nu \rho_a ds$$

Integrating

$$\ln\left(\frac{I_\nu(S)}{I_\nu(0)}\right) = -\int_{s=0}^{S} k_\nu \rho_a ds$$

This is usually written

$$I_\nu(S) = I_\nu(0) \exp\left(-\int_{s=0}^{S} k_\nu \rho_a ds\right) \tag{3.1}$$

The quantity

$$\mathcal{T}_\nu = \exp\left(-\int_{s=0}^{S} k_\nu \rho_a ds\right)$$

is called the **transmittance** and is the fraction of the incident energy that is transmitted through the medium.

In general we cannot simplify equation 3.1 because k_ν will not be constant along the path taken by the radiation. For example, in the atmosphere k_ν is a function of pressure and temperature and the integral can be evaluated only with some difficulty. However in the case in which the path is homogeneous the integral can easily be evaluated because k_ν is constant and may be taken outside the integral, along with ρ_a. Thus equation 3.1 reduces to:

$$I_\nu(S) = I_\nu(0) \exp(-k_\nu \rho_a S) \tag{3.2}$$

This result is known as **Beer's Law** or **Bouguer's Law**.

It can be shown (see problem set) that transmittances are multiplicative, i.e. the transmittance through two layers is equal to the product of the transmittances through the two separate layers. This does not depend on the layers being identical or require either of them to be homogeneous.

WARNING. This result holds for monochromatic radiation (single frequency) and does **NOT** usually hold for the mean transmittance over a finite frequency interval.

We introduce the term **optical path u**, defined by

$$du = \rho_a ds$$

The transmittance may now be written as $\exp(-\int k_\nu du)$. The units of k_ν will be the inverse of the units used to describe u, i.e. $cm^2\ g^{-1}$ or $m^2\ kg^{-1}$. In practice we often use other units to describe the absorber amount u. One commonly used unit is (atm. cm.) and another is molecules cm^{-2} (or more simply cm^{-2}), which corresponds to ρ_a being given as number density in units of cm^{-3}. In this case the absorption coefficient has units cm^2 and is refered to as a **cross section**.

As another example, consider a homogeneous path, e.g. ocean water, the amount of absorber is proportional to the distance, and the transmittance can be expressed as $\exp(-K_\nu S)$ and K_ν has the units of inverse length, cm^{-1} or m^{-1}.

The transmittance may be written in yet another form as

$$exp(-\tau)$$

where τ is the **optical thickness** or **optical depth** defined through the differential relationship

$$d\tau = k_\nu du = k_\nu \rho_a ds$$

When the optical depth is unity the incident radiation has been reduced to $1/e$ of its original intensity.

The calculation of transmittance in an inhomogeneous atmosphere is a major problem in applying radiative transfer theory to real situations and will be taken up in some detail in Chapter 4.

3.2 The Radiative Transfer Equation with a Source Term.

We now generalize the radiative transfer equation by adding a source term:

$$dI_\nu = -k_\nu I_\nu \rho_a ds + \epsilon_\nu \rho_a ds \tag{3.3}$$

The term ϵ_ν is the emission coefficient. Two sources which contribute to the emission coefficient will be important to us:

1. Thermal emission.

2. Scattering into the path of the radiation from a different direction.

We also generalize the concept of k_ν, which now includes the the effect of scattering radiation out of the path of I_ν, as well as absorption. We often refer to k_ν as an extinction coefficient under these circumstances.

Dividing equation 3.3 by $k_\nu \rho_a ds$ we get

$$\frac{dI_\nu}{k_\nu \rho_a ds} = -(I_\nu - \epsilon_\nu / k_\nu) \tag{3.4}$$

We define the source function J_ν by

$$J_\nu = \frac{\epsilon_\nu}{k_\nu}$$

Kirchoff's Law, related to the earlier version, states that in local thermodynamic equilibrium (LTE) the source function for emission is given by the Planck black-body function $B(\nu, T)$. In cases in which scattering is an important factor the source function is usually more difficult to determine and will be considered later in the course.

WARNING. The fact that the source function is the black-body function does not mean that the atmosphere looks like a black-body. This will become clear when we integrate the equation of radiative transfer.

