Introduction to Geomagnetically Trapped Radiation

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Chapter

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Diffusion in pitch angle

Diffusion in the pitch angles of trapped particles is an important redistribution and loss mechanism. Observations of particles with pitch angles inside the loss cone indicate that this process takes place at all L values in which trapping occurs, although the process proceeds more rapidly with increasing L value. At $L \approx 6$, which is the magnetic shell whose field lines connect to the auroral zone, rapid pitch-angle diffusion of electrons is a common occurrence. Electrons are fed into the loss cone by multiple deflections, and their subsequent motion into the atmosphere in both the northern and southern hemispheres supplies energy to the polar aurora.

Electron diffusion by collisions with atmospheric atoms

Collisions of electrons with atmospheric atoms is one cause of pitch-angle diffusion. While collisions are the dominant loss for electrons at only very low L values ($L < 1.3$), they occur at all L for those electrons which mirror at low altitudes. It is a well-understood process, and for this reason it is instructive to derive the diffusion coefficient from the basic formula describing the scattering of electrons by atoms. Because of their greater mass, protons and heavier ions are not scattered appreciably in pitch angle by collisions. The cumulative effect of collisions on ions is to reduce the ion velocity to thermal values while leaving their direction largely unchanged. The following treatment of electron collisions with the atmosphere is appropriate for regions in which the ambient atmosphere has only a small effect during the electron drift period. The diffusion coefficients will therefore be obtained by averaging over a complete drift cycle.

Convenient coordinates for pitch-angle diffusion calculations are E, L and $x = \cos \alpha_{eq}$. The value of L is not altered by a collision, as the guiding center can move at most only two gyroradii in a single collision, and the change in L is usually much less. The electron energy will be almost unchanged when the electron is deflected by the much heavier nucleus of an atom. The loss of electron energy by collisions with bound or free electrons is important and will be included later by adding a term to the standard Fokker–Planck equation. Thus, for the pitch-angle diffusion process x is the only coordinate involved and $\langle(\Delta x)^2\rangle/2$ is the only diffusion coefficient to be evaluated. The equatorial pitch angle, or some other function of α_{eq}, could equally well be used as the variable, but the equations are somewhat simpler using the cosine of the pitch angle. The local pitch angle, or some function of it, would not be a useful variable as it is not constant during the unperturbed bounce motion of the electron.

The cross-section for scattering of electrons by the nucleus of a neutral atom is

$$\sigma(\eta) = \frac{z^2 e^4}{64\pi^2 \varepsilon_0^2 m_0^2 c^4} \frac{1-\beta^2}{\beta^4} \frac{1}{\sin^4 \dfrac{\eta}{2}} \tag{7.1}$$

where z is the atomic number of the atom, ε_0 is the electric permitivity of free space and η is the scattering angle. Because of the $\sin^4 \eta/2$ term in the denominator the cross-section is large for small deflections, and this feature justifies the assumption inherent in the Fokker–Planck equation that the deflection in an individual interaction is small. To calculate $\langle(\Delta x)^2\rangle$ we will first find the average change per unit time in the local pitch-angle cosine due to collisions. With this collision average we will compute the equivalent change in the cosine of the equatorial pitch angle. Then this change will be averaged over the bounce motion of the particle, weighted at each portion of the path by the density of scattering centers at that path increment and by the time the particle spends in that increment. Finally, the average over longitude will be computed giving the diffusion coefficient $\langle(\Delta x)^2\rangle/2$.

The geometry of an electron scattering event is illustrated in Figure 7.1, where $\mathbf{v}$ and $\mathbf{v}'$ are the electron velocities before and after the collision. If the local pitch angles before and after the collision are α and α', respectively, and the angle through which the electron is scattered is η, the change in $\cos\alpha$ produced by a particular collision is (see Figure 7.1)

$$\begin{aligned}\Delta\cos\alpha &= \cos\alpha' - \cos\alpha \\ &= \cos\alpha\cos\eta + \sin\alpha\sin\eta\cos\psi - \cos\alpha \\ &= -2\cos\alpha\sin^2\frac{\eta}{2} + (1-\cos^2\alpha)^{1/2}\sin\eta\cos\psi\end{aligned} \tag{7.2}$$

