Introduction to Geomagnetically Trapped Radiation

Martin Walt

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6

Particle diffusion and transport

Introduction

The trapping properties of the geomagnetic field were described in preceding chapters. In particular, the guiding center equations and the adiabatic invariants were obtained, and it was found that energetic ions and electrons with appropriate initial position and velocity conditions were confined by the Earth's magnetic field. If the invariants were rigorously conserved, this confinement would be permanent; a trapped particle would remain trapped forever. However, rigorous conservation of the adiabatic invariants would also prevent other particles, such as those in the solar wind, in cosmic rays or in the ionosphere from ever becoming trapped. The radiation belts would then consist of only those particles which were injected in place by decaying neutrons or other radioactive particles.

There is a large body of experimental evidence showing that the adiabatic invariants are not conserved absolutely. Low-altitude satellites usually observe a flux of particles moving down the field lines and destined to be absorbed in the atmosphere. Also, the drift loss cone usually contains small but measurable fluxes of particles whose invariants must have been altered during their last drift period and which will be lost into the atmosphere during their current drift cycle. The observed time variations of both electron and ion fluxes also illustrate the frequent alteration of particle orbits. Although some of these variations can be attributed to reversible changes in the geomagnetic field and not to changes in the adiabatic invariants of the particles, most of the variations demand substantial modifications to the particle trajectories along with the injection of fresh particles and the loss of previously trapped ones.

The picture of the radiation belts is therefore one in which the magnetic container is an imperfect trap. Some defects are expected since the

conditions required for the adiabatic invariants to hold rigorously are not always present. In the discussion of adiabatic invariants in Chapter 4 it was assumed that the magnetic and electric fields did not change appreciably during the cyclic motion of the particle. Field changes more rapid than any of the three cyclic motions associated with the three adiabatic invariants will lead to a change in the value of that invariant. In the magnetosphere a rich variety of electromagnetic and electrostatic waves are present with frequencies comparable to the gyration and bounce frequencies of the trapped particles. These waves will change the values of the corresponding adiabatic invariants and may remove the particles from the trapping regions. Magnetic activity, with its large-scale time variations in the electric and magnetic fields, also leads to a breakdown in one or more of the adiabatic invariants, usually the third invariant. The next three chapters will deal with changes in adiabatic invariants resulting from time variations in the fields and will show how field changes affect the particle distributions.

The equations for particle motion developed in the preceding chapters were deterministic in that the electric and magnetic fields were specified and the particle trajectories were then calculated. In treating the motion of charged particles in fluctuating fields a different approach is needed. In general, the time-dependent fields are not known precisely. Only the statistical properties, such as the power spectra and the geometrical pattern of the distortions, are known. It is therefore not possible to predict where an individual particle will be at some future time. One can only calculate the probabilities of the particle's behavior. When dealing with a large number of particles, these probabilities are quite adequate to describe the time evolution of the entire particle distribution.

Treatment of the time evolution of a distribution of particles whose trajectories are disturbed by innumerable small, random changes is by diffusion theory. However, this application of diffusion theory differs in concept from most other uses. In the usual diffusion applications, such as gases diffusing through porous media or neutrons diffusing through a moderator, the particle motion itself constitutes the diffusion. Any movement of the particles is tabulated as diffusion and is treated as such. If no diffusion occurs, the particles will remain in place. Diffusion of trapped particles is quite different. These particles may gyrate, bounce and drift around the Earth without actually diffusing. Only when one or more of their adiabatic invariants is altered can the particle be said to diffuse. Thus, the major motion is not associated with the diffusion processes to be studied, and one must remove this normal, adiabatic behavior from consideration in describing the diffusion process.

Diffusion equation

To illustrate the standard treatment of diffusion consider the one-dimensional case of particles diffusing through a porous column oriented along the x axis, as in Figure 6.1. In this case the distribution function $f(x, t)$ at time t is the number of particles in unit dx. Assume that at time $t = 0$ the distribution was $f(x, t = 0)$ and that there are no sources or sinks in the interval x_1 to x_2. The container has free boundaries at x_1 and x_2 where the particles can escape, so the distribution must vanish at those positions. The net current of particles diffusing across a position at x is proportional to the negative of the gradient of $f(x, t)$ at that point and is given by Fick's law:

$$\text{Current} = -D\frac{\partial f}{\partial x} \tag{6.1}$$

D being the diffusion coefficient. All the information on the physical mechanisms governing the diffusion process is contained in D. It is always positive and may of course be a function of x. The direction of net flow will be in the positive x direction if $\partial f/\partial x$ is negative. The net number of particles entering a section of length Δx in unit time by diffusing past x and $x + \Delta x$ is then

$$\Delta x\frac{\partial f}{\partial t} = D\frac{\partial f}{\partial x}\bigg|_{x+\Delta x} - D\frac{\partial f}{\partial x}\bigg|_{x} \tag{6.2}$$

In the limit of $\Delta x \to 0$

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left[D\frac{\partial f}{\partial x}\right] \tag{6.3}$$

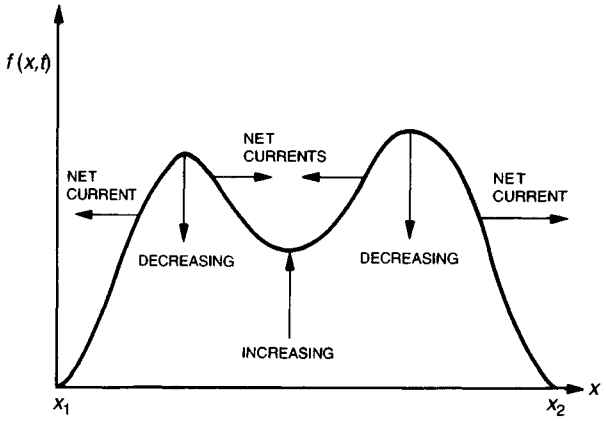


Figure 6.1. Impact of diffusion on a distribution of particles. Diffusion smoothes the distribution by decreasing the density at the peaks and increasing the density in the valleys.

