Introduction to Geomagnetically Trapped Radiation

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Chapter

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Charged particle motion in magnetic and electric fields

Introduction

The fundamental equation describing the motion of a charged particle in magnetic and electric fields is the Lorentz equation

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q(\mathbf{v} \times \mathbf{B} + \mathbf{E}) \tag{2.1}$$

For rationalized MKS units the force $\mathbf{F}$ is in newtons, the charge q is in coulombs, the electric field $\mathbf{E}$ is in volts m^{-1}, the velocity $\mathbf{v}$ is in $\mathrm{m\,s^{-1}}$ and the magnetic field $\mathbf{B}$ is in webers m^{-2} or tesla. A list of symbols used throughout this book is given in the list of symbols (pp. xiii–xvi).

For some simple field geometries, equation (2.1) can be integrated directly to give the trajectory of the particle. However, for the geomagnetic field such an integration is not possible, and one must resort to approximations. Fortunately, for radiation belt particles whose energy is so low that the magnetic field appears almost uniform, an efficient approximate theory has been developed. The results of this theory will be presented in stages in the following chapters. First, the motion of a charged particle in simplified magnetic and electric fields will be considered. This discussion will illuminate the fundamental reasons for particle trapping. It will be seen that in general the particle executes a rapid circular motion while at the same time the center of the circle moves through the electric and magnetic fields. Equations for the motion of this so-called 'guiding center' give a quantitative description of the motion of the guiding center and confirm the trapping properties of the geomagnetic field. In Chapter 4 the adiabatic invariance approximation is introduced. This theory describes the long-term trajectory of the guiding center, although it does not give the guiding center velocity or indicate where the guiding center will be at a given time.

In the first part of this chapter we will use equation (2.1) to obtain the particle motion in fields with simple geometries. The extension of this motion to the geomagnetic field will then be easy to understand. Equation (2.1) can be separated into components parallel and perpendicular to the magnetic field giving

$$\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_{\parallel} = q\mathbf{E}_{\parallel} \tag{2.2}$$

and

$$\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_{\perp} = q(\mathbf{v} \times \mathbf{B} + \mathbf{E}_{\perp}) \tag{2.3}$$

Uniform magnetic field

Assume **B** is uniform and constant and that **E** is zero. For these conditions equations (2.2) and (2.3) become

$$\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_{\parallel} = 0 \tag{2.2$'$}$$

$$\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_{\perp} = q(\mathbf{v} \times \mathbf{B}) = q(\mathbf{v}_{\perp} \times \mathbf{B}) \tag{2.3$'$}$$

Integrating (2.2′) gives

$$\mathbf{p}_{\parallel} = \text{constant}$$

indicating that the particle moves parallel to **B** at a constant speed. The momentum change in equation (2.3′) is perpendicular to $\mathbf{v}_{\perp}$. Therefore, $\mathbf{v}_{\perp}$ is constant in magnitude, and the trajectory is a circle of radius ρ when projected on to a plane perpendicular to **B**. The centrifugal force must balance the Lorentz force giving

$$\frac{mv_{\perp}^2}{\rho} = qv_{\perp}B$$

or

$$\rho = \frac{p_{\perp}}{Bq} \tag{2.4}$$

The radius ρ is an important parameter characterizing particle motion. It is frequently called the gyroradius or cyclotron radius. The angular frequency of the gyration motion, the gyrofrequency, is

$$\Omega = 2\pi\frac{v_{\perp}}{2\pi\rho} = \frac{Bq}{m}\ \text{radians s}^{-1} \tag{2.5}$$

Note that in the non-relativistic case (m = constant), Ω is independent of particle energy. Thus, in a uniform magnetic field with no electric field the particle describes a helix, the circular motion in the plane perpendicular to **B** being superimposed on a uniform motion parallel to **B**. The pitch angle of the helix is the angle between the particle velocity and the magnetic field and is given by $\alpha = \tan^{-1}(v_\perp/v_\parallel)$. Particles with large pitch angles near 90° move essentially in circles. If the pitch angle is near 0°, the helix is more open and the particle motion is predominently parallel to **B**.

The helical motion described above is the primary motion of trapped particles in the geomagnetic field because the non-uniformities in the field are small over distances the length of the gyroradius. However, even weak gradients in the geomagnetic field introduce deviations in the particle motion, and these deviations lead to particle trapping.

Uniform magnetic and electric fields

If $\mathbf{E}_\parallel$ is constant, the parallel equation (2.2) leads to uniform acceleration along a field line

$$\mathbf{p}_\parallel(t) = \mathbf{p}_\parallel(t = 0) + q\mathbf{E}_\parallel t \tag{2.6}$$

Such parallel fields are rarely found in the trapping region of the magnetosphere, although they are important in accelerating particles in the aurora.