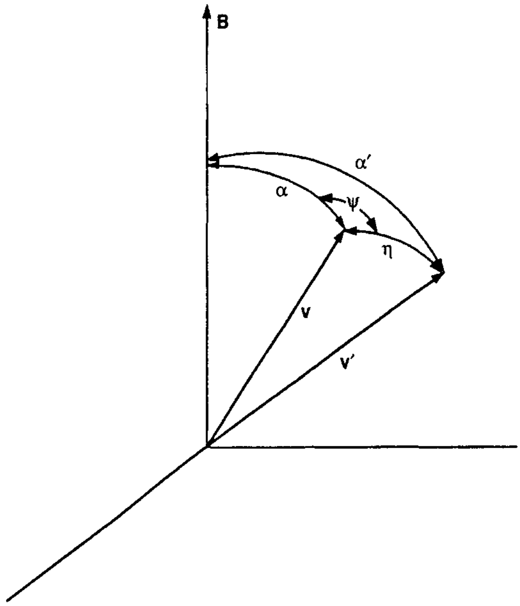


Figure 7.1. Geometry of particle scattering through angle η which changes the velocity from $\mathbf{v}$ to $\mathbf{v}'$, and changes the pitch angle from α to α'.

The average time rate of change in $\cos\alpha$ is the value of $\Delta\cos\alpha$ from equation (7.2) multiplied by the probability of that change occurring in unit time and integrated over all possible $\Delta\alpha$. The probability per unit time of an electron scattering through angle η into unit solid angle is $Nv\sigma(\eta)$, where N is the density of scattering centers. If the sum over all scattering possibilities is denoted by the use of curly brackets, the time rate of change of any quantity $g(\eta, \psi)$ by collisions is given by

$$\{g\} = \int_0^{2\pi} d\psi \int_0^{\pi} g(\eta, \psi) N v \sigma(\eta) \sin\eta \, d\eta \qquad (7.3)$$

The collision average time rate of change in $(\Delta\cos\alpha)^2$ is obtained by squaring $\Delta\cos\alpha$ in (7.2) and substituting $(\Delta\cos\alpha)^2$ for $g(\eta, \psi)$ in (7.3). The cross-term in the square of (7.2) contains $\cos\psi$ and vanishes after integrating over ψ. The two remaining terms can be compared for relative value by noting that $\sin\eta\, d\eta = 4\sin(\eta/2)\, d(\sin\eta/2)$.

The square of the first term of (7.2) when inserted in (7.3) has a η dependence of $\int_0^{\pi}\sin(\eta/2)\, d(\sin(\eta/2)$, which is of order 1. The square of the second term contains a divergent factor $\int_0^{\pi} d(\sin\eta/2)/(\sin\eta/2)$ and will therefore be the dominant term. The logarithmic divergence results from the coulomb cross-section (7.1), which becomes infinite for $\eta = 0$. The cross-section expression was derived for scattering of an electron by an unshielded nucleus of charge ze. In reality, the charge is shielded by the orbital electrons and there exists a minimum scattering angle η_{min} corresponding to a maximum impact parameter at the shielding boundary. Therefore, the lower limit of the integral in (7.3) over η should not be zero but should be some minimum value η_{min}.

After converting $v(1-\beta^2)\beta^{-4}$ to an expression in kinetic energy the collision average of the rate of change in $(\Delta \cos \alpha)^2$ is

$$\{(\Delta \cos \alpha)^2\} = \frac{e^4 c(E + m_0 c^2)}{4\pi\varepsilon_0^2 E^{3/2}(E + 2m_0c^2)^{3/2}} Nz^2 \ln \frac{2}{\eta_{min}}(1 - \cos^2 \alpha) \quad (7.4)$$

