Introduction to Geomagnetically Trapped Radiation

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Chapter

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Adiabatic invariants

Introduction

The guiding center equations of motion developed in Chapter 3 are an enormous improvement over the Lorentz force equation for describing the long-term behavior of particles in inhomogeneous magnetic and electric fields. When applied to the Earth's field, they clarify the cause of magnetic trapping. However, the drift and mirroring force equations do not allow long-range prediction of particle location, particularly in fields without axial symmetry. For example, without numerically integrating the guiding center equations over many bounces and over many degrees of longitudinal drift, a procedure likely to introduce errors, it is not possible to predict where a particle launched on a field line over Africa will be when it has drifted over the United States.

Missing in the theory described thus far are 'constants of motion' analogous to the conservation of energy, momentum and angular momentum in mechanical systems. Adiabatic invariants fill the role of the required constants of motion, and their use is essential in research on trapped radiation.

Fortunately, in mechanical systems undergoing periodic motion in which the forces change very slightly over a period, approximate constants do exist. These are called 'adiabatic invariants', implying that their values are constant provided the forces directing the motion are altered infinitely slowly. The concept of an adiabatic invariant is illustrated in the following heuristic example of an frictionless particle confined in a potential well whose shape undergoes slow variations with time. Let the one-dimensional potential well be described by curve 1 in Figure 4.1, where $V(x)$ is the potential energy as a function of the spatial variable x. A particle with total energy $\varepsilon = V(a) = V(b)$ will then oscillate between turning points a

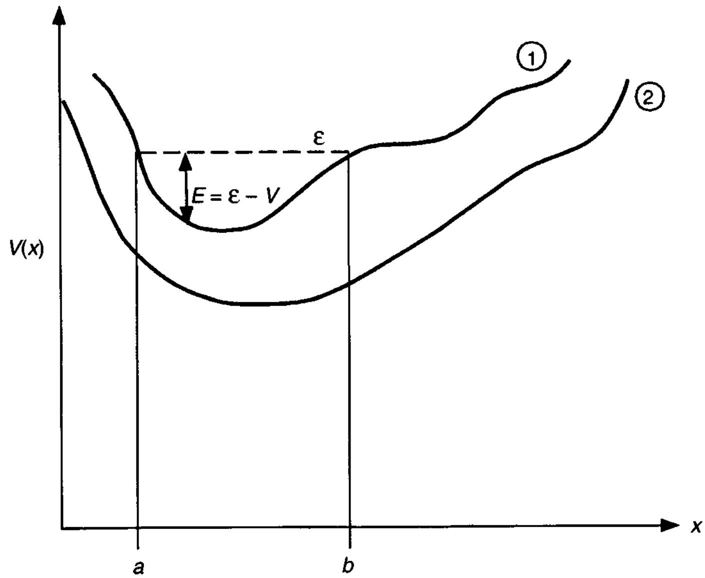


Figure 4.1. Frictionless particle confined in a potential well in which the shape of the potential changes slowly from curve 1 to curve 2.

and b, its kinetic energy at any point in its periodic motion being $E = \varepsilon - V(x)$.

The equation of motion of the particle is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{\mathrm{d}V}{\mathrm{d}x} \tag{4.1}$$

The total energy ε and velocity are

$$\left.\begin{aligned} \varepsilon &= \tfrac{1}{2}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + V(x) \\ v(x) &= \sqrt{(2E/m)} \end{aligned}\right\} \tag{4.2}$$

Suppose, now, that the shape of the potential well changes slowly with time, the change being very small during a single period of the particle's motion. However, the cumulative change can be large, eventually altering the shape of the well to curve 2 in Figure 4.1. Since the moving walls of the well can either add or subtract energy from the particle, the total particle energy, kinetic plus potential, is not a constant. The issue is, then, what will the energy of the particle be when the shape of the well is at curve 2? Are there other quantities which remain almost constant during the variations in $V(x)$? It will be seen that the classical action integrals fill this role.

Let the potential function be parameterized with a time variation as $V(x, a(t))$. As mentioned above, ε will not be constant, but at any time t, the average energy during a complete cycle can be defined as

$$\bar{\varepsilon} = \oint \mathrm{d}t \left[\frac{m}{2} \left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 + V(x, a(t)) \right] \Big/ \oint \mathrm{d}t \tag{4.3}$$

The classical action variable is defined in terms of an integral of the momentum $p(x)$ over a complete cycle

$$J = \oint p(x)\,\mathrm{d}x = \oint \sqrt{[2m(\bar{\varepsilon} - V(x, a))]}\,\mathrm{d}x \tag{4.4}$$

and its time derivative is

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \frac{\partial J}{\partial \bar{\varepsilon}} \frac{\mathrm{d}\bar{\varepsilon}}{\mathrm{d}t} + \frac{\partial J}{\partial a} \frac{\mathrm{d}a}{\mathrm{d}t} \tag{4.5}$$

But

$$\frac{\partial J}{\partial \bar{\varepsilon}} = \oint \frac{\mathrm{d}x}{\sqrt{[2(\bar{\varepsilon} - V)/m]}} = \oint \frac{\mathrm{d}x}{v} = \oint \mathrm{d}t \tag{4.6}$$

and

$$\begin{aligned} \frac{\mathrm{d}\bar{\varepsilon}}{\mathrm{d}t} &= \oint \mathrm{d}t \left[\left(m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\partial V}{\partial x} \right) \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial V}{\partial a} \frac{\mathrm{d}a}{\mathrm{d}t} \right] \Big/ \oint \mathrm{d}t \\ &= \oint \mathrm{d}t \frac{\partial V}{\partial a} \frac{\mathrm{d}a}{\mathrm{d}t} \Big/ \oint \mathrm{d}t \quad \text{using (4.1)} \end{aligned} \tag{4.7}$$

Also,

$$\frac{\partial J}{\partial a} = \oint \frac{-\mathrm{d}x}{\sqrt{[2(\bar{\varepsilon} - V)/m]}} \frac{\partial V}{\partial a} = -\oint \frac{\mathrm{d}x}{v} \frac{\partial V}{\partial a} \tag{4.8}$$

Therefore, substituting (4.6), (4.7) and (4.8) into (4.5) gives

$$\begin{aligned} \frac{\mathrm{d}J}{\mathrm{d}t} &= \oint \mathrm{d}t \frac{\partial V}{\partial a} \frac{\mathrm{d}a}{\mathrm{d}t} - \oint \frac{\mathrm{d}x}{v} \frac{\partial V}{\partial a} \frac{\mathrm{d}a}{\mathrm{d}t} \\ &= 0 \end{aligned} \tag{4.9}$$

This example indicates that slow variations in the shape of the potential well do not change the value of J. However, the proof is not rigorous and several questions remain. For example, variations in the end points of the integral as $V(x, a)$ is changed have been ignored, and no estimate has been made of the error introduced by using an average $\bar{\varepsilon}$ for energy in defining the action integral. Nevertheless, the example suggests that J should remain nearly constant, even if large changes occur in V and $\bar{\varepsilon}$, provided that $a(t)$ is nearly constant during a complete period of the motion. A more rigorous analysis confirms that J is an adiabatic invariant and remains nearly constant for large alterations in $V(x)$ if these changes are made infinitely slowly.