Moderate electric fields perpendicular to a uniform **B** result in a drift motion perpendicular to both **B** and **E** (Figure 2.1). This effect can be

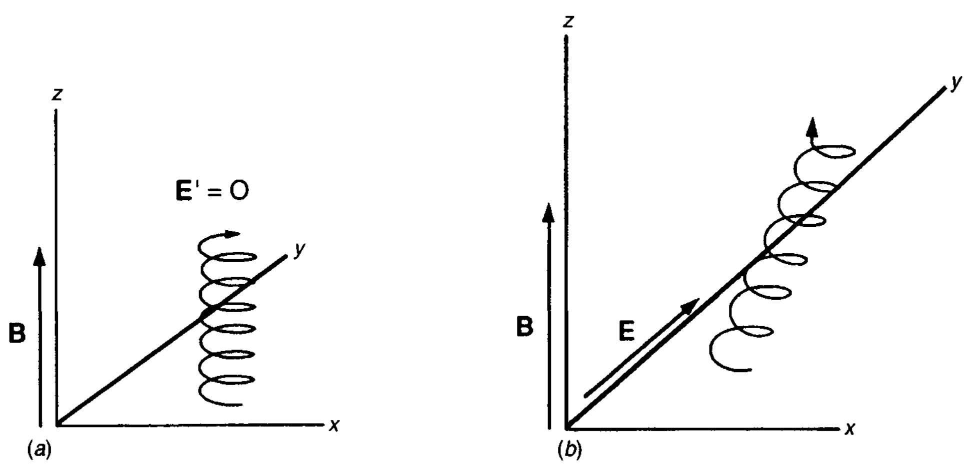

Figure 2.1. Motion of a charged particle in perpendicular electric and magnetic fields. (*a*) Particle motion as observed in a frame of reference moving in the *x* direction with velocity $\mathbf{V}_E = \mathbf{E} \times \mathbf{B} / B^2$ such that $\mathbf{E}' = 0$ in the moving frame. (*b*) Particle motion as observed in a stationary frame of reference in which an electric field **E** is present.

understood most easily by using a Lorentz coordinate transformation to eliminate the electric field (see Appendix A).

Let $\mathbf{B} = B\hat{\mathbf{e}}_z$, $\mathbf{E} = E\hat{\mathbf{e}}_y$. If the primed quantities denote values measured in a reference frame moving at some arbitrary velocity $\mathbf{V}$ perpendicular to $\mathbf{B}$, then the electric field in the moving system is

$$\mathbf{E}' = \mathbf{E} + \mathbf{V} \times \mathbf{B} \tag{2.7}$$

To eliminate the electric field in the moving frame, $\mathbf{V}$ is chosen such that $\mathbf{E}' = 0$. The vector product of (2.7) and $\mathbf{B}$ (setting $\mathbf{E}' = 0$) gives

$$\begin{aligned} 0 &= \mathbf{B} \times \mathbf{E} + \mathbf{B} \times (\mathbf{V} \times \mathbf{B}) \\ &= \mathbf{B} \times \mathbf{E} + (\mathbf{B} \cdot \mathbf{B})\mathbf{V} - (\mathbf{B} \cdot \mathbf{V})\mathbf{B} \end{aligned}$$

Because $\mathbf{B} \cdot \mathbf{V} = 0$, the required frame velocity is

$$\mathbf{V} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \equiv \mathbf{V}_{\mathrm{E}} \tag{2.8}$$

In a frame moving at velocity $\mathbf{V}_{\mathrm{E}}$ the electric field vanishes and the particle executes the helical motion described earlier. In a stationary frame the motion is a deformed gyromotion drifting at velocity $\mathbf{V}_{\mathrm{E}}$ in the x direction. The reason for the drift can be traced to a distortion of the circular gyromotion by the electric field.

In its gyromotion a positive particle has greatest energy and largest gyroradius when it is at maximum excursion in the y direction (Figure 2.2). Viewed in the $x-y$ plane the trajectory accumulates displacement in

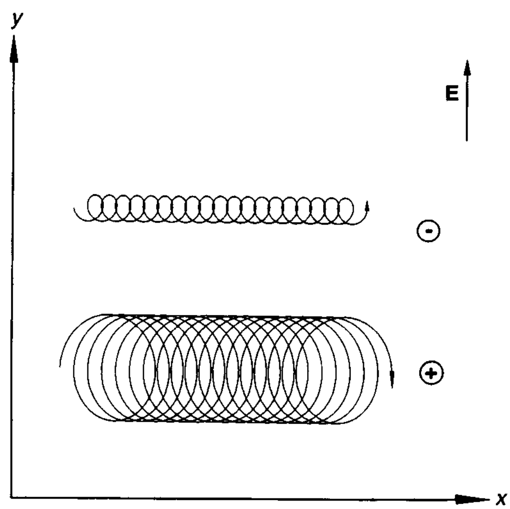

Figure 2.2. Explanation of $\mathbf{E} \times \mathbf{B}/B^2$ drift mechanism. Radius of curvature increases as particle kinetic energy is increased.

the x direction. A negative particle circles in the opposite sense and has its largest gyroradius while at minimum y, thus drifting in the positive x direction also. As is apparent from equation (2.8), all charged particles drift in the $\mathbf{E} \times \mathbf{B}$ direction with a velocity depending only on $\mathbf{E}$ and $\mathbf{B}$ and independent of particle charge, mass or velocity. Note also that the drift is perpendicular to $\mathbf{E}$ so that, on average, no energy is gained or lost during the drift.