Equation 3.4 now becomes:

$$\frac{dI_\nu}{k_\nu \rho_a ds} = -(I_\nu - J_\nu) \tag{3.5}$$

This is a differential equation of the form:

$$y' + f(x)y = g(x)$$

and can be solved by the use of an integrating factor:

$$\exp\left(\int f(x)dx\right)$$

giving:

$$\exp\left(\int f(x)dx\right) y' + f(x)\exp\left(\int f(x)dx\right) y = g(x)\exp\left(\int f(x)dx\right)$$

or

$$\frac{d}{dx}\left(y\exp\left(\int f(x)dx\right)\right) = g(x)\exp\left(\int f(x)dx\right)$$

In equation 3.5 we have the following:

$x = s \qquad y = I_\nu \qquad f(x) = k_\nu \rho_a \qquad g(x) = k_\nu \rho_a J_\nu$

Substituting in equation 3.5

$$\frac{d}{ds}\left(I_\nu \exp\left(\int_0^s k_\nu \rho_a ds\right)\right) = k_\nu \rho_a J_\nu \exp\left(\int_0^s k_\nu \rho_a ds\right)$$

Integrating along the path from s=0 to S we obtain

$$I_\nu(S)\exp\left(\int_0^S k_\nu \rho_a ds\right) - I_\nu(0) = \int_0^S k_\nu \rho_a J_\nu \exp\left(\int_0^s k_\nu \rho_a ds\right) ds$$

Now divide both sides by $\exp\left(\int_0^S k_\nu \rho_a ds\right)$ to get:

$$I_\nu(S) = I_\nu(0)\exp\left(-\int_0^S k_\nu \rho_a ds\right) + \int_0^S k_\nu \rho_a J_\nu \exp\left(-\int_s^S k_\nu \rho_a ds\right) ds \quad (3.6)$$

This equation is not quite so complicated as it seems at first if we examine it term by term.

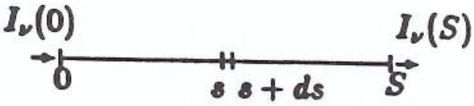


Figure 3.2: Path taken by radiation

The first term on the right side is exactly the same as we got in the derivation of Beer's Law, i.e. the amount of the incident radiation which passes through the medium to emerge out the other side. It is the incident specific intensity multiplied by the transmittance all the way through the medium.

The second term involves an integral. Now

$$k_\nu \rho_a J_\nu ds = \epsilon_\nu \rho_a ds$$

which is the specific intensity of the radiation emitted by the element ds. Note also that

$$\exp\left(-\int_s^S k_\nu \rho_a ds\right)$$

is the transmittance along the path from the point s to S. Thus

$$k_\nu \rho_a J_\nu ds \exp\left(-\int_s^S k_\nu \rho_a ds\right)$$

represents the specific intensity of radiation emited by the element ds which reaches the boundary S. The integration sums the contributions of all elements ds from $s = 0$ to S.

Equation 3.6 may also be written in terms of the optical thickness τ. We defined

$$d\tau = k_\nu \rho_a ds = k_\nu du$$

Integrating from $s = 0$ to s

$$\int_{s=0}^{s} d\tau = \int_0^{s} k_\nu \rho_a ds$$

Since $\tau_0 = 0$ we may write this as

$$\tau_s = \int_0^s k_\nu \rho_a ds$$

Equation 3.6 becomes

$$I_\nu(S) = I_\nu(0)\exp(-\tau_S) + \int_0^{\tau_S} J_\nu \exp(-(\tau_S - \tau))d\tau \qquad (3.7)$$

The integral form of the equation of radiative transfer is most commonly written in this form.

Example.

Consider an isothermal atmosphere temperature T_0 bounded below by a black-body surface whose temperature is also T_0. Find the specific intensity of long-wave radiation directed upward at any point in the atmosphere. What is the long-wave radiative flux at top of the atmosphere? Is the atmosphere being heated or cooled by long-wave radiation?

This an extermely important example, since many similar problems may be solved using the same techniques, including those assigned to you in problem sets.

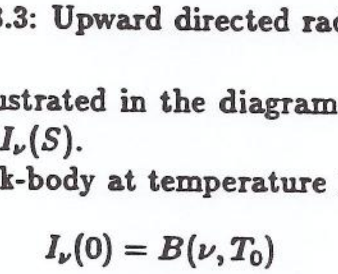


Figure 3.3: Upward directed radiation.

Consider the point illustrated in the diagram and use equation 3.7 to find the specific intensity $I_\nu(S)$.
Since the surface is a black-body at temperature T_0, we have

$$I_\nu(0) = B(\nu, T_0)$$

We will assume that we are dealing with a case in which scattering is not important, so that $J_\nu = B(\nu, T_0)$.