The more formal Hamilton–Jacobi theory defines action-angle variables for periodic motion. If p_i and q_i are the momenta and its conjugate coordinate, then $J_i = \oint p_i \, \mathrm{d}q_i$, where the integral is taken over the periodic orbit.

In the case of a charged particle in a magnetic field, the canonical momentum **P** is

$$\mathbf{P} = m\mathbf{v} + q\mathbf{A} = \mathbf{p} + q\mathbf{A} \tag{4.10}$$

where **A** is the vector potential of the magnetic field. The adiabatic invariants of the particle motion are then given by integrals of **P** over the appropriate periodic orbits. For charged particles in the geomagnetic field, three periodicities are readily apparent. These cycles correspond to the rapid gyration about the field lines, the north–south oscillation between magnetic mirroring points and the slow longitudinal drift about the Earth. Calculating the action integral associated with each of these periodicities leads to the three adiabatic invariants of the particle motion.

First adiabatic invariant

The so-called first adiabatic invariant is obtained by integrating **P** from equation (4.10) around the gyration orbit, where **dl** is an element of the particle path around the orbit.

$$\begin{aligned} J_1 &= \oint [\mathbf{p} + q\mathbf{A}] \cdot \mathrm{d}\mathbf{l} \\ &= p_\perp \cdot 2\pi\rho + q\oint \mathbf{A} \cdot \mathrm{d}\mathbf{l} \\ &= p_\perp \cdot 2\pi \frac{p_\perp}{Bq} + q\oint \nabla \times \mathbf{A} \cdot \mathrm{d}\mathbf{S} \end{aligned} \tag{4.11}$$

where **dS** is an element of the area enclosed by the path. Therefore,

$$\begin{aligned} J_1 &= \frac{2\pi p_\perp^2}{Bq} + q\oint \mathbf{B} \cdot \mathrm{d}\mathbf{S} \\ &= \frac{2\pi p_\perp^2}{Bq} - qB\pi\rho^2 \\ &= \frac{2\pi p_\perp^2}{Bq} - \frac{\pi p_\perp^2}{Bq} = \frac{\pi p_\perp^2}{qB} \end{aligned} \tag{4.12}$$

The second term in (4.12) is negative because **dS** as defined by the particle orbit points in the opposite direction to **B**.

Rather than the above expression for J_1 the first invariant is taken to be $p_\perp^2/2m_0B$, which is equal to J_1 except for constant factors. The quantity

$$\mu = \frac{p_\perp^2}{2m_0B} \tag{4.13}$$

is often called the magnetic moment since in the non-relativistic limit it is equal to the current around the particle orbit times the area of the loop.

$$\begin{aligned}\mathcal{M} = I \cdot S &= \left(\frac{v_\perp}{2\pi\rho}\right)q \cdot \pi\rho^2 \\ &= \tfrac{1}{2}q\frac{v_\perp m v_\perp}{Bq} \\ &= \tfrac{1}{2}\frac{p_\perp^2}{mB}\end{aligned}$$

The constancy of μ or of $p_\perp^2/B$ can be shown directly for simple geometries. For example, consider a particle in circular motion in a uniform field which increases with time. (See Figure 4.2.) The magnetic field is assumed to be symmetric about the center of the particle orbit so that the induced **E** is equal at all points in the orbit. If **B** is uniform and increases, the integral of Maxwell's equation over the area of the orbit gives

$$\oint \nabla \times \mathbf{E} \cdot d\mathbf{S} = \oint \mathbf{E} \cdot d\mathbf{l} = -\oint \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\pi\rho^2 \frac{\partial B}{\partial t} \tag{4.14}$$

The energy change in one revolution or in one gyroperiod τ_g is therefore

$$\Delta W = -q\oint \mathbf{E} \cdot d\mathbf{l} = q\pi\rho^2 \frac{\partial B}{\partial t} \tag{4.15}$$

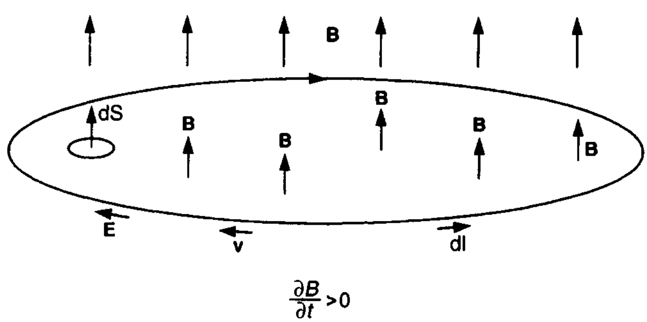


Figure 4.2. Charged particle with velocity **v** gyrating in a uniform magnetic field which is increasing slowly with time. Magnetic moment of particle remains constant.

Hence

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\Delta W}{\tau_g} = q\pi\rho^2 \frac{\partial B}{\partial t} \cdot \frac{Bq}{2\pi m}$$
$$= \frac{p_\perp^2}{2mB}\frac{\partial B}{\partial t} \quad (4.16)$$

Also,

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m_0 c^2) = m_0 c^2 \frac{\mathrm{d}\gamma}{\mathrm{d}t} \quad (4.17)$$

where

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[1 + \frac{p_\perp^2}{m_0^2 c^2}\right]^{1/2}$$
$$= \frac{1}{2m_0^2 c^2 \gamma}\frac{\mathrm{d}p_\perp^2}{\mathrm{d}t} \quad (4.18)$$

Equate (4.16) and (4.17) using (4.18) for $\mathrm{d}\gamma/\mathrm{d}t$ to obtain

$$\frac{1}{B}\frac{\partial B}{\partial t} = \frac{1}{p_\perp^2}\frac{\mathrm{d}p_\perp^2}{\mathrm{d}t}$$

Therefore

$$\frac{p_\perp^2}{B} = \text{constant} \quad (4.19)$$

and it follows that $\mu = p_\perp^2/2m_0B$ is also constant.

The adiabatic or slow change constraint enters with the assumption that the orbit is circular. In fact, the gyration radius decreases as the acceleration takes place so the circular radius is only an approximation, valid for small changes in B during a single revolution.

