Introduction to Geomagnetically Trapped Radiation

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Chapter

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Appendix A

Summary of frequently used formulas

Numbers indicate where formula is located in text.

Motion of a charged particle in electric and magnetic fields

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B} + \mathbf{E}) \tag{2.1}$$

Gyroradius:

$$\rho = \frac{p_\perp}{Bq} = \frac{\gamma m_0 v_\perp}{Bq} \tag{2.4}$$

Gyrofrequency:

$$\Omega = \frac{qB}{\gamma m_0}$$

$$\Omega\ (\text{radians s}^{-1}) = 1.758 \times 10^{11} \frac{B(T)}{\gamma} \quad \text{(electrons)}$$

$$= 9.581 \times 10^{7} \frac{B(T)}{\gamma} \quad \text{(protons)}$$

Guiding center drift velocities

Electric field drift:

$$\mathbf{V}_\mathrm{E} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{2.8}$$

Gradient B drift:

$$\mathbf{V}_\mathrm{G} = \frac{m v_\perp^2}{2qB^3} (\mathbf{B} \times \nabla B) \tag{2.23}$$

Curvature B drift:

$$\mathbf{V}_\mathrm{c} = \frac{m v_\parallel^2}{qR_\mathrm{c}} \frac{\mathbf{n} \times \mathbf{B}}{B^2} \tag{2.25}$$

In regions where $\nabla \times \mathbf{B} = 0$ the $\mathbf{V}_\mathrm{c}$ is given by

$$\mathbf{V}_c = \frac{mv_\parallel^2}{qB^3}(\mathbf{B} \times \nabla B) \tag{2.27}$$

Mirroring force:

$$\mathbf{F}_z = -\frac{mv_\perp^2}{2B}\frac{\partial B}{\partial z}\hat{\mathbf{e}}_z \tag{2.31}$$

$$= -\frac{\mu}{\gamma}\frac{\partial B}{\partial z}\hat{\mathbf{e}}_z \tag{4.20}$$

Dipole field equations ($B_0 = 3.12 \times 10^{-5}$ T)

$$B_r = -2B_0\left(\frac{\mathrm{R_E}}{r}\right)^3 \cos\theta \tag{3.13}$$

$$B_\theta = -B_0\left(\frac{\mathrm{R_E}}{r}\right)^3 \sin\theta \tag{3.14}$$

$$|B| = B_0\left(\frac{\mathrm{R_E}}{r}\right)^3 \sqrt{(1 + 3\cos^2\theta)} \tag{3.15}$$

Equation for dipole field line:

$$r = R_0 \sin^2\theta \tag{3.17}$$

$$= R_0 \cos^2\lambda$$

Particle motion in geomagnetic field

Bounce period:

$$\tau_b(\mathrm{s}) = 0.117\left(\frac{R_0}{\mathrm{R_E}}\right)\frac{1}{\beta}[1 - 0.4635(\sin\alpha_{eq})^{3/4}] \tag{4.28}$$

Drift period:

$$\tau_d(\mathrm{s}) = C_d\left(\frac{\mathrm{R_E}}{R_0}\right)\frac{1}{\gamma\beta^2}[1 - 0.333(\sin\alpha_{eq})^{0.62}] \tag{4.47}$$

where

$$C_d = 1.557 \times 10^4 \text{ for electrons}$$

$$= 8.481 \text{ for protons}$$

Adiabatic invariants

$$J_1 = \mu = \frac{p_\perp^2}{2m_0B} \tag{4.13}$$

$$J_2 = J = \oint \mathbf{p} \cdot d\mathbf{s} \tag{4.31}$$

$$J_3 = \Phi = q\int_S \mathbf{B} \cdot d\mathbf{S} \tag{4.50}$$

Relationship of flux j(α, E) to phase space density F(q, p)

$$F(\mathbf{q}, \mathbf{p}) = j(\alpha, E)/p^2$$

Electric and magnetic field transformations

Lorentz transformations of electric and magnetic fields are given for a primed system moving at velocity **V** with respect to an unprimed system. Parallel quantities are measured parallel to **V**:

$$B'_{\parallel} = B_{\parallel} \qquad E'_{\parallel} = E_{\parallel}$$

$$\mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \frac{\mathbf{V} \times \mathbf{E}}{c^2}\right)_{\perp} \qquad \mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{V} \times \mathbf{B})_{\perp}$$

where

$$\gamma = (1 - V^2/c^2)^{-1/2}$$

In magnetospheric applications $V < 10^5\ \mathrm{m\,s^{-1}}$, $10^{-8}\ \mathrm{T} < \mathrm{B} < 10^{-5}\ \mathrm{T}$, $\mathrm{E} < 0.1\ \mathrm{vm^{-1}}$. Therefore, to an excellent approximation $\gamma = 1$, $B \gg \mathbf{V} \times \mathbf{E}/c^2$ and it is permissible to use

$$B'_{\parallel} = B_{\parallel} \qquad E'_{\parallel} = E_{\parallel}$$

$$\mathbf{B}'_{\perp} = \mathbf{B}_{\perp} \qquad \mathbf{E}'_{\perp} = \mathbf{E}_{\perp} + (\mathbf{V} \times \mathbf{B})$$

Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \cdot \mathrm{E} = \frac{\rho}{\varepsilon_0} - \frac{\nabla \cdot \boldsymbol{\rho}}{\varepsilon_0}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{i} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

where **i** is the current density, ρ is charge density and $\boldsymbol{\rho}$ is the polarization of the medium.

Vector identities

A, **B** and **C** are vector fields; d**S** is a surface element of **S**; d**l** is a line element of a contour around **S**; dV is an element of volume V:

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$$

$$\nabla \times (\nabla \psi) = 0 \qquad \nabla \cdot (\nabla \times \mathbf{F}) = 0$$

$$\iint_S (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} = \oint_c \mathbf{A} \cdot \mathrm{d}\mathbf{l}$$

$$\iiint_V (\nabla \cdot \mathbf{A})\, \mathrm{d}V = \iint_S \mathbf{A} \cdot \mathrm{d}\mathbf{S}$$

$$\mathbf{A} \cdot \nabla \mathbf{B} = (\mathbf{A} \cdot \nabla)\, \mathbf{B} = \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z}\right)\mathbf{B}$$

In spherical polar coordinates

$$\nabla \cdot \psi = \hat{\mathbf{e}}_r \frac{\partial \psi}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial \psi}{\partial \theta} + \hat{\mathbf{e}}_\phi \frac{1}{r \sin\theta}\frac{\partial \psi}{\partial \phi}$$

$$\nabla^2 \psi = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}\right]$$

Relativistic relations

$\mathbf{v}$ = velocity, Tm_0c^2 = kinetic energy, W = kinetic energy + rest mass energy

$$\beta = \frac{v}{c} \qquad \gamma = (1 - \beta^2)^{-1/2} = T + 1$$

$$m = m_0\gamma \qquad \mathbf{p} = m\mathbf{v} = m_0\gamma\mathbf{v}$$

$$W = (T + 1)\, m_0c^2 = (m_0^2c^4 + c^2p^2)^{1/2}$$

$$\beta^2 = \frac{T(T + 2)}{(T + 1)^2}$$