It is now necessary to convert the local collision average to the change in x, the cosine of the equatorial pitch angle, and average these changes over the gyration and bounce motion of the particle. The gyration phase does not enter into equation (7.2) and need not be considered further. However, $\{(\Delta \cos \alpha)^2\}$ must be averaged over the bounce motion as α and N depend on the electron's location on the field line. The relation between $\Delta \cos \alpha$ and $\Delta \cos \alpha_{eq} = \Delta x$ is obtained from the fact that the magnetic moment is constant along the bounce:

$$1 - \cos^2 \alpha = (1 - x^2)\left(\frac{B(s)}{B_{eq}}\right) \quad (7.5)$$

Differentiating (7.5) leads to

$$\cos \alpha \, \mathrm{d}(\cos \alpha) = x \, \mathrm{d}x\left(\frac{B(s)}{B_{eq}}\right) = x \, \mathrm{d}x \frac{(1 - \cos^2 \alpha)}{(1 - x^2)}$$

and

$$\begin{aligned}\{(\Delta x)^2\} &= \{(\Delta \cos \alpha)^2\}(\mathrm{d}x/\mathrm{d}\cos \alpha)^2 \\ &= \{(\Delta \cos \alpha)^2\}\frac{\cos^2 \alpha}{x^2}\frac{(1 - x^2)^2}{(1 - \cos^2 \alpha)^2} \quad (7.6)\end{aligned}$$

The bounce average is obtained by integrating (7.6) between mirror points weighting each element of the path by the time a particle spends in that increment:

$$\langle(\Delta x)^2\rangle = \int_{s_m}^{s'_m}\{(\Delta x)^2\}\frac{\mathrm{d}s}{v \cos \alpha}\Big/\int_{s_m}^{s'_m}\frac{\mathrm{d}s}{v \cos \alpha} \quad (7.7)$$

$$\langle(\Delta x)^2\rangle = \frac{e^4 c(E + m_0 c^2)}{4\pi\varepsilon_0^2 E^{3/2}(E + 2m_0c^2)^{3/2}}\int_{s_m}^{s'_m}\frac{\mathrm{d}s}{v \cos \alpha}\frac{\cos^2 \alpha}{x^2}$$

$$\times \frac{(1-x^2)^2}{(1-\cos^2\alpha)} N(s) z^2 \ln \frac{2}{\eta_{\min}} \times \frac{1}{\displaystyle\int_{s_m}^{s'_m} \frac{ds}{v\cos\alpha}} \tag{7.8}$$

If the atmosphere contains a mixture of elements, each with a different z_i, N_i and $\eta_{i,\min}$, then $N(s)z^2 \ln(2/\eta_{\min})$ in equation (7.8) is replaced by $\sum_i N_i z_i^2 \ln(2/\eta_{i,\min})$. Because $N(s)$ increases rapidly as s increases and the electron samples the lower atmosphere, $\langle(\Delta x)^2\rangle$ will be a strong function of x. In general, equation (7.8) must be computed numerically using an atmospheric model for $N(s)$.

Finally, the value of $\langle(\Delta x)^2\rangle$ must be averaged over the drift in longitude. If the geomagnetic field were a centered dipole, this averaging would not be necessary. However, there is considerable distortion of the field at low altitudes where the atmosphere is important to trapped particles. Therefore, in longitude or drift averaging, one must take account of the fact that the particle 'sees' a different atmosphere at each longitude, and the northern and southern halves of its bounce trajectory pass through different air densities. Furthermore, the angular drift velocity varies with longitude. These factors are usually accounted for by constructing an 'average' atmosphere based on the atmospheric density along traces of constant B, L about the Earth, the density at each point being weighted inversely with drift rate. This average atmosphere is then used to evaluate the integral of equation (7.8).

The energy loss which electrons experience in collisions with free and bound electrons can be included in equation (6.33) by adding an additional term to the Fokker–Planck equation

$$\left.\frac{\partial f}{\partial t}\right|_{\text{energy loss}} = -\frac{\partial}{\partial E}\langle\Delta E\rangle f \tag{7.9}$$

where $\langle\Delta E\rangle$ is the time rate of energy loss by the electron as it collides with free electrons and with the orbital electrons of neutral atoms. The loss of energy as an electron penetrates material, the so-called dE/dx (in this expression dx is the differential of length and not $\cos\alpha_{eq}$) is well known. The average time rate of energy loss is therefore