The expression for the magnetic moment or first adiabatic invariant occurs naturally in many of the equations for particle motion. The mirror force equation (2.31) is given by

$$F_\parallel = -\tfrac{1}{2}qv_\perp\rho\nabla_\parallel B = -\frac{\mu}{\gamma}\frac{\partial B}{\partial s} \quad (4.20)$$

This equation is familiar as it describes the force on a dipole magnet of moment μ/γ in an inhomogeneous magnetic field. When the dipole field opposes the applied field, as is always the case when the magnetic moment is produced by a particle circling in the applied field, the force is repulsive, as given in equation (4.20).

Although the proof of the invariance of $p_\perp^2/B$ or $p_\perp^2/2m_0B$ was shown for a time-dependent magnetic field, the invariance also applies if the particle moves into a region of different B, either by following a field line or by drifting across field lines. The motion parallel to field lines is particularly important, and it is in this case that the first invariant is most useful. Let α be the angle between the particle velocity and the magnetic field. This angle is customarily called the pitch angle:

$$\tan\alpha = \frac{v_\perp}{v_\parallel} \tag{4.21}$$

Therefore, from equation (4.19),

$$\frac{p_\perp^2}{B} = \frac{p^2\sin^2\alpha}{B} = \text{constant} \tag{4.22}$$

This equation is one of the most frequently used equations in radiation belt physics. Consider a particle whose pitch angle is α_{eq} at the equator where $B = B_{eq}$ (see Figure 4.3). As the particle moves along the magnetic field line towards the Earth, the field increases (see equation (3.21)). By (4.22), $p_\perp^2$ must also increase, and since p^2 is constant (in the absence of electric fields), $\sin^2\alpha$ must increase. In the diagram $\alpha_2 > \alpha_1 > \alpha_{eq}$. When α reaches 90° the particle will be reflected, will return to the equator and will then repeat the trajectory in the opposite hemisphere. Equation (4.22) allows one to compute the pitch angle at any position of the trajectory, provided B is known at that position. It also specifies the magnetic field at the mirror point in terms of the field and the pitch angle at any other position. For example, if a particle has pitch angle α at B it will mirror at B_m where

$$\frac{p^2\sin^2 90^\circ}{B_m} = \frac{p^2\sin^2\alpha}{B} \tag{4.23}$$

or $B_m = B/\sin^2\alpha$ if $\mathbf{E} = 0$ and therefore p = constant.

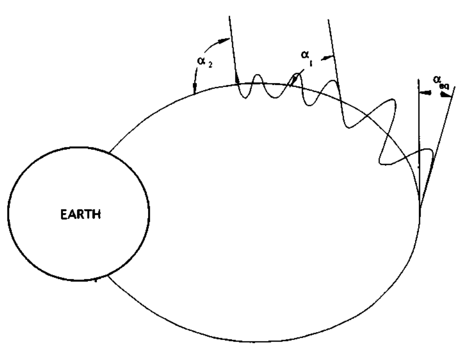


Figure 4.3. Conservation of magnetic moment results in an increase in pitch angle as the particle moves down the geomagnetic field line into the more intense field.

Note that the mirror point field is independent of particle momentum or charge and does not depend on the polarity of the magnetic field. The pitch angle at any point along a field line is given in terms of the equatorial values B_{eq}, α_{eq} or the mirror values B_m, α_m:

$$\sin\alpha(s) = \sqrt{\left[\frac{B(s)}{B_{eq}}\right]}\sin\alpha_{eq} = \sqrt{\left[\frac{B(s)}{B_m}\right]} \tag{4.24}$$

Equation (4.22) applies even when a parallel electric field $E_\parallel$ accelerates particles along a field line. Knowing a particle's energy, and therefore the momentum, as a function of B is sufficient to calculate B_m. The details of the helical trajectory through the electric field are not needed to find the mirror field.

At any location with field intensity B there will be few particles with $\alpha < \alpha_{LC} = \sin^{-1}\sqrt{(B/B_a)}$, where B_a is the field intensity at the top of the sensible atmosphere (~ 100 km). Particles with $\alpha < \alpha_{LC}$ will strike the atmosphere during each bounce and will be rapidly removed from the trapping region. The quantity α_{LC} is called the bounce loss cone angle. Because of the north–south asymmetry in the geomagnetic field the value of B_a on a given field line may be different at the two hemispheres. In such cases α_{LC} will be defined for the lower value of B_a and, therefore, for the larger value of α_{LC}.

Knowledge of the pitch angle as a function of position allows computation of such quantities as the time required to move between positions on a field line. For example, the time for a complete cycle of motion between mirror points s_m and s'_m is the bounce time τ_b where

$$\tau_b = 2\int_{s_m}^{s'_m}\frac{ds}{v_\parallel(s)} = \frac{2}{v}\int_{s_m}^{s'_m}\frac{ds}{\cos\alpha(s)} = \frac{2}{v}\int_{s_m}^{s'_m}\frac{ds}{\sqrt{\left[1-\frac{B(s)}{B_m}\right]}} = \frac{2}{v}\int_{s_m}^{s'_m}\frac{ds}{\sqrt{\left[1-\frac{B(s)}{B_{eq}}\sin^2\alpha_{eq}\right]}} \tag{4.25}$$

By changing the variable of integration from s to λ and using equations (3.19) and (3.21) the bounce time integral for a particle in a dipole field can be expressed in terms of the equatorial pitch angle α_{eq}, and the equatorial crossing distance R_0:

$$\tau_b = \frac{4R_0}{v} \int_0^{\lambda_m} \frac{\sqrt{(1 + 3\sin^2\lambda)}\cos\lambda\,d\lambda}{\left[1 - \frac{\sin^2\alpha_{eq}}{\cos^6\lambda}\sqrt{(1 + 3\sin^2\lambda)}\right]^{1/2}} \tag{4.26}$$

The helical distance traveled during a complete bounce is

$$S_b = v\tau_b = 2\int_{s_m}^{s'_m} \frac{ds}{\sqrt{\left[1 - \frac{B(s)}{B_m}\right]}} \tag{4.27}$$

Unfortunately, in a dipole field equation (4.26) cannot be integrated in closed form. However, an approximate formula good to about 0.5% is

$$\tau_b = 0.117\left(\frac{R_0}{R_E}\right)\frac{1}{\beta}[1 - 0.4635(\sin\alpha_{eq})^{3/4}]\ \text{s} \tag{4.28}$$

where R_0 is the distance from the center of the dipole to the equatorial crossing of the field line and $\beta = v/c$. Note the insensitivity of τ_b to the equatorial pitch angle. A particle mirroring near the equator has a bounce time of about half that of a particle which mirrors at the limiting distance near the dipole.