Equation (6.3) is the diffusion equation which describes the time evolution of the distribution function. If D is independent of x, the instantaneous growth or decay of the particle density is proportional to the local value of $\partial^2 f/\partial x^2$. Where $f(x, t)$ is concave upward ($\partial^2 f/\partial x^2 > 0$), $\partial f/\partial t$ is positive and $f(x, t)$ will grow. Similarly, f will decrease at local maxima where $\partial^2 f/\partial x^2 < 0$. The diffusion process reduces peaks and increases valleys, thereby smoothing the distribution function (see Figure 6.1).

If $f(x, t)$ is known at some time, say $t = 0$, the value of f at a later time can be found by solving (6.3) as an initial value problem. In the general case where $D = D(x)$ the solution is obtained in the usual way by separation of variables. Let $f(x, t)$ be represented as the product of a function of x and a function of t:

$$f(x, t) = X(x)T(t) \tag{6.4}$$

Substituting (6.4) into (6.3) and dividing by $X(x)T(t)$ gives

$$\frac{1}{T}\frac{\partial T}{\partial t} = \frac{1}{X}\frac{\partial}{\partial x}\left[D\frac{\partial X}{\partial x}\right] \tag{6.5}$$

Because the left-hand side is independent of x and the right-hand side is independent of t, each term must be equal to a constant, which is designated $-\lambda_n$. Equation (6.4) can then be separated to give

$$\frac{1}{T}\frac{\partial T}{\partial t} = -\lambda_n \tag{6.6}$$

$$\frac{1}{X}\frac{\partial}{\partial x}\left[D\frac{\partial X}{\partial x}\right] = -\lambda_n \tag{6.7}$$

The time dependence is obtained by integrating (6.6) to give

$$T(t) = a_n \mathrm{e}^{-\lambda_n t} \tag{6.8}$$

where a_n is a constant of integration. The x dependence of f is more difficult to obtain since equation (6.7) with $f(x_1) = f(x_2) = 0$ is an eigenfunction equation, having solutions for only specific values of λ_n. The function $X(x)$ will be expressed as a sum of eigenfunctions $g_n(x)$, where the g_n satisfy

$$\frac{\partial}{\partial x}\left[D\frac{\partial g_n}{\partial x}\right] = -\lambda_n g_n \tag{6.9}$$

with boundary conditions

$$g_n(x_1) = g_n(x_2) = 0$$

and are ordered such that $\lambda_1 < \lambda_2 < \lambda_3 \ldots$. These boundary conditions represent free escape or absorption at the boundaries where the particle

density must vanish. If particles are reflected at the boundaries, different boundary conditions, such as $dg_n(x_1)/dx = dg_n(x_2)/dx = 0$ must be used. There are an infinite number of g_n, although in any practical problem n will be limited. Although the detailed shape of $g_n(x)$ depends upon $D(x)$, the general character follows the pattern illustrated in Figure 6.2 for the first three eigenfunctions. The lowest eigenfunction g_1 vanishes only at the boundaries and is always positive between x_1 and x_2. The next higher term becomes zero once in the interval x_1 to x_2, and subsequent g_n cross the horizontal axis $n - 1$ times between boundaries. Thus, higher n eigenfunctions have higher spatial frequencies and show more structure.

The eigenfunctions of (6.9) are orthogonal and can be normalized to give

$$\int_{x_1}^{x_2} g_n g_m \, dx = \delta_{nm} \tag{6.10}$$

where δ_{nm} is zero unless $n = m$, in which case $\delta_{nm} = 1$.

The general solution of (6.7) is a sum of the eigenfunctions, and the solution of (6.3) in the form of (6.4) is therefore

$$f(x, t) = \sum_n a_n e^{-\lambda_n t} g_n(x) \tag{6.11}$$

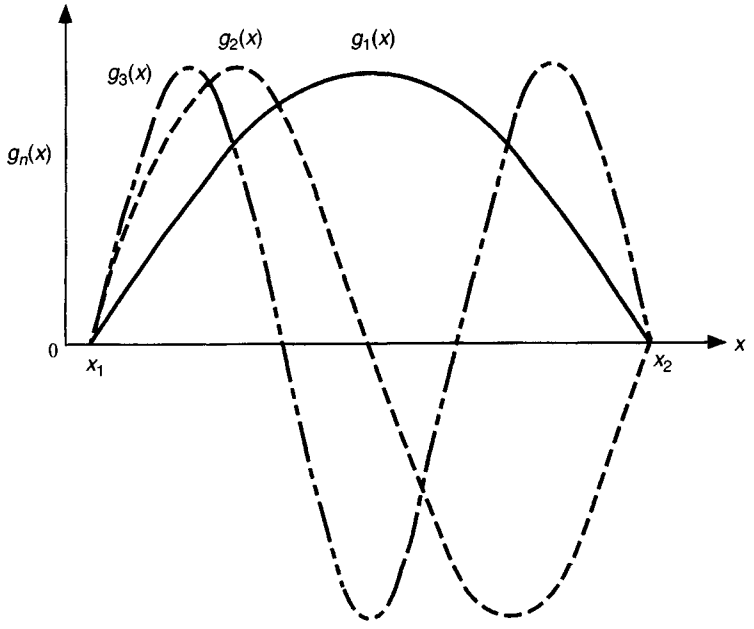


Figure 6.2. First three eigenfunctions of equation (6.9).