Equation (2.8) is valid as long as $|\mathbf{V}_E|/c \ll 1$. If the electric field is so large that $|\mathbf{V}_E|/c$ is appreciable, relativistic equations must be used to calculate the particle motion and the description used here does not apply. In the Earth's magnetosphere electric fields are never so large that (2.8) cannot be used.

Inhomogeneous magnetic field

The most interesting effects from the standpoint of trapping occur when $\mathbf{B}$ is not uniform. Even for electrons and protons of many Mev energy, $\rho \ll R_E$, and the geomagnetic field experienced by the particle during a gyration is almost uniform. Nevertheless the slight deviations from helical motion which are produced by ∇B accumulate over time and lead to important perturbations in the otherwise helical motion of the particle.

One is generally not interested in the individual gyrations of the particle but wishes to follow its path over an extended trajectory very much larger than the gyroradius. This motivation leads to the concept of a 'guiding center' in which one separates the particle behavior into the circular motion about the 'guiding center' and the motion of the guiding center itself. The derivation of the equations of motion for the guiding center is sketched here for $\mathbf{E} = 0$, $\partial\mathbf{B}/\partial t = 0$ and for non-relativistic particles, as this case will illustrate the approximations involved. For the more general case and for more details see Northrop, 1963.

Express the position $\mathbf{r}$ of a particle in terms of its instantaneous gyroradius $\boldsymbol{\rho}$ and the center of gyration $\mathbf{R}$. Thus $\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}$. Expand the magnetic field in the vicinity of $\mathbf{R}$ in a Taylor series about $\mathbf{R}$

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{R}) + \boldsymbol{\rho} \cdot \nabla\mathbf{B}(\mathbf{R}) + \ldots \tag{2.9}$$

where

$$\boldsymbol{\rho} \cdot \nabla\mathbf{B} = \left(\rho_x \frac{\partial}{\partial x} + \rho_y \frac{\partial}{\partial y} + \rho_z \frac{\partial}{\partial z}\right)\mathbf{B}$$

Substitute (2.9) into (2.1), with $\mathbf{E} = 0$ and denote the time derivatives by dots above the quantity.

$$m(\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}) = q(\dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}) \times [\mathbf{B}(\mathbf{R}) + \boldsymbol{\rho} \cdot \nabla \mathbf{B}(\mathbf{R}) + \ldots] \tag{2.10}$$

The basic assumption that $\rho(|\nabla \mathbf{B}|/B) \ll 1$ allows one to neglect the higher-order terms in the Taylor expansion. Let $\hat{\mathbf{e}}_1$ be a unit vector in the direction of the magnetic field at $\mathbf{R}$; the unit vectors $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ then form an orthogonal coordinate system such that $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3$ (see Figure 2.3). The gyroradius $\boldsymbol{\rho}$ will be in the $\hat{\mathbf{e}}_2$–$\hat{\mathbf{e}}_3$ plane and can be expressed as

$$\boldsymbol{\rho} = \rho(\hat{\mathbf{e}}_2 \sin \Omega t + \hat{\mathbf{e}}_3 \cos \Omega t) \tag{2.11}$$

Repeated differentiations with respect to time give

$$\dot{\boldsymbol{\rho}} = \Omega\rho(\hat{\mathbf{e}}_2 \cos \Omega t - \hat{\mathbf{e}}_3 \sin \Omega t) + \sin \Omega t \frac{\mathrm{d}}{\mathrm{d}t}(\rho \hat{\mathbf{e}}_2) + \cos \Omega t \frac{\mathrm{d}}{\mathrm{d}t}(\rho \hat{\mathbf{e}}_3) \tag{2.12}$$

$$\begin{aligned} \ddot{\boldsymbol{\rho}} = {} & \Omega^2\rho(-\hat{\mathbf{e}}_2 \sin \Omega t - \hat{\mathbf{e}}_3 \cos \Omega t) + \dot{\Omega}\rho(\hat{\mathbf{e}}_2 \cos \Omega t - \hat{\mathbf{e}}_3 \sin \Omega t) \\ & + 2\Omega \cos \Omega t \frac{\mathrm{d}}{\mathrm{d}t}(\rho \hat{\mathbf{e}}_2) - 2\Omega \sin \Omega t \frac{\mathrm{d}}{\mathrm{d}t}(\rho \hat{\mathbf{e}}_3) + \sin \Omega t \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho \hat{\mathbf{e}}_2) \\ & + \cos \Omega t \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho \hat{\mathbf{e}}_3) \end{aligned} \tag{2.13}$$

Equations (2.11), (2.12) and (2.13) for $\boldsymbol{\rho}$, $\dot{\boldsymbol{\rho}}$ and $\ddot{\boldsymbol{\rho}}$ are now substituted into equation (2.10) and the resulting equation is averaged over time, integrating over a complete cyclotron period with t going from 0 to $2\pi/\Omega$. Because