The constancy of μ is also useful in tracing paths for equatorially trapped particles ($\alpha_{eq} = 90°$). In the absence of electric fields, μ = constant requires the particle to drift along a contour of constant B, as expected since the gradient drift is perpendicular to ∇B. In the geomagnetic field which is more compressed on the sunward side the drift path will necessarily bring the particle closer to the Earth on the night side.

Graphs of the bounce periods for electrons and protons in the dipole approximation of the Earth's field are presented in Appendix B.

Second adiabatic invariant

The second mode of periodic motion of a geomagnetically trapped particle is the bounce motion between mirror points. If the longitudinal drift is small during a single bounce, the action variable associated with the bounce motion would be expected to be an invariant.

Returning to equation (4.10) for the canonical momentum, the action integral over a bounce is

$$J_2 = \oint(\mathbf{p} + q\mathbf{A})\cdot d\mathbf{s} \tag{4.29}$$

where $d\mathbf{s}$ is the element of length along a field line. The second term can be changed to a surface integral over the area **S** enclosed by the bounce path

$$\begin{aligned}\oint q\mathbf{A}\cdot\mathrm{d}\mathbf{s} &= q\int\nabla\times\mathbf{A}\cdot\mathrm{d}\mathbf{S}\\ &= q\int\mathbf{B}\cdot\mathrm{d}\mathbf{S}\\ &= 0\end{aligned}\tag{4.30}$$

since the integration path along the field line encloses a negligible area and no magnetic flux.

Therefore

$$J_2 = \oint\mathbf{p}\cdot\mathrm{d}\mathbf{s} = \oint p\cos\alpha\,\mathrm{d}s = \oint p_\parallel\,\mathrm{d}s = \text{constant}\tag{4.31}$$

That J_2 in (4.31) is an adiabatic invariant can be seen from the following argument. From the equation of parallel motion for the non-relativistic case ($\gamma = 1$), equation (4.20) gives

$$F_\parallel = m\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = -\mu\frac{\partial B}{\partial s}\tag{4.32}$$

This equation can be integrated to give

$$\tfrac{1}{2}mv_\parallel^2 + \mu B = \mathscr{E}'\tag{4.33}$$

where $\mathscr{E}'$ is a constant of integration. Equation (4.33) is equivalent to (4.2) with $\mathscr{E}'$ corresponding to total parallel energy and μB corresponding to a potential energy. The parallel velocity is then

$$v_\parallel = \sqrt{[2(\mathscr{E}' - \mu B)/m]}\tag{4.34}$$

If B varies slowly with time, either by an explicit field change or by the particle drifting on to different field lines, one can define an average value $\bar{\mathscr{E}}'$ by integrating equation (4.33) over a bounce period as was done in equation (4.3). The value of J_2 expressed in terms of $\bar{\mathscr{E}}'$ is then

$$J_2 = \oint p_\parallel\,\mathrm{d}s = \oint\sqrt{\{2m[\bar{\mathscr{E}}' - \mu B(s, a(t))]\}}\,\mathrm{d}s\tag{4.35}$$

where $a(t)$ denotes a parameter which allows B to change slowly with time. That is,

$$\frac{\partial B}{\partial t} \ll \frac{B}{\tau_\mathrm{b}}\tag{4.36}$$

where τ_b is the bounce period. With $\mu B(s, a(t))$ taking the role of the potential $V(x, a(t))$ the argument for $J_2 = $ constant is the same as developed in equations (4.4)–(4.9).

The second adiabatic invariant is often called the integral invariant. It is usually designated by J rather than by J_2 and this convention will be followed in the remainder of the book. To remove the particle momentum from the definition and express the invariant of a location entirely in terms of the magnetic field geometry, a related quantity, I, is often used as the integral invariant coordinate. For a point in space, I is defined in terms of J for a particle of momentum p mirroring at that point:

$$I = J/2p$$
$$= \tfrac{1}{2}\oint \cos\alpha \, ds \quad (4.37)$$

or

$$I = \int_{s_m}^{s'_m} \sqrt{\left[1 - \frac{B(s)}{B_m}\right]} \, ds \quad (4.38)$$

Again, s_m and s'_m are the locations of the mirroring points along a field line.

The primary use of the second or integral invariant is to define drift paths and the surfaces mapped out by the bouncing and drifting particle. In an axisymmetric magnetic field where $\mathbf{E} = 0$, this surface will also be axisymmetric since the gradient and curvature drifts are everywhere perpendicular to $\mathbf{B}$ and $\nabla_\perp B$. If the Earth possessed this idealized field, a drifting particle would circle the Earth and return to the initial field line. In a distorted field it is not clear from the drift equations that the guiding center drift path is closed. However, the invariance of J_2 or I ensures that the particle will return to the original field line.

Figure 4.4 illustrates a drift path in an asymmetric field. A particle initially on curve 1 on the right-hand side will drift to curve 2 on the left-hand side and return to 1, mirroring at B_m in both northern and southern hemispheres throughout the drift. At each longitude there is only one curve between mirror values of B_m having the required J value, because at a given longitude the value of J for given B_m increases monotonically with distance from the Earth.

Although the magnetic field is more compressed on the sunward side, the drift shell is closer to the Earth on the night side. For the same (r, θ) B is smaller on the night side, and the particle will move closer to the Earth to keep J constant. This result is similar to that for equatorial particles discussed on page 44.

In a distorted field there is no requirement that particles initially on the same field line but having different pitch angles will follow identical drift paths. Consequently, in their longitudinal drift, particles which started on the same field line may trace out different shells before returning to the

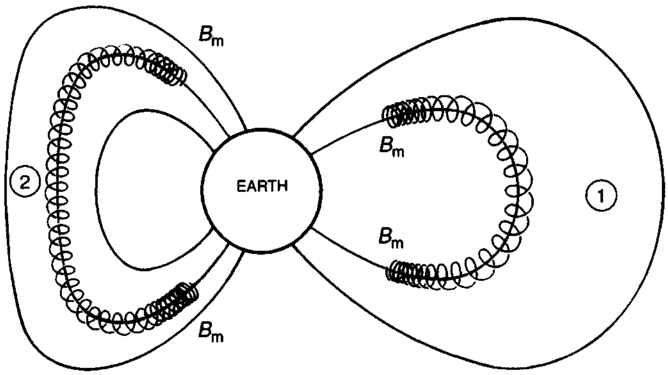


Figure 4.4. Trace of particle drifting and mirroring in the geomagnetic field. If the field is static, the particle will be reflected at fixed B_m and at each longitude will select the field line on which its motion between mirroring points conserves the second invariant.

initial line. This condition, called L-shell splitting, becomes important for field lines extending more than $\sim 4R_E$ from the Earth.