The constants a_n must be chosen to fit the initial conditions. If $f(x, t = 0)$ is the distribution at $t = 0$, each a_m can be obtained by multiplying both sides of equation (6.11) by g_m and integrating over x from x_1 to x_2. Because of the orthonormality of the eigenfunctions only the term containing g_m on the right-hand side survives. The coefficient a_m is then given by

$$a_m = \int_{x_1}^{x_2} f(x, t = 0) g_m(x)\, dx \tag{6.12}$$

The magnitude of the coefficient a_m depends on the complexity of the x dependence of $f(x, t = 0)$. If the initial distribution has a single maximum, as does g_1, then a_1 will be large and coefficients for $n > 1$ will be smaller. Values of a_n for large n will be appreciable only if $f(x, t = 0)$ contains high spatial frequencies. Computing the a_n is equivalent to decomposing $f(x, t = 0)$ into its eigenfunction modes. The solution to the diffusion problem as represented by equation (6.11) is a sum of eigenfunctions with initial amplitudes, each decaying exponentially on a time scale of λ_n^{-1}. Since λ_n increases with n, the modes with high n decay more rapidly and the distribution eventually consists largely of g_1, which then decays as $\exp(-\lambda_1 t)$. The more rapid decay of higher modes leads to smoothing of the distribution function described earlier.

Diffusion in more than one dimension can be treated in a similar manner, although separation of the spatial variables is sometimes difficult. If one can choose a coordinate system in which one or more coordinates are constant on a boundary surface, the solution is obtained more easily. For example, spherical polar coordinates are appropriate for treating diffusion inside a ball. Cartesian coordinates are best for a rectangular block, and cylindrical coordinates simplify the treatment of diffusion in a right circular cylinder.

Particle diffusion in the radiation belts

As described previously, radiation belt particles undergo large-scale motion even if no diffusion is taking place. It is therefore necessary to cast the diffusion equation in a form which excludes normal adiabatic motion. The obvious coordinates for such a description are the adiabatic invariants themselves; a distribution function $f(\mu, J, \Phi)$ would remain unchanged if only adiabatic motion occurred. However, a distribution function in these coordinates is difficult to visualize and setting up a diffusion equation in these variables is not straightforward. One must resort to a general approach to diffusion theory which is valid for arbitrary coordinate

systems. The Fokker–Planck equation provides such an approach as it is a prescription for deriving a diffusion equation in terms of the time rate of change in the coordinates of the particles.

A restricted form of the Fokker–Planck equation is derived as follows where $f(\mathbf{x}, t)$ is the number of particles per unit $\mathbf{x}$ at time t. The quantity $\mathbf{x}$ is a vector so that the following derivation is valid for diffusion in more than one dimension. Let $\Psi(\mathbf{x} - \Delta\mathbf{x}, \Delta\mathbf{x}, \Delta t)$ be the probability that a particle at $\mathbf{x} - \Delta\mathbf{x}$ will have its coordinates increased by $\Delta\mathbf{x}$ in time Δt. The characteristics of the mechanisms for producing the diffusion are contained in the probability Ψ. In the spirit of diffusion it is assumed that the processes driving the diffusion will change $\mathbf{x}$ by only a small increment during each elemental interaction. The probability $\Psi(\Delta\mathbf{x})$ will therefore be large only for small $\Delta\mathbf{x}$. In this concept the time interval Δt must be long compared to the time interval between the individual perturbations, yet short enough that major changes in the distribution function do not take place in Δt. At time $t + \Delta t$

$$f(\mathbf{x}, t + \Delta t) = \int \mathrm{d}(\Delta x) f(\mathbf{x} - \Delta\mathbf{x}, t)\Psi(\mathbf{x} - \Delta\mathbf{x}, \Delta\mathbf{x}, \Delta t) \qquad (6.13)$$

where the integral is taken over all values of the increment $\Delta\mathbf{x}$, although contributions to the integral will be small for large $\Delta\mathbf{x}$. Expand the left-hand side in a Taylor series about t and the right-hand side in a Taylor series about $\mathbf{x}$:

$$f(\mathbf{x}, t) + \frac{\partial f}{\partial t}\Delta t + \ldots = \int \mathrm{d}(\Delta\mathbf{x})\left\{ f(\mathbf{x})\Psi(\mathbf{x}, \Delta\mathbf{x}, \Delta t) - \left(\frac{\partial}{\partial x_i}(\Psi f)\right)\Delta x_i + \frac{1}{2}\left(\frac{\partial^2}{\partial x_i \partial x_j}\Psi f\right)(\Delta x_i \Delta x_j) + \ldots \right\} \qquad (6.14)$$

where the x_i are the components of $\mathbf{x}$. In this and the following chapters when an index is repeated within a single term, the term is to be summed over that index. The first term on the right-hand side of equation (6.14) is equal to $f(\mathbf{x}, t)$ since $f(\mathbf{x})$ is not a function of $\Delta\mathbf{x}$ and the integral over Ψ is unity. In the remaining terms on the right the order of differentiation and integration can be interchanged since $\Delta\mathbf{x}$ is not a function of x_i or x_j. With these changes equation (6.14) becomes

$$\frac{\partial f}{\partial t}\Delta t + \ldots = -\frac{\partial}{\partial x_i}\int \mathrm{d}(\Delta\mathbf{x})(\Psi f)(\Delta x_i) + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\int \mathrm{d}(\Delta\mathbf{x})(\Psi f)(\Delta x_i \Delta x_j) + \ldots \qquad (6.15)$$

$f(\mathbf{x})$ is not a function of $\Delta\mathbf{x}$ and can be brought outside the integrals. If one denotes the averaged time rate of changes in the increments by brackets,

$$\langle \Delta x_i \rangle = \frac{1}{\Delta t}\int \Delta x_i \Psi(\mathbf{x}, \Delta\mathbf{x}, \Delta t)\, \mathrm{d}(\Delta\mathbf{x})$$

$$\langle \Delta x_i \Delta x_j \rangle = \frac{1}{\Delta t}\int \Delta x_i \Delta x_j \Psi(\mathbf{x}, \Delta\mathbf{x}, \Delta t)\, \mathrm{d}(\Delta\mathbf{x}) \tag{6.16}$$

one obtains the Fokker–Planck equation as

$$\frac{\partial f}{\partial t} + \ldots = -\frac{\partial}{\partial x_i}\langle \Delta x_i \rangle f + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\langle \Delta x_i \Delta x_j \rangle f + \ldots \tag{6.17}$$