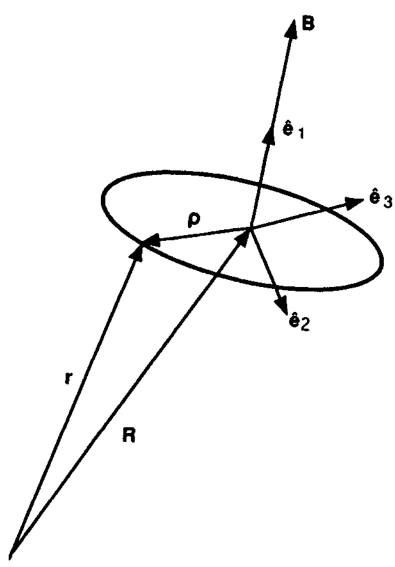

Figure 2.3. Diagram defining vector coordinate system for particle gyration in an inhomogeneous magnetic field.

all terms in $\boldsymbol{\rho}$, $\dot{\boldsymbol{\rho}}$ and $\ddot{\boldsymbol{\rho}}$ contain either $\sin \Omega t$ or $\cos \Omega t$ as factors, the averages of these quantities are zero:

$$\langle \boldsymbol{\rho} \rangle = \langle \dot{\boldsymbol{\rho}} \rangle = \langle \ddot{\boldsymbol{\rho}} \rangle = 0$$

After time averaging, equation (2.10) becomes

$$m\ddot{\mathbf{R}} = q[\dot{\mathbf{R}} \times \mathbf{B}(\mathbf{R})] + q\frac{\rho^2\Omega}{2}[\hat{\mathbf{e}}_2 \times (\hat{\mathbf{e}}_3 \cdot \nabla)\mathbf{B} - \hat{\mathbf{e}}_3 \times (\hat{\mathbf{e}}_2 \cdot \nabla)\mathbf{B}] + \ldots \tag{2.14}$$

Additional, somewhat tedious, vector algebra reduces this expression to

$$m\ddot{\mathbf{R}} = q[\dot{\mathbf{R}} \times \mathbf{B}(\mathbf{R})] - q\frac{\rho^2\Omega}{2}\nabla B + \ldots \tag{2.15}$$

where B is the magnitude of the magnetic field.

Equation (2.15) is the basic equation of motion for the guiding center. The higher-order terms which have been neglected are generally not important for radiation belt studies, and these additional terms will not be indicated in subsequent equations. However, it is as well to recognize that the equations derived here and on the pages immediately following contain approximations which become less valid as the gyration radius increases. The more useful forms of equation (2.15) are obtained by separating the equation into perpendicular and parallel components. The perpendicular component is obtained by taking the vector product of (2.15) with $\hat{\mathbf{e}}_1$:

$$m\ddot{\mathbf{R}} \times \hat{\mathbf{e}}_1 = q(\dot{\mathbf{R}} \times \mathbf{B}) \times \hat{\mathbf{e}}_1 - \frac{q\rho^2\Omega}{2}\nabla B \times \hat{\mathbf{e}}_1$$

$$= q\{(\hat{\mathbf{e}}_1 \cdot \dot{\mathbf{R}})\hat{\mathbf{e}}_1 B - B\dot{\mathbf{R}}\} - \frac{q\rho^2\Omega}{2}\nabla B \times \hat{\mathbf{e}}_1$$

or

$$Bq\{\dot{\mathbf{R}} - (\hat{\mathbf{e}}_1 \cdot \dot{\mathbf{R}})\hat{\mathbf{e}}_1\} = Bq\dot{\mathbf{R}}_\perp = m(\hat{\mathbf{e}}_1 \times \ddot{\mathbf{R}}) + \frac{q\rho^2\Omega}{2}\hat{\mathbf{e}}_1 \times \nabla B$$

Hence

$$\dot{\mathbf{R}}_\perp = \frac{m}{Bq}(\hat{\mathbf{e}}_1 \times \ddot{\mathbf{R}}) + \frac{\rho^2\Omega}{2B}\hat{\mathbf{e}}_1 \times \nabla B \tag{2.16}$$

To the approximation required here,

$$\ddot{\mathbf{R}} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{R}}_\perp + \dot{\mathbf{R}}_\parallel) \approx \frac{\mathrm{d}\dot{\mathbf{R}}_\parallel}{\mathrm{d}t} = \hat{\mathbf{e}}_1\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} + v_\parallel^2\frac{\partial \hat{\mathbf{e}}_1}{\partial s} \tag{2.17}$$

where s is the distance measured along the field line, which need not be straight. With this expression for $\ddot{\mathbf{R}}$ inserted into (2.16) the perpendicular velocity then becomes

$$\dot{\mathbf{R}}_\perp = \hat{\mathbf{e}}_1 \times \left(\frac{\rho^2 \Omega}{2B} \nabla B + \frac{m}{Bq} v_\parallel^2 \frac{\partial \hat{\mathbf{e}}_1}{\partial s} \right)$$

$$= \hat{\mathbf{e}}_1 \times \left(\frac{m v_\perp^2}{2qB^2} \nabla B + \frac{m}{Bq} v_\parallel^2 \frac{\partial \hat{\mathbf{e}}_1}{\partial s} \right) \tag{2.18}$$

For obvious reasons the first term in (2.18) is called the gradient drift and the second term the curvature drift. A more transparent interpretation of these quantities will be given shortly.