In the geomagnetic field distorted by anomalies and by the off-center dipole the mirroring altitude of a drifting particle will change with longitude, the altitude being lowest in the regions where the surface magnetic field is low. On each drift shell, the smallest pitch angle that still allows particles to drift completely round the Earth without striking the atmosphere defines a drift loss cone angle. At those longitudes where B_{max} is well above the atmosphere, particles mirroring below B_{max} can exist locally but are said to be in the drift loss cone. During their next drift orbit around the Earth, they will strike the atmosphere. The drift loss cone regions are in the 'magnetic shadow' of the Earth or its atmosphere and will contain only those trapped particles which have been diverted into that trajectory within their most recent drift period.

The rate at which trapped particles drift in longitude in a dipole field with $\nabla \times \mathbf{B} = 0$ is obtained by using equation (2.33) for the instantaneous drift velocity and averaging over a complete bounce period. From Figure 4.5 it is apparent that the instantaneous change in longitude ϕ is

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{V_\perp(r, \theta)}{r \sin\theta} \tag{4.39}$$

The change in ϕ during a complete bounce along a field line which crosses the equator at R_0 is

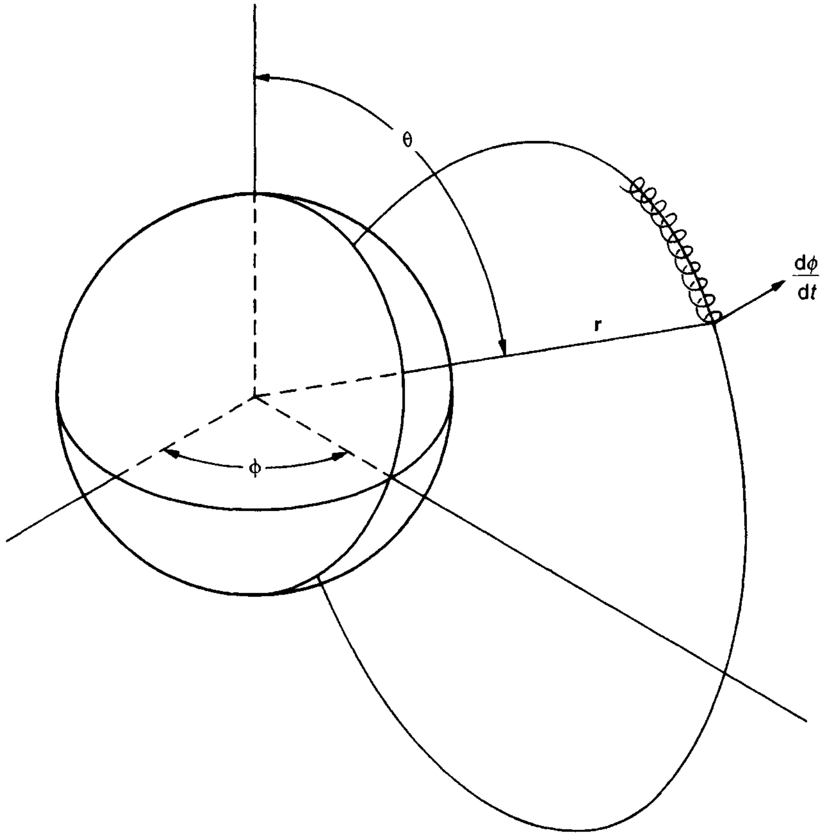


Figure 4.5. Coordinate system for calculating the longitudinal drift velocity of a trapped particle.

$$\Delta\phi = 4\int_0^{s_m} \frac{V_\perp(\theta)}{R_0 \sin^3\theta}\frac{ds}{v_\parallel} \tag{4.40}$$

where ds is the element of arc length along a magnetic field line measured from the equator and s_m is the distance along the field line from the equator to the mirror point. After changing the variable of integration from s to θ the time rate of change of longitude averaged over a bounce is

$$\left\langle\frac{d\phi}{dt}\right\rangle = \frac{\Delta\phi}{\tau_b} = \frac{4}{\tau_b}\int_{\theta_m}^{\pi/2} \frac{V_\perp(\theta)}{R_0 \sin^3\theta}\left(\frac{ds}{d\theta}\right)\frac{d\theta}{v_\parallel} \tag{4.41}$$

Factors in the integrand of equation (4.41) can easily be computed for a dipole field. From equations (3.19) and (2.33):

$$\frac{ds}{d\theta} = R_0 \sin\theta[1 + 3\cos^2\theta]^{1/2} \tag{4.42}$$

$$V_\perp = \frac{m}{qB^3}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\mathbf{B} \times \nabla_\perp B \quad (\text{if } \nabla \times \mathbf{B} = 0) \tag{4.43}$$

The parallel and perpendicular velocities can be expressed in terms of the equatorial pitch angle α_{eq} as

$$\left.\begin{aligned} v_\perp^2(\theta) &= v^2 \sin^2\alpha_{eq}\frac{(1+3\cos^2\theta)^{1/2}}{\sin^6\theta} \\ v_\parallel^2(\theta) &= v^2\left[1 - \sin^2\alpha_{eq}\frac{(1+3\cos^2\theta)^{1/2}}{\sin^6\theta}\right] \end{aligned}\right\} \tag{4.44}$$

Inserting these expressions into equation (4.41) results in an integral expression for the angular drift velocity:

$$\left\langle\frac{d\phi}{dt}\right\rangle = \frac{4}{\tau_b}\cdot\frac{3mvR_0^2}{qB_0R_E^3}\cdot\int_{\theta_m}^{\pi/2}\frac{\sin^3\theta(1+\cos^2\theta)\left[1-\frac{1}{2}\sin^2\alpha_{eq}\frac{(1+3\cos^2\theta)^{1/2}}{\sin^6\theta}\right]}{(1+3\cos^2\theta)^{3/2}\left[1-\sin^2\alpha_{eq}\frac{(1+3\cos^2\theta)^{1/2}}{\sin^6\theta}\right]^{1/2}}\,d\theta \tag{4.45}$$

This equation cannot be integrated analytically, and values of the angular drift velocity and drift period $\tau_d = 2\pi/\langle d\phi/dt\rangle$ must be obtained numerically. For most purposes it is adequate to use an empirical fit which approximates the values of equation (4.45). An expression for the drift period accurate to $\sim 0.5\%$ is

$$\tau_d = \frac{2\pi qB_0R_E^3}{mv^2}\frac{1}{R_0}[1 - 0.3333(\sin\alpha_{eq})^{0.62}] \tag{4.46}$$

This approximation can be simplified by collecting all constant factors to give

$$\tau_d = C_d\cdot\left(\frac{R_E}{R_0}\right)\frac{1}{\gamma\beta^2}[1 - 0.3333(\sin\alpha_{eq})^{0.62}] \tag{4.47}$$

where

$$C_d = 1.557 \times 10^4 \text{ s for electrons}$$

and

$$C_d = 8.481 \text{ s for protons}$$

Note that as R_0 increases, the drift period decreases in spite of the larger drift path. Also as the particle velocity or energy increases, the drift period decreases. For non-relativistic particles such as protons below

50 MeV, the drift period is inversely proportional to energy. Equatorial particles drift more rapidly than those mirroring at higher latitudes, although this variation is not large.