For this equation to be useful, the higher-order terms in the Taylor series expansion must be negligible, a characteristic governed by the mechanisms driving the diffusion. If the incremental steps in the diffusion process are small, these higher-order terms are not important. The coefficients $\langle \Delta x_i \rangle$ and $\langle \Delta x_i \Delta x_j \rangle$ are sometimes called the first and second Fokker–Planck coefficients. In six-dimensional phase space there will be 42 of these coefficients. However, in the three-dimensional space of the adiabatic invariants the number is reduced to 12. Furthermore, only nine of these are independent since $\langle \Delta x_i \Delta x_j \rangle$ and $\langle \Delta x_j \Delta x_i \rangle$ are equal. In actual practice, some mechanisms affect only one variable; in that case the second coefficient is zero unless $i = j$. In the discussion leading to equation (6.20) it will be shown that the first Fokker–Planck coefficients can be expressed in terms of the second Fokker–Planck coefficients. Also, by selecting the coordinates cleverly, it is possible to further reduce the number of coefficients needed and simplify equation (6.17).

The transformation from six-dimensional phase space to the three-dimensional space of the adiabatic invariants represents a loss of information. The adiabatic invariant representation contains no information on the instantaneous phase of the particle, that is, its position in gyration, latitude or longitude. Such information is not needed if the particles are uniformly distributed in phase. In the remainder of this chapter it will be assumed that there is an efficient phase mixing mechanism which quickly restores phase uniformity after each elemental perturbation. The distribution function will therefore represent a value averaged over the phases of the variables.

The Fokker–Planck equation as derived here is linear in that the coefficients do not depend on $f(\mathbf{x}, t)$. This condition obtains if the probability of changes Ψ is independent of $f(\mathbf{x}, t)$. If one were considering mechanisms in which the particles act on themselves, for example by scattering from each other, this linearity would be lost and a more

complex approach would be needed. The great virtue of the Fokker–Planck equation in this application is that it provides a prescription for a diffusion equation, even if the coordinates and geometry are difficult to interpret. One only needs to calculate the coefficients by applying equation (6.16) to obtain a valid equation. Although it is sometimes difficult to obtain $\langle \Delta x_i \rangle$, this coefficient is not necessary, as will be seen next.

In Chapter 5 an equation for the evolution of the phase space distribution function $F(\mathbf{p}, \mathbf{q})$ was obtained in proving Liouville's theorem. This equation, generalized to three spatial dimensions and three conjugate momenta, is

$$\frac{\partial F}{\partial t} + \dot{q}_i \frac{\partial F}{\partial q_i} + \dot{p}_i \frac{\partial F}{\partial p_i} = \frac{\mathrm{d}F}{\mathrm{d}t} = 0 \tag{6.18}$$

From (6.18) it can be seen that if $\partial F/\partial q_i$ and $\partial F/\partial p_i$ are zero for all q_i and p_i, then the time variation of F, namely $\partial F/\partial t$, must also be zero. Thus, if the distribution is uniform in p_i and q_i, it will remain uniform and constant.

If one uses phase space coordinates for the Fokker–Planck equation (6.17) and postulates that the distribution $F_0(\mathbf{p}, \mathbf{q})$ is uniform in $\mathbf{p}$ and $\mathbf{q}$, one obtains

$$\begin{aligned} \frac{\partial F_0}{\partial t} = 0 &= -\frac{\partial}{\partial x_i} \langle \Delta x_i \rangle F_0 + \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} \langle \Delta x_i \Delta x_j \rangle F_0 \\ &= F_0 \frac{\partial}{\partial x_i} \left[-\langle \Delta x_i \rangle + \frac{1}{2} \frac{\partial}{\partial x_j} \langle \Delta x_i \Delta x_j \rangle \right] \end{aligned} \tag{6.19}$$

The quantity in brackets must be a constant, independent of the mechanisms causing changes, and further considerations show that it is zero. Thus, in phase space coordinates there is a relationship between the first and second Fokker–Planck coefficients:

$$\langle \Delta x_i \rangle = \frac{1}{2} \frac{\partial}{\partial x_j} \langle \Delta x_i \Delta x_j \rangle \tag{6.20}$$

Although this relationship was derived for a uniform F_0, the Fokker–Planck coefficients do not depend on the distribution function and the result ((6.20)) is general. With equation (6.20) substituted for $\langle \Delta x_i \rangle$ in equation (6.17) and neglecting the higher-order terms the Fokker–Planck equation takes on the diffusion form

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial x_i} \left[\frac{\langle \Delta x_i \Delta x_j \rangle}{2} \frac{\partial F}{\partial x_j} \right] \tag{6.21}$$

where $\frac{1}{2} \langle \Delta x_i \Delta x_j \rangle = D_{ij}$ is the diffusion matrix.

The simplicity of equation (6.21) is a direct result of working in phase

space variables. In any other set of variables, say y_i where the distribution $Y(\mathbf{y})$ is the number of particles per unit $\Delta\mathbf{y}$, the diffusion equation ((6.21)) would become

$$\frac{\partial Y(\mathbf{y}, t)}{\partial t} = \frac{\partial}{\partial y_i}\left[\frac{\langle \Delta y_i \Delta y_j \rangle}{2} \cdot \mathcal{J} \frac{\partial}{\partial y_j}\left(\frac{Y(\mathbf{y}, t)}{\mathcal{J}}\right)\right] \tag{6.22}$$

where

$$\mathcal{J} = \frac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)} \tag{6.23}$$

is the Jacobian for transforming from phase space variables $\mathbf{x}$ to any other coordinates $\mathbf{y}$. There must exist a one-to-one mapping between the two coordinates systems, otherwise the Jacobian will be zero. Since $\mathcal{J}$ enters in both the numerator and denominator of (6.22), the constant factors in $\mathcal{J}$ cancel and need not be considered in making this transformation.