Equation 3.7 becomes:

$$I_\nu(S) = B(\nu, T_0)\exp(-\tau_s) + \int_0^{\tau_s} B(\nu, T_0)\exp(-(\tau_s - \tau))\,d\tau$$

Now $B(\nu, T_0)$ is independent of position along the path and can be taken outside the integral, giving

$$\begin{aligned} I_\nu(S) &= B(\nu,T_0)e^{-\tau_s} + B(\nu,T_0)\int_0^{\tau_s} \exp(\tau-\tau_s)\, d\tau \\ &= B(\nu,T_0)e^{-\tau_s} + B(\nu,T_0)\exp(\tau-\tau_s)\Big|_0^{\tau_s} \\ &= B(\nu,T_0)e^{-\tau_s} + B(\nu,T_0)(1-e^{-\tau_s}) \\ &= B(\nu,T_0) \end{aligned}$$

We have shown that for upward directed radiation the specific intensity remains the same at all points in the atmosphere and is equal to the Planck black-body function.

At the top of the atmosphere the downward component of long-wave flux is assumed to be zero. (This is not strictly true but is a very, very good approximation.) So the net flux is just the upward component σT_0^4.

At the surface the upward component of flux is still σT_0^4. The downward component is no longer zero because of the emission by the atmosphere. Net flux F = upward component $F\uparrow$ - downward component $F\downarrow$

$$F = F\uparrow - F\downarrow = \sigma T_0^4 - F\downarrow < \sigma T_0^4$$

So the net flux F is larger at the top of the atmosphere than it is at the surface, i.e. $\Delta F/\Delta z > 0$. Hence the atmosphere is being cooled by long-wave radiative processes.

This result shows that the atmosphere as a whole is being cooled by the long-wave radiative transfer of energy. It is in fact easy to show that every point in this atmosphere is being cooled. To do this we will show that at every point $dF/dz > 0$. We will start by finding the intensity of downward directed radiation.

$$I_\nu(S) = 0 \cdot e^{-\tau_s} + B(\nu,T_0)(1-e^{-\tau_s}) = B(\nu,T_0)(1-e^{-\tau_s})$$

Now τ_s increases as we go down in the atmosphere. So $1-\exp(-\tau_s)$ increases and hence I_ν increases as we go down. Integrating over all angles and frequencies we see that $F\downarrow$ increases as we go down. So at all levels

$$F = F\uparrow - F\downarrow = \sigma T_0^4 - F\downarrow$$

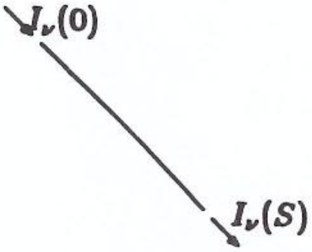


Figure 3.4: Downward directed radiation.

increases as we go up in our atmosphere, i.e $dF/dz > 0$ at all levels, and so the atmosphere is cooling at every level.

Notice that in this example $F\uparrow$ was independent of altitude and of atmospheric transmittance. $F\downarrow$ however depended on both of these parameters. In the case of an atmosphere which is not isothermal, or is isothermal with an underlying surface which is at a different temperature, then $F\uparrow$ will also depend on these two parameters.

Example: Determine if the surface of the earth in the previous example is being heated or cooled by long-wave radiation.

This type of problem is solved by considering the difference between the radiant energy being emited and that being absorbed by the surface.

Flux of energy being emited = $\epsilon \sigma T_0^4 = \sigma T_0^4$

Flux of energy being absorbed = $a F\downarrow = F\downarrow$

Now for downward directed radiation

$$I_\nu(S) = B(\nu, T_0)\left[1 - e^{-\tau_s}\right]$$
$$< B(\nu, T_0)$$

Integrating over angle and frequency we have the inequality:

$$F\downarrow < \sigma T^4$$

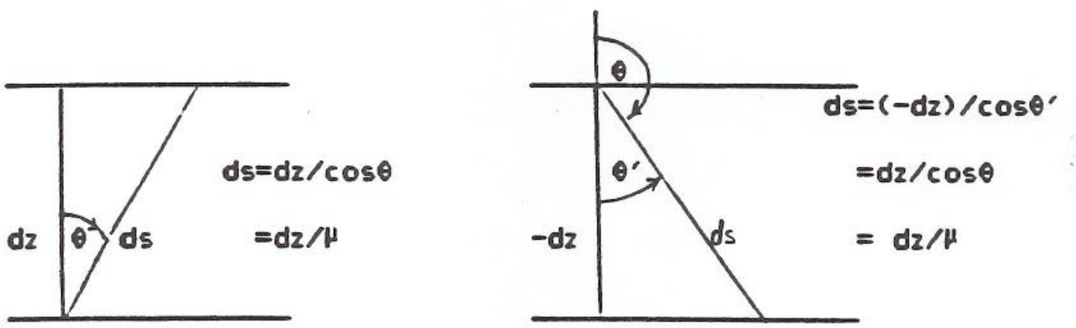
So more energy is being emited than absorbed, and hence the surface is being cooled by long-wave radiation.