Graphs of the drift periods of electrons and protons are plotted in Appendix B. Figures B.2 and B.3 give the drift periods for electrons and protons which mirror at the equatorial plane. The drift periods are plotted as a function of particle energy and equatorial crossing distance given by the parameter L, which is defined in the section on p. 53. For particles with equatorial pitch angles other than 90°, one must use Figure B.5. In Figure B.2 note the limiting value of τ_b as electron energy becomes relativistic. This condition occurs because the bounce period is simply the helical distance divided by the particle velocity. As the velocity approaches c, no further reduction in τ_b is possible.

Third adiabatic invariant

The third periodic motion of a geomagnetically trapped particle is the longitudinal drift about the Earth. In a static field, conservation of the first and second invariants will ensure that the particle returns to its original field line and will specify the field line occupied by the particle at each longitude. In a slowly changing magnetic field, μ and J are still conserved, but p may change in a way which depends on the details of the trajectory and the magnetic field changes. Hence, an additional constant of motion is needed to define trajectories in slowly changing magnetic fields.

The third invariant is derived as before by integrating the canonical momentum over the periodic trajectory:

$$J_3 = \oint (\mathbf{p} + q\mathbf{A}) \cdot \mathrm{d}\mathbf{l} \tag{4.48}$$

where in this case $\mathrm{d}\mathbf{l}$ is the increment of longitudinal drift path, usually taken at the equator. The first term in (4.48) is neglected because the average momentum $\mathbf{p}$ in the direction of $\mathrm{d}\mathbf{l}$ is small, $\mathbf{v}_\mathrm{d}$ being orders of magnitude less than the actual particle velocity. Again, using Stoke's theorem,

$$J_3 = q \oint (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} \tag{4.49}$$

where $\mathrm{d}\mathbf{S}$ is an element of the surface enclosed by the equatorial drift path. Since $\nabla \times \mathbf{A} = \mathbf{B}$ the third invariant is

$$J_3 = q \oint \mathbf{B} \cdot \mathrm{d}\mathbf{S} = q\Phi \tag{4.50}$$

The quantity Φ is the magnetic flux enclosed by the drift path. Since the north–south oscillations are along field lines, the value of Φ does not depend on the latitude of **dl** as long as the drift path encompasses the Earth on the shell containing the guiding center trajectory. For this reason the third invariant is often called the flux invariant and is usually denoted as Φ, omitting the charge q.

Because of the magnetic singularity at the center of the Earth it is inconvenient to calculate the flux enclosed by the drift path. However, in the Earth's field the *net* flux inside the drift path is equal to the flux outside the path. Thus, the flux outside the integration path is usually computed to find Φ. In a dipole field the value of Φ for a particle at equatorial distance R_0 is

$$\begin{aligned}\Phi &= \int_{R_0}^{\infty} B_0\left(\frac{R_E}{r}\right)^3 2\pi r\,dr \\ &= 2\pi B_0 \frac{R_E^3}{R_0} \qquad (4.51)\end{aligned}$$

Note that, as R_0 increases, the net flux enclosed decreases.

A non-relativistic proof of the invariance of Φ for a simple case is given below (see Figure 4.6). Assume that the particle is in the equatorial plane ($p_\parallel = 0$) and the magnetic field, which was initially B_1, slowly changes to a smaller value B_2. All vectors except **B** and d**S**, the element of area in the surface enclosed by the drift path, lie in a plane. The particle will move

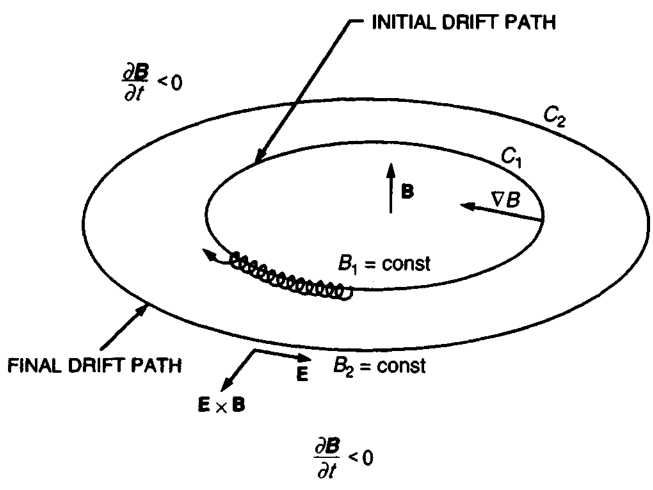


Figure 4.6. Conservation of the third adiabatic invariant. Magnetic flux enclosed by a particle drifting perpendicular to **B** and to ∇B remains constant during slow changes in the magnetic field.

from curve C_1 to C_2, being driven by the $\mathbf{E} \times \mathbf{B}/B^2$ force of the electric field induced by the changing magnetic field. Any increment to Φ resulting from $\partial B/\partial t$ is produced by a change in the flux density $\mathbf{B}$ within the original curve and by a change in the area enclosed by the drift path. Thus

$$\Delta\Phi = \Delta\Phi_B + \Delta\Phi_A \qquad (4.52)$$

The change in Φ attributed to a change in $\mathbf{B}$ during a drift period τ_d is

$$\Delta\Phi_B = \tau_d \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \qquad (4.53)$$

where the integral is over the entire surface inside the particle drift path.