The parallel component of equation (2.15) is extracted by forming the scalar product with $\hat{\mathbf{e}}_1$.

$$m\ddot{\mathbf{R}} \cdot \hat{\mathbf{e}}_1 = -\frac{q\rho^2\Omega}{2} (\nabla B) \cdot \hat{\mathbf{e}}_1$$

or

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = -\frac{1}{2} \frac{v_\perp^2}{B} (\nabla B)_\parallel \tag{2.19}$$

Equation (2.19) shows that for motion parallel to $\mathbf{B}$ the guiding center of a particle is accelerated in a direction opposite to the gradient of the magnetic field. If the particle is moving into a stronger field, it will be repelled, regardless of the sign of the particle's charge or the direction of the magnetic field.

Equations (2.8), (2.18) and (2.19) give the guiding center drifts of primary interest to radiation belt physics. As mentioned before, they contain approximations which may become important as the particle energy and gyration radius increases. In particular, the equation for parallel motion (2.19) is less exact than the equation for guiding center motion perpendicular to the magnetic field (2.18). Whenever these equations are used together when numerically tracking a particle trajectory, it is necessary to use a more accurate version of (2.19).

Additional terms neglected in equation (2.17) may be important if there are large electric fields or if the magnetic field changes direction with time.

Simple, physical interpretations for the gradient and curvature drifts of equation (2.18) and the 'mirroring' forces in (2.19) can be given. These drifts are analogous to the electric field drift in that the gyroradius ρ varies during the circular motion. This effect can be seen in Figure 2.4 where ∇B is in the y direction and $\mathbf{B}$ is in the $-\hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_y$ direction. The trajectory of a positive particle illustrates how the smaller gyroradius at larger y (and larger B) leads to a drift in the $\mathbf{B} \times \nabla B$ direction. The magnitude of the drift can be estimated directly as follows. Because the trajectory is symmetric about a line parallel to the y axis and passing through the point

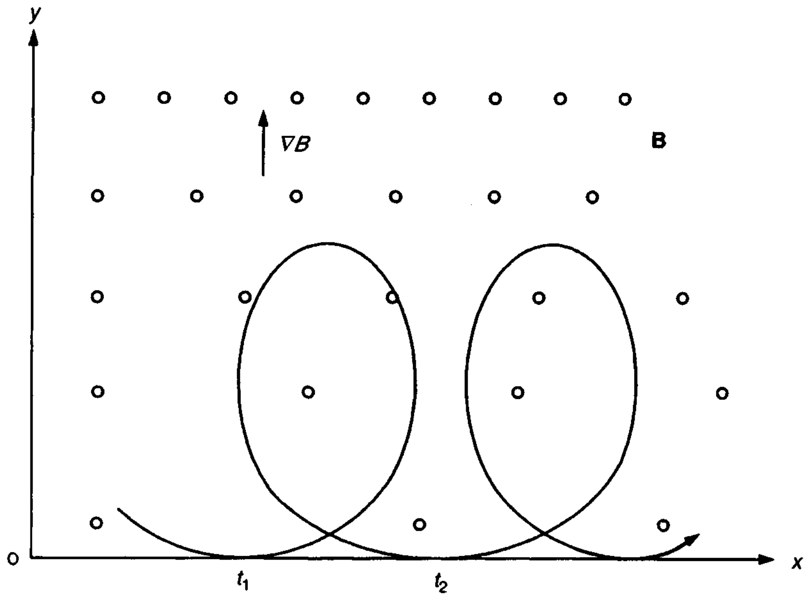

Figure 2.4. Drift motion perpendicular to **B** and to ∇B in an inhomogeneous magnetic field.

where the trajectory crosses itself, there is no net motion in the y direction. Hence, averaging the force over a gyration period should result in no net force in the y direction. The times t_1 and t_2 denote minimum y positions (taken as $y = 0$) and therefore the start and end of a cycle. If F_y is the force in the y direction,

$$\int_{t_1}^{t_2} F_y \, \mathrm{d}t = 0 = \int_{t_1}^{t_2} q \frac{\mathrm{d}x}{\mathrm{d}t} B(y) \, \mathrm{d}t \tag{2.20}$$

Because $B(y)$ does not change appreciably in a gyroradius, $B(y)$ can be approximated by the first two terms of a Taylor series:

$$B(y) = B_0 + y \frac{\partial B}{\partial y} \tag{2.21}$$

where B_0 is the value of the field at $y = 0$ and $\partial B/\partial y$ is a constant. Equation (2.21) is substituted into equation (2.20), giving

$$0 = q \int_{t_1}^{t_2} \frac{\mathrm{d}x}{\mathrm{d}t} B_0 \, \mathrm{d}t + q \int_{t_1}^{t_2} \frac{\mathrm{d}x}{\mathrm{d}t} y \frac{\partial B}{\partial y} \, \mathrm{d}t$$