The most logical variables for radiation belt calculations are the adiabatic invariants μ, J and Φ. Since it is possible to transform from phase space variables to μ, J and Φ by a series of canonical transformations whose Jacobians are constants, equation (6.21) can immediately be written with the adiabatic invariants as independent variables. The transformed equation is

$$\frac{\partial F(\mu, J, \Phi)}{\partial t} = \frac{\partial}{\partial x_i}\left(D_{ij} \frac{\partial}{\partial x_j} F(\mu, J, \Phi)\right) \tag{6.24}$$

where x_i with $i = 1, 2$ or 3 denotes the adiabatic invariants μ, J and Φ, and $F(\mu, J, \Phi)\,\mathrm{d}\mu\,\mathrm{d}J\,\mathrm{d}\Phi$ is the number of particles in the elemental volume $\mathrm{d}\mu\,\mathrm{d}J\,\mathrm{d}\Phi$. Transformations to other variables can be made using equation (6.22) and the appropriate Jacobian.

In treating diffusion in the radiation belts, a shrewd choice of coordinates based on the mechanism causing diffusion can greatly simplify the calculation. If an adiabatic invariant is not changed by the process considered, that invariant is a good one to use as a coordinate since the diffusion terms containing it will vanish. Possible choices of the three variables in addition to (μ, J and Φ) are (E, α_{eq}, L), (μ, B_{m}, L) and many others. In all cases it is essential that the Jacobian relating the adiabatic invariants to these new variables does not vanish.

For example, if one is dealing with diffusion in Φ only, the adiabatic invariants μ, J and Φ are a reasonable choice since μ and J are constant and the only diffusion coefficient required is $\langle(\Delta\Phi)^2\rangle/2$. On the other hand, if only the pitch angles change in the diffusion process, then (E, α_{eq}, L) would be a good choice as the only term in (6.22) would be the one containing $\langle(\Delta\alpha_{\mathrm{eq}})^2\rangle$. In each of these cases the equation to be solved has only one term in the sum on the right-hand side.

Constructing the Jacobian for a given transformation is straightforward but sometimes tedious. The procedure is to express the adiabatic invariants in terms of the desired variables and then carry out partial differentiation and evaluate the Jacobian determinant. Each partial derivative operation is done keeping the other two independent variables constant. As an example, if one wishes to treat changes in Φ but prefers to use the set of variables (μ, J, L) (since L is more easily associated with experiments than Φ), the Jacobian is

$$\mathcal{J} = \begin{vmatrix} \frac{\partial\mu}{\partial\mu} & \frac{\partial\mu}{\partial J} & \frac{\partial\mu}{\partial L} \\ \frac{\partial J}{\partial\mu} & \frac{\partial J}{\partial J} & \frac{\partial J}{\partial L} \\ \frac{\partial\Phi}{\partial\mu} & \frac{\partial\Phi}{\partial J} & \frac{\partial\Phi}{\partial L} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{L^2} \end{vmatrix} = \frac{1}{L^2} \tag{6.25}$$

For diffusion in L the diffusion equation ((6.22)) then becomes

$$\frac{\partial f(\mu, J, L)}{\partial t} = \frac{\partial}{\partial L}\left[D_{LL}\frac{1}{L^2}\frac{\partial}{\partial L}(L^2 f(\mu, J, L))\right] \tag{6.26}$$

where $D_{LL} = \langle(\Delta L)^2\rangle/2$.

A more complex example is the case of pitch-angle scattering in which α_{eq} is changed but the energy and L are kept constant. A good set of independent variables is E = kinetic energy, L and $x = \cos\alpha_{eq}$. For these variables, the adiabatic invariants can be written as

$$\mu = \frac{E(E + 2m_0c^2)(1 - x^2)}{2m_0c^2 B_{eq}} \tag{6.27}$$

$$J = \frac{\sqrt{[E(E + 2m_0c^2)]}}{c} 4\int_0^{s_m} \cos\alpha(x, s)\,ds$$

$$= \frac{\sqrt{[E(E + 2m_0c^2)]}}{c} LN_1(x) \tag{6.28}$$

$$\Phi = L^{-1} \tag{6.29}$$

In evaluating the Jacobian, note that B_{eq} in (6.27) is a function of L. Also, the derivative of $N_1(x)$ is obtained by expressing $\cos\alpha(x, s)$ in terms of x and differentiating the integral, leading to

$$\frac{dN_1(x)}{dx} = \frac{x}{(1 - x^2)}\{N_2(x) - N_1(x)\} \tag{6.30}$$

where

$$N_2(x) = \frac{4}{L}\int_0^{s_m} \frac{ds}{\cos\alpha} = \frac{\tau_b v}{L} \tag{6.31}$$

is $1/L$ times the distance along the helix between mirror points. The resulting Jacobian, omitting constant terms, is

$$\mathcal{J} = \sqrt{[E(E + 2m_0c^2)]}\,(E + m_0c^2)L^2xN_2(x) \tag{6.32}$$

After canceling terms with no x dependence the diffusion equation for changes in $x = \cos\alpha_{eq}$ becomes

$$\frac{\partial f(E, x, L)}{\partial t} = \frac{\partial}{\partial x}\left[D_{xx}xN_2(x)\frac{\partial}{\partial x}\left(\frac{f(E, x, L)}{xN_2(x)}\right)\right] \tag{6.33}$$

The remaining task is to relate the distribution function in whatever variables are chosen to the experimentally accessible quantities, namely the particle fluxes. This relationship can often be found directly using $f(x_1, x_2, x_3)$ to represent the number of particles in unit $dx_1\,dx_2\,dx_3$ and calculating the number of particles in an energy interval which cross a unit area in the equatorial plane at α_{eq} per unit time. This result can be used to obtain flux. A more systematic method is to use the relation (equation 5.16) between flux and phase space density. Since phase space density is equal (within constant factors) to the density of particles in the elemental volume defined by the coordinates (μ, J and Φ), the phase space density is related to the chosen distribution function by the Jacobian transforming that distribution to the space defined by the adiabatic invariants. Thus, the flux $j(E, a)$ is related to the phase space density by

$$j(E, \alpha) = p^2F(\mathbf{p}, \mathbf{q}) \propto p^2F(\mu, J, \Phi) \propto p^2\frac{f(x_1, x_2, x_3)}{\mathcal{J}} \tag{6.34}$$

For example, in the L diffusion case the particle flux is obtained directly from $f(\mu, J, L)$ and can be expressed in several ways:

$$\begin{aligned} j(E, \alpha) = p^2f(\mu, J, L)L^2 &= \frac{E(E + 2m_0c^2)}{c^2}L^2f(\mu, J, L) \\ &= \frac{\mu 2m_0B_0}{L(1 - x^2)}f(\mu, J, L) \end{aligned} \tag{6.35}$$

The Jacobians for several selections of coordinates are given in Table 6.1.