The increase in Φ associated with the change in the area enclosed by the drift path is

$$\Delta\Phi_A = \mathbf{B} \cdot \Delta\mathbf{S} = \tau_d \oint \mathbf{B} \cdot (\mathbf{V}_E \times d\mathbf{l}) \qquad (4.54)$$

where $\mathbf{V}_E$ is the $\mathbf{E} \times \mathbf{B}/B^2$ drift velocity and $d\mathbf{l}$ is the element of drift path. Therefore,

$$\Delta\Phi_A = \tau_d \oint \mathbf{B} \cdot \left(\left(\frac{\mathbf{E} \times \mathbf{B}}{B^2} \right) \times d\mathbf{l} \right) \qquad (4.55)$$

Expanding the triple vector product and noting that $\mathbf{B} \cdot d\mathbf{l} = 0$ results in

$$\begin{aligned} \Delta\Phi_A &= \tau_d \oint \frac{\mathbf{B} \cdot (\mathbf{E} \cdot d\mathbf{l})\mathbf{B}}{B^2} \\ &= \tau_d \oint \mathbf{E} \cdot d\mathbf{l} \\ &= \tau_d \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} \\ &= -\tau_d \int_S \frac{d\mathbf{B}}{dt} \cdot d\mathbf{S} \end{aligned} \qquad (4.56)$$

Thus

$$\Delta\Phi = \Delta\Phi_B = \Delta\Phi_A = 0 \qquad (4.57)$$

The third or flux invariant is most useful in describing drift paths during slow changes in the geomagnetic field. For example, slow compressions or expansions of the geomagnetic field will cause trapped particles to move inward or outward as required to conserve the magnetic flux exterior to their orbits. Similarly, the very slow secular decay of the geomagnetic field results in an imperceptible inward motion of the radiation belts. The overall effect on trapped radiation of these slow changes is reversible; restoration of the field will return the particles to their original condition. Rapid changes in B, that is, $\partial B/\partial t \gtrsim B/\tau_d$, will cause permanent changes in Φ, as will be discussed in Chapter 6.

Geomagnetic coordinate system based on adiabatic invariants – the *L*-shell parameter

The lack of symmetry in the irregular geomagnetic field greatly complicates the tabulation of trapped radiation fluxes as a function of position, and in geographic coordinates a three-dimensional grid would be required for a complete description of flux values in space. Furthermore, a spatial coordinate system based on geographic coordinates loses the simplicity of the dipole formulas and does not lead to insights into the relationships of fluxes at different locations. What is needed is a coordinate system based on trapped particle motion which will have identical values for the coordinates of magnetically equivalent positions. By utilizing the near symmetry of the Earth's field to some extent the coordinate system would also eliminate the need for a longitude coordinate to describe the long-term trapped populations of particles.

The adiabatic invariants suggest such a system. The scalar value of the magnetic field is a useful coordinate since particles mirroring at a given B will mirror at the same value of B throughout their longitudinal drift. The second invariant, or, rather, its related function I, could be used as the second coordinate, since two positions in space with the same B and I values are magnetically equivalent from the standpoint of a trapped particle. Particles mirroring at a given value of B, I will drift around the Earth, mirroring at identical values of B and I in both hemispheres. Unfortunately, the quantity I is not an easy coordinate to interpret and does not vary linearly with any familiar variable. A more serious limitation is that it is not readily apparent from the values of B and I at several positions whether these locations lie near the same magnetic drift shell.

These difficulties are circumvented by a coordinate system devised by McIlwain. He recognized the convenience in a dipole field of the parameter, R_0, the distance from the dipole center to the equatorial crossing or minimum B value of a field line. In a dipole, R_0 defines a field line as well as a drift shell and is readily visualized. During particle bounce and drift motion the particle remains on field lines having the same R_0. In a dipole field, if the B and I values of a location are known, the equatorial crossing point of the field line passing through that point can be determined. Thus,

$$R_0 = f(I_D, B_D, \mathscr{M}_D) \tag{4.58}$$

where $f(I_D, B_D, \mathscr{M}_D)$ denotes a function of the dipole magnetic field value B_D, the integral invariant function I_D and the magnetic moment of the central dipole $\mathscr{M}_D$.

For the Earth's field, one then defines a new variable, L, in terms of the actual geomagnetic field. The L value of a location is based on values of B and I at that location calculated from the true, distorted geomagnetic field, but uses the same functional relationship relating the equatorial crossing distance to B and I for a dipole field. Thus,

$$LR_E = f(I, B, \mathscr{M}_E) \tag{4.59}$$

where $\mathscr{M}_E$ is the value of the dipole term for the Earth's field.

The variable L will be used as the second spatial coordinate. The distance LR_E is roughly equal to the distance from the center of the Earth's tilted, off-center, equivalent dipole to the equatorial crossing of the field line. Positions around the world having the same B and I will, by definition, have the same L values. Positions on the same field line in the distorted Earth's field will have very nearly the same values of L. Thus, a particle which bounces and drifts around the Earth will be very near to an L = constant shell and can be assumed to follow an L = constant path. In the distorted field L defines magnetic drift shells, and the value of L denotes the distance in Earth radii from the center of the equivalent dipole to the equatorial crossing for that drift shell.

It is frequently convenient to represent fluxes in the easily understood $r - \lambda$ polar coordinate system based on the B, L values of positions in the distorted Earth's field. For this purpose the coordinates r and λ of a point are defined implicitly by the B and L values of that point by

$$\left.\begin{aligned} r &= L\cos^2\lambda \\ B &= \frac{B_0}{r^3}\left(4 - \frac{3r}{L}\right)^{1/2} \end{aligned}\right\} \tag{4.60}$$

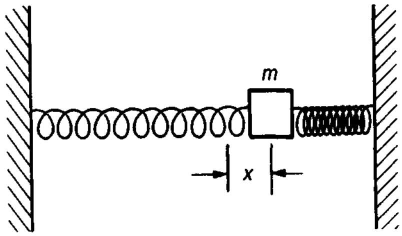
with r given in units of Earth radii. The values of B in terms of r and λ have the familiar dipole relationship of equation (3.15).

Computer programs which represent the Earth's distorted field usually include features which will compute the value of L for any position in space. Although the functional relationship of a position to its L coordinate is quite involved, in practice the conversion is routine.

A convenient way to interpret the B, L coordinate system for trapped particles is an follows. Imagine a dipole field with $\mathscr{M}$ equal to the dipole term in the Earth's field. If one removes trapped particles from the Earth's distorted field and places them in the imaginary dipole field at positions which preserve their mirroring B and J values, then one has a description of the radiation belts in the B, L coordinate system.

Problems

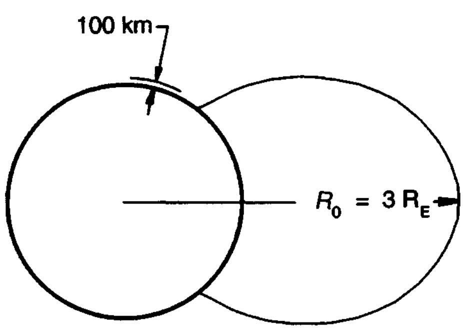
1. The one-dimensional, frictionless spring oscillator sketched below obeys the differential equation.



$$m\frac{d^2x}{dt^2} = -kx$$

(a) If the maximum amplitude of oscillation is A, find the expression for the x coordinate as a function of time.
(b) Find the expression for the adiabatic invariant of the system.
(c) If, initially, the displacement is A and the springs are slowly strengthened to increase the frequency, what is the new amplitude when the frequency is double the initial value?