The two problems that we have worked are artificial in the sense that we don't encounter isothermal atmospheres in nature. The reason for making the isothermal assumption was to simplify the evaluation of the integrals to determine the specific intensities. The processes of determining the heating and cooling of the atmosphere and the underlying surface are the same in more realistic cases, but the calculation of specific intensities and fluxes are much more complicated, requiring the use of a computer. However the results that we have obtained are qualitatively the same: the surface is just about always being cooled by long-wave radiation and most parts of the atmosphere are also being cooled.

The major exception to the latter is the very coldest regions of the atmosphere, e.g. the mesopause.

3.3 The Transfer Equation in a Plane Parallel Atmosphere.

The transfer equation written in either of the forms ~~3.2.4~~ 3.6 or ~~3.2.5~~ 3.7 leads us to some notation difficulties because the distance s which the beam travels in the medium depends on the path taken and will vary with zenith angle θ. For atmospheric applications we would like to write it in terms of the altitude z. We will assume that we have a plane parallel atmosphere (no spherical effects) and that the zenith angle θ remains constant over the path taken by the photons (no refraction).



Upward directed — Downward directed

For both upward and downward directed radiation we have

$$ds = dz/\mu \qquad \text{where } \mu = cos\theta$$

Define

$$\tau_z = \int_z^\infty k_\nu \rho_a dz$$

Physically τ_z is the optical thickness for a vertical path from altitude z to the top of the atmosphere. It does of course depend on frequency. The transmittance from z to the top of the atmosphere is $exp(-\tau_z/\mu)$. It is easy to show that the transmittance between altitude z_1 and z_2 is $exp(-(\tau_1 - \tau_2)/\mu)$, a result that holds for both upward and downward directed radiation.

We now substitute in Equation 3.7 to get:

Upward: $\mu > 0$

$$I_\nu(\tau) = I_\nu(\tau_g)\exp(-(\tau_g - \tau)/\mu) + \int_\tau^{\tau_g} J_\nu(t)\exp(-(t - \tau)/\mu)dt/\mu$$

Downward: $\mu < 0$

$$I_\nu(\tau) = I_\nu(0)\exp(\tau/\mu) + \int_\tau^0 J_\nu(t)\exp(-(t - \tau)/\mu)dt/\mu \qquad (3.8)$$

$I_\nu(\tau)$ is the specific intensity at optical depth τ. The subscript g refers to the value at the ground. $J_\nu(\tau)$ is the source function at optical depth τ. We have dropped the subscript z on τ, it being understood that τ is a function of z.

If we consider thermal radiation without scattering in an atmosphere above a surface which is a black-body, equation 3.8 may be rewritten:

Upward: $\mu > 0$

$$I_\nu(\tau) = B(\nu, T)\exp(-(\tau_g - \tau)/\mu) + \int_\tau^{\tau_g} B(\nu, T)\exp(-(t - \tau)/\mu)dt/\mu$$

Downward: $\mu < 0$

$$I_\nu(\tau) = \int_\tau^0 B(\nu, T)\exp(-(t - \tau)/\mu)dt/\mu \qquad (3.9)$$

We have also assumed that no long-wave radiation is incident on the top of the atmosphere.

We can also write F_ν and F in terms of μ.
From equation 2. .2 we have

$$F_\nu = 2\pi \int_0^\pi I_\nu \cos\theta \sin\theta d\theta$$

Now $\mu = \cos\theta$ and $d\mu = -\sin\theta d\theta$ Changing the variable of integration from θ to μ

$$F_\nu = 2\pi \int_{+1}^{-1} I_\nu \, \mu(-d\mu) = 2\pi \int_{-1}^{+1} I_\nu \mu \, d\mu$$

To obtain the net flux F we integrate over frequency

$$F = 2\pi \int_{\nu=0}^{\infty} \int_{\mu=-1}^{1} I_\nu \mu d\mu d\nu \tag{3.10}$$

For hemispheric fluxes the limits of integration of μ should be changed to:

0 to 1 for upward directed radiation
-1 to 0 for downward directed radiation.

The evaluation of fluxes involves three separate integrations:

1. Along the path of the radiation (Equation 3.9)
2. Over all zenith angles (Equation 3.10)
3. Over all frequencies (Equation 3.10)

This would seem to be a relatively simple and easy task, but in fact it is a major stumbling block in the quantitative solution of radiative transfer problems. The difficulty lies in the nature of the transmittance functions, which are usually very rapidly varying functions of frequency. Furthermore, the absorption coefficient is a function of temperature and often pressure also, making the integration along the path of the radiation a complicated task, particularly in an inhomogeneous atmosphere. The nature of the transmittance functions and ways of simplifying the integrations is the subject of the next chapter.