The last selection of variables is suitable for treating simultaneous diffusion in pitch angle at constant energy and diffusion in L at constant mirror latitude. The scattering process only changes x and the diffusion in L only changes L. The variable z is fixed in both processes for equatorial particles and is nearly constant for $\alpha_{eq} \neq 90°$. The diffusion equation will therefore have only two terms and two diffusion coefficients, D_{LL} and D_{xx}.

Table 6.1. *Frequently used coordinates and corresponding Jacobians*

Variables	Jacobian is proportioned to
μ, J, L	L^{-2}
$L, E, x = \cos \alpha_{eq}$	$\sqrt{[E(E + 2m_0c^2)](E + m_0c^2)}L^2 x N_2(x)$
$\mu, L, \xi = (1 - x^2) = \sin^2 \alpha_{eq}$	$\mu^{1/2}\xi^{-3/2}L^{-5/2}N_2(\xi)$
$x, L, z = \dfrac{(E + 2m_0c^2)EL^3}{m_0^2c^4}$	$L^{-5/2}x N_2(x)z^{1/2}$

As a generalization of Fick's law, the transformed equation (6.22) shows that the particle current across any coordinate y_i is given by

$$\text{current across } y_i = -\frac{\langle \Delta y_i \Delta y_j \rangle}{2} \mathcal{J} \frac{\partial}{\partial y_j}\left(\frac{Y(\mathbf{y}, t)}{\mathcal{J}}\right) \tag{6.36}$$

When only the diagonal elements $\langle (\Delta x_i)^2 \rangle$ of the diffusion matrix are not zero, the direction of flow will be given by the signs of the derivative factors. The value of this feature can be appreciated from the following one-dimensional example.

Using (μ, J, L) coordinates and treating diffusion in L, the derivative term in (6.36) for equatorial mirroring particles becomes (see equations (6.26) and (6.35))

$$\frac{\partial}{\partial L}(L^2 f) \propto \frac{\partial}{\partial L}\left(L^3 j\left(E, \alpha_{eq} = \frac{\pi}{2}, L\right)\right)_{\mu,J} \tag{6.37}$$

where the last relation in equation (6.35) is used to express flux in terms of $f(\mu, J, L)$. Because μ and x are constant for equatorial particles which diffuse in L, keeping μ and J constant, only the L dependence of $\mathcal{J}$ is retained.

A plot of $L^3 j(\alpha_e = \pi/2)$ as a function of L, where j is evaluated at constant μ (not constant energy), will immediately show whether particles are diffusing inwards towards the Earth or outwards towards the magnetopause. This flow direction is determined entirely by the particle distribution and is independent of the mechanism causing the diffusion. The magnitude of the diffusion is, of course, proportional to the diffusion coefficient and is therefore influenced by the diffusion mechanism (see Figure 6.3).

It was stated earlier that a careful choice of coordinates could greatly reduce the difficulty of working with the general diffusion equation. Three

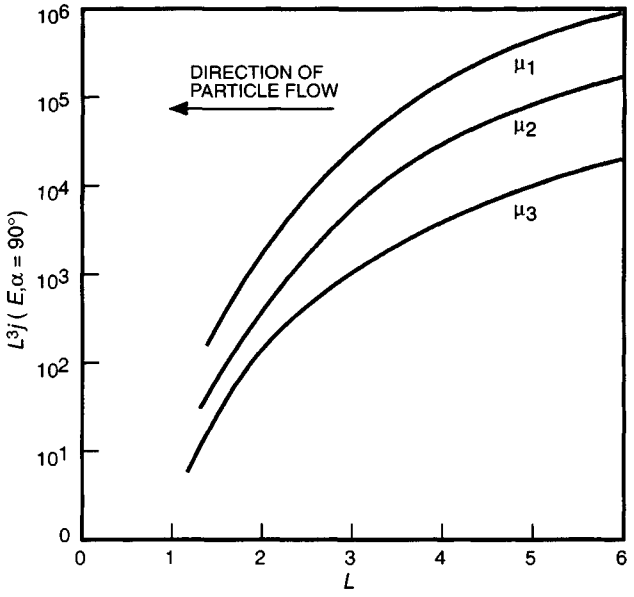


Figure 6.3. Illustration of how the particle flow direction is related to the derivative of the distribution function.

coordinates are necessary to describe a particle's condition (after averaging out the three cyclic phase variables) and the diffusion matrix therefore contains nine elements. This number can usually be reduced by choosing one, or at most two, variables which are not altered by the diffusion process. If two of the variables are constants, the diffusion matrix has only one non-zero element, and the diffusion equation is one-dimensional, similar to equation (6.3). Even if two coordinates are active during the diffusion, it may be possible to choose variables in which the changes are uncorrelated, in which case the off-diagonal terms $\langle \Delta x_i \Delta x_j \rangle i \neq j$ vanish.

Finally, the choice of coordinates can be influenced by the ease of calculation of the diffusion coefficients. The radial coordinate L is a convenient variable for evaluating diffusion across magnetic shells as the change in L by magnetic and electric field fluctuations is readily calculated (see Chapter 8). Diffusion in the pitch angles of trapped electrons by collisions with atmospheric atoms is simplified by using $x = \cos \alpha_{eq}$ as the basic variable, as will be shown in the next chapter.