Therefore,

$$\int_{x(t_1)}^{x(t_2)} \mathrm{d}x = -\frac{1}{B_0} \frac{\partial B}{\partial y} \int_{x(t_1)}^{x(t_2)} y \, \mathrm{d}x \tag{2.22}$$

The right-hand integral is the negative of the area enclosed by the curve, and if the drift in a gyroperiod is small, this area is equal to $\pi\rho^2$. The distance traveled in the x-direction during one gyration is therefore

$$\Delta x = \frac{1}{B_0}\frac{\partial B}{\partial y}\pi\rho^2$$

and the time Δt required for this cycle is $2\pi\rho/v_\perp$. Therefore, with appropriate substitutions and setting $B_0 = B$ the gradient drift term is

$$\mathbf{V}_G = \frac{\Delta \mathbf{x}}{\Delta t} = \frac{mv_\perp^2}{2qB^3}(\mathbf{B} \times \nabla B) \tag{2.23}$$

in agreement with the first term in equation (2.18).

Note that the gradient drift term is in a direction perpendicular to **B** and to ∇B. Hence, this drift will carry particles along a line of constant B. This characteristic will be useful later in tracing the drift paths of particles near the Earth's equatorial plane. In contrast to the electric field drift, the gradient drift velocity depends on the particle energy and charge. In the non-relativistic case the gradient drift velocity is proportional to the perpendicular energy. Negative particles and positive particles drift in opposite directions. The drifts therefore produce electric currents, even in neutral plasmas.

The curvature drift term (the second term in equation (2.18)) depends on the magnetic field changing direction with distance s along the field line. A heuristic derivation of this term follows from the assumptions that the guiding center 'almost' follows a field line and the field line curvature therefore exerts a centrifugal force on the particle. The force is perpendicular to **B** and lies in the plane of curvature. The geometry is given in Figure 2.5, where **n** is a unit vector in the direction of the radius of curvature. The guiding center motion parallel to **B** exerts a centrifugal force

$$\mathbf{F} = \frac{mv_\parallel^2}{R_c}\mathbf{n} \tag{2.24}$$

where R_c is the radius of curvature of the field line. The force on the particle is equivalent to that from an electric field of magnitude $\mathbf{E}_c = mv_\parallel^2\mathbf{n}/qR_c$. Such an electric field results in a drift velocity

$$\mathbf{V}_c = \frac{\mathbf{E}_c \times \mathbf{B}}{B^2} = \frac{mv_\parallel^2}{qR_c} \cdot \frac{\mathbf{n} \times \mathbf{B}}{B^2} \tag{2.25}$$

This result is the same as the last term in (2.18) because

$$\frac{\partial \hat{\mathbf{e}}_1}{\partial s} = \frac{\partial \hat{\mathbf{e}}_1}{\partial \theta} \cdot \frac{\partial \theta}{\partial s} = -\mathbf{n}\left(\frac{1}{R_c}\right)$$

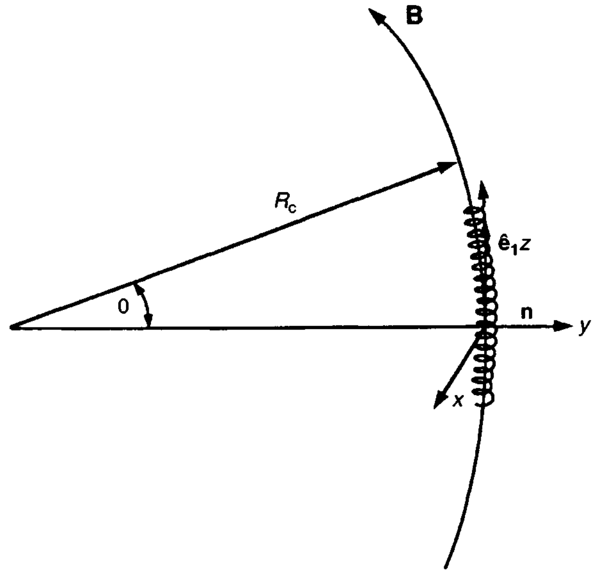

Figure 2.5. Geometry for curvature drift calculation. Drift is perpendicular to **B** and to the field line radius of curvature R_c.

If the region of space under consideration does not contain electric currents, a more convenient expression for $-\mathbf{n}/R_c$ can be derived. With the geometry shown in Figure 2.5 and utilizing $\partial B_z/\partial y = \partial B_y/\partial z$ obtained from $\nabla \times \mathbf{B} = 0$ (only valid if $\mathbf{J} = 0$ and $\partial \mathbf{E}/\partial t = 0$),

$$\nabla_\perp B = \mathbf{n}\frac{\partial B_z}{\partial y} = \mathbf{n}\frac{\partial B_y}{\partial z} = -\mathbf{n}\frac{B}{R_c} \tag{2.26}$$

The curvature drift term in (2.18) thus reduces to

$$\mathbf{V}_c = \frac{mv_\parallel^2}{qB^3}(\mathbf{B} \times \nabla B) \quad \text{if } \nabla \times \mathbf{B} = 0 \tag{2.27}$$

Note the similarity between equations (2.23) and (2.27) in that both drifts are in the same direction and have the same dependence on B and q. They differ, however, in their pitch-angle dependence. Particles with large pitch angles respond primarily to the gradient drift, while the curvature term is more important for particles with large $v_\parallel$.