2. A satellite experiment is designed to produce artificial aurora by deflecting trapped electrons so that they enter the atmosphere. The satellite is in the equatorial plane at $R_0 = 3R_E$, and it generates electromagnetic fields which can deflect electrons:
(a) Assuming that the Earth's field is a centered dipole and that electrons can be trapped only if they mirror above 100 km, what is the smallest pitch angle that a trapped electron can have at the satellite position?
(b) To make sure that most of the electrons impact the atmosphere it is desired that the pitch angles at the 100 km altitude be 45°. For these electrons what must their pitch angle be at the equatorial plane? What is the minimum deflection angle that the satellite must supply to the trapped electrons to achieve this result?
(c) If the satellite were crossing the $R_0 = 3R_E$ field line at a latitude of 45° what is the minimum required deflection?



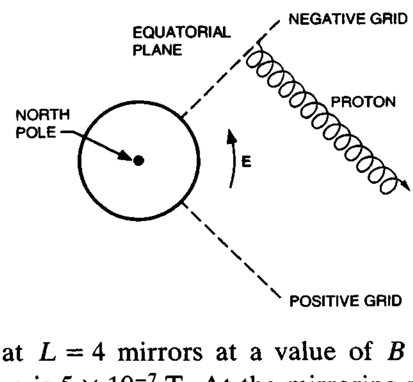
3. An electron of total momentum p is traveling along a field line between mirror points of strength B_m. In the northern hemisphere a weak D.C. electric field occurs, accelerating the electron downward to a new total momentum p_1. What is the magnetic field value at the new mirroring point? Assuming that the electron does not strike the atmosphere and that the electric field is maintained, what will be the mirroring field value in the southern hemisphere?

4. In a dipole field a charged particle of momentum p starts at $R_0 = 2R_E$ with second invariant $J = 0$. As it drifts to the other side of the Earth it encounters a weak electric field in the ϕ direction which slowly increases its momentum to 1.2p. On what R_0 value will the particle now be? What will be its new J value?

5. Assume the particle in Problem 4 has an equatorial pitch angle of 45°. After being accelerated by the azimuthal electric field to 1.2p, will the mirroring latitude be increased or decreased? Explain why.

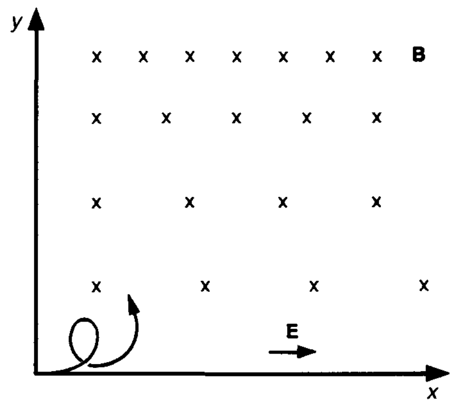
6. A proton of momentum p is drifting around the Earth at $L = 2$ with an equatorial pitch angle of 90°. If the magnitude of the Earth's dipole moment slowly increases by 50%, at what distance from the dipole will the proton guiding center be at the end of this increase? What will its new momentum be?

7. An advanced civilization wishes to remove the radiation belts around its planet by constructing an electrostatic grid of wires in meridian planes extending several planet radii. As protons drift between these wires the $\mathbf{E} \times \mathbf{B}$ drift moves the protons outward. Repeated passages are intended to remove the protons to the region beyond the grids. Why won't this work using constant voltages on the grids? Would it work using time-dependent voltages? (Assume that the potential differences are much less than the energy of the protons.)



8. A trapped proton at $L = 4$ mirrors at a value of $B = 4 \times 10^{-5}$ T, while the equatorial field value is 5×10^{-7} T. At the mirroring point the proton collides with an oxygen atom and has its pitch angle changed by 5° with no loss in energy.
   (a) Find the B value of the new mirroring point.
   (b) Find the new value of the equatorial pitch angle.
   (c) How much was the equatorial pitch angle altered by the collision?

9. A non-relativistic proton with 90° pitch angle is drifting in crossed electric and magnetic fields starting at $x = y = 0$. The electric field **E** is in the x direction and is uniform. The magnetic field is directed into the paper and has a flux density which varies with y as $B = B_0 e^{\alpha y}$. The guiding center will move under combined $\mathbf{E} \times \mathbf{B}/B^2$ and gradient B drifts.



Will the energy of the proton increase or decrease? If the initial momentum is p_0, what is the x, y position when the momentum has changed to p. (Note how readily the answer is obtained using conservation of the first adiabatic invariant.)

10. At $L = 4$, stably trapped electrons of 5 keV fill all pitch angles except for the loss cone (assume a dipole).
    (a) What is the value of the loss cone angle if there are no electric fields?
    (b) If an electric field directed upward from the Earth has an overall voltage drop of 1 keV between the equator at $L = 4$ and the atmosphere, what will be the equatorial loss cone angle for the 5 keV electrons?

11. A 5 MeV proton with second adiabatic invariant $J = 0$ is drifting on the $L = 2$ shell:
    (a) What must the energy be of a He^{++} ion (doubly charged He ion) so that it will drift at the same velocity on the same L shell? (Neglect relativistic effects.)
    (b) If the proton and helium ion are not confined to the equatorial plane but have the same pitch angle and the energies established in part (a), will they still drift at the same velocity?
    (c) Given the conditions in part (b), do the proton and He ion have the same J value?

12. A synchronous satellite in orbit at $L = 6.6$ is monitoring the loss cone while a TV camera on the ground is observing the atmosphere at the end of the field line passing through the satellite. (Assume a dipole field and neglect the thickness of the atmosphere.)

(a) At what magnetic latitude does the auroral observer locate his equipment in order to observe the light produced by particles intersecting the atmosphere?

(b) What is the angle of the loss cone as measured by the satellite at the equator?

(c) The satellite measures a sudden burst of electrons with energies ranging from 10 keV to 100 keV moving parallel to the field line. If the 100 keV electrons arrive at the top of the atmosphere 0.308 s after being detected by the satellite, how much later will the 10 keV electrons arrive?

13. A non-relativistic proton is trapped in the equatorial plane of a centered dipole field, and a uniform electric field **E** extends in the dawn to dusk direction. If the proton has velocity v_0 when crossing the noon–midnight meridian, show that the equation of its guiding center is

$$\frac{mv_0^2}{2}r^3 + q\mathbf{E}r^4 \sin\phi = \text{constant}$$

where r is the distance from the dipole and ϕ is longitude.