Because of the special advantages of phase space coordinates many authors prefer to use a distribution function proportional to the density in phase space even if the coordinates in the diffusion equation are not phase

space variables. In this case equation (6.22) would be

$$\frac{\partial F(\mathbf{y}, t)}{\partial t} = \frac{1}{\mathcal{J}} \frac{\partial}{\partial y_i} \left[\frac{\langle \Delta y_i \Delta y_j \rangle}{2} \mathcal{J} \frac{\partial F(\mathbf{y}, t)}{\partial y_j} \right] \tag{6.38}$$

where $F(\mathbf{y}, t)$ is the number of particles per unit volume of phase space at $\mathbf{y}$ and t. The diffusion equation for phase space density corresponding to equation (6.26) for L diffusion at constant μ, J is

$$\frac{\partial F(\mu, J, L)}{\partial t} = L^2 \frac{\partial}{\partial L} \left[D_{LL} \frac{1}{L^2} \frac{\partial F(\mu, J, L)}{\partial L} \right] \tag{6.39}$$

Injection of protons by cosmic ray albedo neutrons

In this chapter the diffusion of particles through the Earth's magnetic trapping region has been described. This diffusion, which will be treated in greater detail in Chapters 7 and 8, must occur if the solar and ionospheric plasmas supply particles to the radiation belts. However, one source of radiation belt protons and electrons does not require non-adiabatic motion. This source is the spontaneous decay within the trapping region of energetic neutrons produced by cosmic ray collisions with the Earth's atmosphere.

When a cosmic ray strikes the nucleus of an atmospheric atom, the products include high energy neutrons. Some of these neutrons escape the atmosphere immediately; others escape after further collisions. Thus, in the region around the Earth there is a flux of outward moving neutrons, a small fraction of which will decay within the magnetosphere. A neutron decays into a proton, an electron and a neutrino, the half life of the neutron being about 630 s. Because the proton mass is much larger than that of the electron or the neutrino, a proton from neutron decay will move initially with almost the same velocity as the parent neutron. If the decay takes place in the geomagnetic field and the newly born proton has a pitch angle outside the loss cone, the proton will be trapped. Electrons from the decay are emitted isotropically in the center of mass system and will also be trapped if their pitch angles are outside the loss cone. The method of calculating the proton source strength is sketched below for equatorially trapped protons.

Figure 6.4 illustrates, on an exaggerated scale, the gyration of a trapped proton in the equatorial plane at distance $L\mathrm{R_E}$ from the center of the Earth. Each differential element of its path may be described by a vector $\mathbf{dr}$ whose sense is the same as the velocity of the proton. If $\mathbf{dr}$ extended backward intersects the Earth's atmosphere, then a neutron originating at

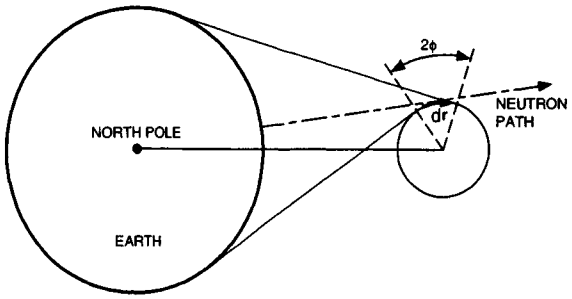


Figure 6.4. Geometry of the source of protons from decay of cosmic ray albedo neutrons.

that intersection point can decay while traversing the element **dr** and thus add a proton to the proton flux in **dr**. If the extension of **dr** does not intersect the atmosphere, injection is not possible in that element.

If $j_n(E, \alpha)$ is the differential, directional neutron flux in the direction **dr**, then the source of trapped proton flux at that position and direction due to neutron decays in a 1 cm^2 column of length dr is given by

$$\frac{\mathrm{d}j_p(E, \alpha)}{\mathrm{d}t}\,\mathrm{d}r = j_n(E, \alpha)\frac{\mathrm{d}r}{v} \cdot \frac{1}{\gamma\tau_n} \tag{6.40}$$

where τ_n is the mean life of the neutron and γ is the relativistic time dilatation factor. The factor $\mathrm{d}r/v$ is the time a neutron of velocity v spends in an element of length dr.

The average flux increase over the gyration cycle of the proton is obtained by integrating both sides of equation (6.38) over a complete gyration, noting that $j_n(E, \alpha)$ vanishes when the backward extension of dr does not intersect the Earth. In the equatorial plane with gyration radius $\rho \ll R_E$ the geometry of Figure 6.4 shows that neutrons contribute to j_p only over the fraction ϕ_m/π of the gyration, where $\sin\phi_m = 1/L$. Thus, if j_n is independent of the zenith angle, the growth in proton flux is given by

$$\frac{\mathrm{d}j_p(E, \alpha_{eq} = \pi/2)}{\mathrm{d}t} = \frac{1}{\pi} j_n\left(E, \alpha_{eq} = \frac{\pi}{2}\right)\frac{1}{v\gamma t_n}\sin^{-1}\frac{1}{L} \tag{6.41}$$

If $j_n(E, \alpha)$ is not independent of the zenith angle, this dependence must be included in the integration of equation (6.38). For the general case of injection off the equator, where neutrons emitted at all latitudes may contribute, the latitude dependence of the albedo neutron flux must also be included.

Extensive numerical calculations of the neutron decay injection show that the source of high-energy protons is extremely small. However,

within the inner zone ($L \approx 1.5$) the loss rates of high-energy protons are very low, and the equilibrium flux which can be produced by this source is comparable to the observed proton flux above 50 Mev. The observed fluxes of low-energy protons (< 10 Mev) and electrons are too large to be produced by this weak, but well understood, source.

Problems

1. A distribution of particles obeys the diffusion equation in one dimension

$$\frac{\partial f(x, t)}{\partial t} = \frac{\partial}{\partial x}\left(D(x)\frac{\partial f}{\partial x}\right)$$

At $x = 0$ a source injects particles so that the net current in the x direction is S_0. At x_1 a sink absorbs all particles reaching x_1:

(a) If $D(x) = D_0 = a$ constant, find the steady-state distribution $f(x)$.
(b) If $D(x) = D_0 e^{ax}$ find the steady-state distribution.