In the parallel motion equation ((2.19)) the effect of the gradient parallel to **B** also has a simple interpretation. With the geometry of Figure 2.6 the magnetic field is in the z direction with a gradient in the $-z$ direction. A particle executing a circle about the z axis will experience a small component of **B** parallel to its gyroradius. When the particle crosses

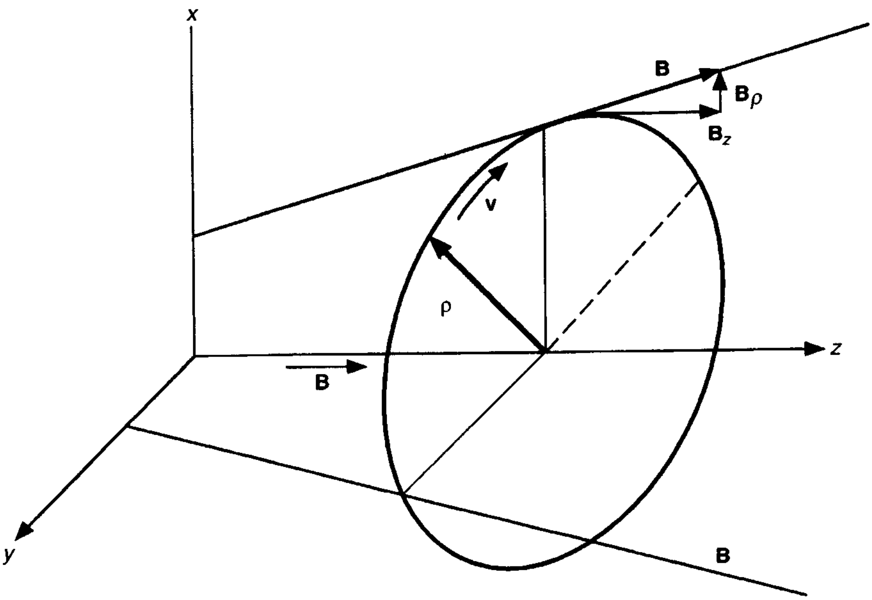

Figure 2.6. Mirror effect produced by a converging magnetic field. Gyrating particle senses a magnetic field component in the ρ-direction which deflects the particle away from the ∇B direction.

the $y = 0$ plane, the component of **B** in the ρ or x direction is

$$B_\rho = \rho \frac{\partial B_x}{\partial x} \tag{2.28}$$

At $x = 0$

$$B_\rho = \rho \frac{\partial B_y}{\partial y} \tag{2.29}$$

Since B_ρ must be constant around the particle orbit

$$B_\rho = \frac{\rho}{2}\left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right] = -\frac{\rho}{2}\frac{\partial B_z}{\partial z} \tag{2.30}$$

where use is made of the Maxwell equation $\nabla \cdot \mathbf{B} = 0$.

The force in the z direction will be given by

$$\begin{aligned} \mathbf{F}_z &= q(\mathbf{v} \times \mathbf{B}_\rho) = -qv_\perp \frac{\rho}{2}\frac{\partial B_z}{\partial z}\hat{\mathbf{e}}_z \\ &= -\frac{mv_\perp^2}{2B}\frac{\partial B}{\partial z}\hat{\mathbf{e}}_z \end{aligned} \tag{2.31}$$

and since $\partial B/\partial z < 0$ is negative, the force is in the positive z direction. Because the force tends to reflect a particle out of a region with high magnetic field, it is called a mirroring force, and a region of high magnetic field is called a magnetic mirror. The force is independent of charge; both positive and negative particles will be reflected. Furthermore, the magnitude of the electric charge does not enter the expression for $\mathbf{F}_z$, the charge dependence of the Lorentz force being exactly canceled by the charge dependence of ρ. Particles with smaller q will have larger ρ and experience a larger B_ρ to compensate for the lower q in (2.31).

The mirroring force is also independent of the direction of B and independent of $v_\parallel$. The change in $v_\parallel$ due to the mirroring force will also affect $v_\perp$ since $v^2 = v_\perp^2 + v_\parallel^2$ is a constant of motion in the presence of magnetic forces only.

Equations (2.8), (2.23) and (2.27) can be combined to give the drift velocity perpendicular to the magnetic field.

$$\mathbf{V}_\perp = \mathbf{V}_E + \mathbf{V}_G + \mathbf{V}_c = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{mv_\perp^2}{2qB^3}(\mathbf{B} \times \nabla B) + \frac{mv_\parallel^2}{qB^2}\left(\mathbf{B} \times \frac{\partial \hat{\mathbf{e}}_1}{\partial s}\right) \tag{2.32}$$

When $\nabla \times \mathbf{B} = 0$,

$$\mathbf{V}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{m}{qB^3}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\mathbf{B} \times \nabla B \tag{2.33}$$

The drift and mirror equations derived in this chapter are the essential elements which lead to particle trapping in the Earth's magnetic field. Although other effects are important, such as time variations of the electric and magnetic fields, the three magnetic effects of gradient drift, curvature drift and mirroring are the primary controlling factors leading to long-term trapping. The electric field drift term, which applies equally to all particles, is of most interest for low-energy particles, for which the magnetic drift terms are smaller. Because the electric field drift is in a direction perpendicular to $\mathbf{E}$, the particle will move along an equipotential surface and thus conserve energy. However, if magnetic curvature or gradient drifts are also present these forces will in general carry the particle across electric equipotentials and alter the particle energy.

The drift terms derived here allow one to understand geomagnetic trapping. The scale size of the magnetosphere is so large compared to the gyroradii of trapped particles that the magnetic field experienced by the particle during a gyration is almost uniform. Thus, an energetic particle introduced into the geomagnetic field circles about the field direction while moving parallel to the field line. The parallel motion will take the particle towards the poles of the Earth, where the increased magnetic field

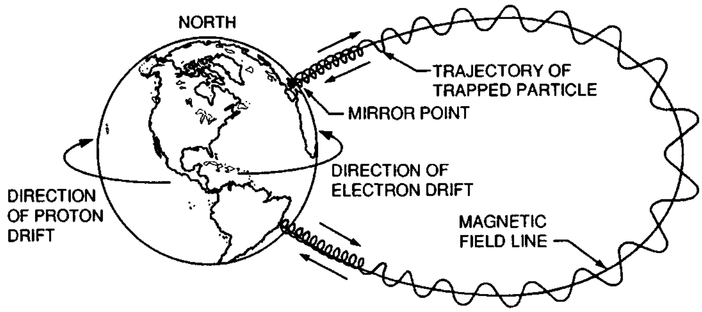

Figure 2.7. Trajectory of trapped electrons and protons experiencing magnetic mirroring and gradient and curvature drifts in the geomagnetic field.

intensity causes the particle to be reflected. The bounce motion between mirrors in the polar regions is superimposed on the much slower curvature and gradient drifts, which are perpendicular to the magnetic field, and for ($\nabla \times \mathbf{B} = 0$) are perpendicular to the gradient of B in the plane perpendicular to the magnetic field. If the Earth's field were symmetric about the polar axis, these drifts would be entirely in the longitudinal direction. However, distortion in the geomagnetic field alters this simple result, and the drifts will have components in the latitude and altitude directions. For the Earth, the gradient and curvature drifts are eastward for electrons and westward for protons. The overall motion is sketched in Figure 2.7. Note, however, that for clarity the size of the gyroradius is greatly exaggerated in this diagram.

Problems

1. An electron of rest mass m_0 moving at velocity v has a total energy $W = \gamma m_0 c^2$, where $\gamma = 1/\sqrt{(1-\beta^2)}$ and $\beta = v/c$. If T is the kinetic energy in rest mass units, show that

$$\beta = \frac{\sqrt{(T(T+2))}}{T+1}$$

   Find the velocity of a 5.1 keV electron, a 51 keV electron, a 510 keV electron and a 94 MeV proton.

2. The guiding center expression for the gradient drift velocity is valid only if $\rho|\nabla B|/B \ll 1$. Assuming that the Earth's magnetic field in the equatorial plane is $B(r) = 3 \times 10^{-5}(R_E/r)^3$ tesla, where $R_E = 6.37 \times 10^3$ km, find the value of r at which $\rho|\nabla B|/B = 0.1$ for a 94 MeV proton moving $\perp$ to $\mathbf{B}$ in the equatorial plane.

3. A magnetic field parallel to the z axis varies as $B(z) = B_0 \exp(az)$. A proton of velocity v moving in the $+z$ direction starts at $t = 0$ with a pitch angle of 45°. Using the equation for $F_{\parallel}$ find the time it takes for the particle to be reflected. Express the answer in terms of v.

4. A low-energy proton with energy of 0.1 eV is above the atmosphere in the equatorial plane of the Earth. Its velocity is entirely in the direction perpendicular to the magnetic field. Neglecting the curvature and gradient B magnetic drifts, find the direction and magnitude of the drift caused by the Earth's gravitational field. Assume a flat Earth, and neglect the variation of gravity with altitude. The geomagnetic field points towards the north and has a magnitude of 3×10^{-5} tesla.

5. A particle of mass m and charge q is at rest in a uniform magnetic field $\mathbf{B}$. At $t = 0$ a uniform electric field perpendicular to $\mathbf{B}$ is switched on. Transform to a moving coordinate system to remove $\mathbf{E}$ and describe the motion of the particle in the moving coordinate system. By transforming back to the original frame show that the maximum energy that the particle can acquire is $2m(E/B)^2$.