2. (a) The radial diffusion equation for equatorially trapped particles is

$$\frac{\partial f(\mu, J, L)}{\partial t} = \frac{\partial}{\partial L}\left[\frac{D_{LL}}{L^2}\frac{\partial}{\partial L}(L^2 f)\right]$$

Let $D_{LL} = D_0 L^{10}$ and assume that there is a source of particles at $L = 7$. All particles are absorbed by the atmosphere at $L = 1$. If $f(L = 7) = f_0$, find the steady-state L dependence of the distribution function. Show that the net current is diffusing inward.

(b) With the same conditions as above, except that $D_{LL} = D_0 L^n$, find the L dependence of $f(L)$.
(c) If $n = 1$ for the conditions given in part (b) find $f(L)$. Sketch $f(L)$.

3. The diffusion equation in a two-dimensional cartesian coordinate system is

$$\frac{\partial f(x, y, t)}{\partial t} = \frac{\partial}{\partial x} D\frac{\partial f}{\partial x} + \frac{\partial}{\partial y} D\frac{\partial f}{\partial y}$$

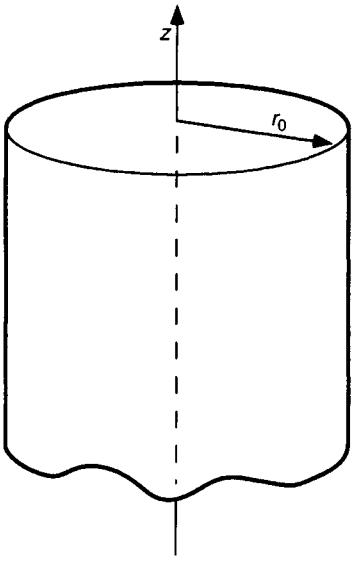
Let $f = X(x)Y(y)T(t)$, separate the variables and write the general solution for the equation where $D = D_0 =$ constant and

$$f = 0 \text{ at } x = 0 \text{ and at } y = 0$$

$$f = 0 \text{ on the lines } x = x_1 \text{ and } y = y_1$$

4. Consider the diffusion of particles in an infinite cylindrical medium. Assume that there is no z or ϕ dependence of the distribution. The diffusion equation is

$$\frac{\partial f(r, t)}{\partial t} = \nabla \cdot D\nabla f$$



Assume $D = D_0 =$ constant. Write the diffusion equation in cylindrical coordinates. Separate the time and spatial variables and find the lowest-order eigenfunction and eigenvalue. Assume that all particles are absorbed at the surface $r = r_0$.

5. A group of 5 kev protons with $\alpha_{eq} = 90°$ is injected at $L = 7$ in a dipole field. If they diffuse with $\mu =$ constant and $J =$ constant, find the energy of the protons when they reach $L = 5$, $L = 3$ and $L = 1.5$.

6. The radial diffusion equation is given below where f is the number of particles per unit L, per unit $\mu =$ magnetic moment, and per unit $J =$ integral invariant:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial L}\left[\frac{D_{LL}}{L^2}\frac{\partial}{\partial L}(L^2 f)\right]$$

Assume that the plasma sheet is injecting a steady net electron source S_0 (electrons per unit $\Delta\mu$ per unit ΔJ) into the radiation belt at $L = 7$ and assume that there are no pitch-angle scattering losses. Also assume that D_{LL} is constant ($D_{LL} = D_0$) everywhere except in the interval $3 < L < 4$ where D_{LL} is infinite:

(a) Find the expression for f in the interval $1 < L < 3$.
(b) Find the expression for f in the interval $3 < L < 4$.
(c) What is the value of f at $L = 7$?

Answers should be in terms of D_0, S_0 and L.

7. One source of high-energy protons in the radiation belt is the decay of neutrons which are produced in the atmosphere by collisions of cosmic rays with oxygen and nitrogen nuclei. Each neutron then decays into a proton, an

electron, and a neutrino, and if this decay occurs while the neutron is moving through the trapping region, the electron and proton may become trapped. In the decay process momentum is conserved so the proton moves with approximately the velocity of the parent neutron. The electron is emitted with an energy of several hundred kilovolts in a random direction:

(a) The half-life of a neutron (which is the time interval in which half the neutrons in an initial group decay) is approximately 10 minutes. Therefore, the probability that a neutron will not have decayed by time t is given by

$$P(t) = \exp(-\lambda t)$$

Show that $\lambda = 1.16 \times 10^{-3}\ \mathrm{s}^{-1}$.

(b) If a 98 MeV neutron is emitted from the top of the atmosphere on the equator in the zenith direction, what is the probability that the neutron will decay inside the $L = 5$ shell? What will be the J value of the trapped proton? (Neglect the relativistic time dilation factor.)

(c) If the neutron decay occurs at $L = 2$, what is the probability that the resulting electron will be in a trapped orbit? (Assume a dipole field and neglect the atmosphere.)

8. Neglecting the phase coordinates, three independent coordinates are needed to specify a trapped particle's trajectory, the most fundamental being the three adiabatic invariants μ, J, and Φ. Which of the following sets are suitable to completely define a trapped particle's trajectory in a dipole field? Assume that you know the particle species:

$$(\cos\alpha_{eq},\ E,\ L),\ (B_{eq},\ B_m,\ \alpha_{eq})$$
$$(B_{eq},\ \cos\alpha_{eq},\ \mathrm{p}),\ (B_{eq}/B_m,\ E,\ L)$$
$$(E,\ p,\ L),\ (L,\ J,\ p),\ (L,\ \Phi,\ \mu)$$
$$(\alpha_{eq},\ B_m,\ \Phi),\ (L,\ J,\ \tau_b),\ (E,\ \mu,\ \Phi)$$

where

α_{eq} = equatorial pitch angle
B_{eq} = equatorial magnetic field of guiding center path
B_m = magnetic field at mirror point
p = scalar momentum
E = kinetic energy