$$\{\Delta E\} = v \cdot \frac{dE}{dx} = -\frac{e^4}{4\pi\varepsilon_0^2 m_0 c\beta} \sum_i z_i N_i \ln \frac{E(E/m_0c^2+2)^{1/2}}{I_i} \tag{7.10}$$

where I_i is the mean ionization potential for an atom of the ith species. Again it is necessary to perform bounce and drift averages of the atmosphere to obtain an 'average' atmosphere which the trapped electron will experience. The trajectory average of $\{\Delta E\}$ is then

$$\langle \Delta E \rangle = \frac{1}{\tau_d}\int_0^{2\pi}\frac{d\phi}{\dot{\phi}} \cdot \frac{2}{\tau_b}\int_{s_m}^{s'_m}\frac{\{\Delta E(s, \phi)\}\, ds}{v\cos\alpha(s)} \tag{7.11}$$

where $\dot{\phi}$ is the longitudinal drift rate (a function of ϕ), and $\{\Delta E\}$ from equation (7.10) is a function of longitude and latitude through the densities N_i.

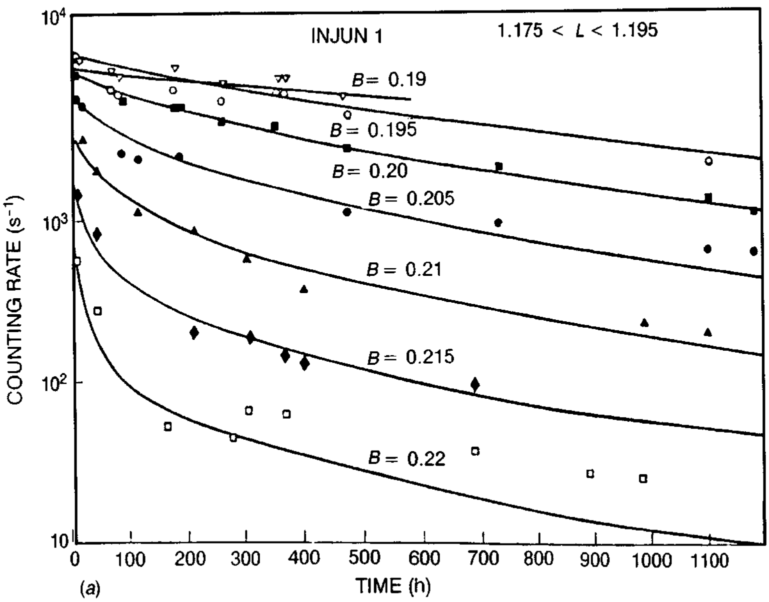
In the coordinates E, x and L the Fokker–Planck equation, including the energy loss term, is (from 6.33)

$$\frac{\partial f(E, x, L)}{\partial t} = -\frac{\partial}{\partial E}\langle \Delta E \rangle f + \frac{\partial}{\partial x}\left[\frac{\langle(\Delta x)^2\rangle}{2} x N_2(x) \frac{\partial}{\partial x}\left(\frac{f}{x N_2(x)}\right)\right] \tag{7.12}$$

where $N_2(x)$ is defined by equation (6.31).

In equation (7.12) the pitch angle and energy variables cannot be separated to allow an eigenfunction solution. The effect of energy loss is to mix the normal modes of the pitch-angle distribution so that they do not decay independently. Hence, an initial distribution in a single mode will evolve into several pitch-angle modes as time passes. This behavior prevents a simple solution by the separation of variables, although approximate solutions by this method have been useful.

Equation (7.12) can be evaluated by finite difference techniques. A straightforward application was the computation of the evolution over time of electrons injected into the magnetosphere by the Starfish nuclear weapon effects test in 1962. Intense fluxes of electrons produced by the beta decay of fission fragments were distributed between $L = 1.12$ and $L = 7$, although the major portion was confined below $L = 2$. Since these electrons were of higher energy than most of the electrons of natural origin, and the fluxes were more intense, it was possible to measure the intensity and distribution of bomb produced electrons for many months. This experiment thus offered a unique opportunity to compare the observed loss rate of trapped electrons with the value expected from atmospheric scattering. Some of the results are shown in Figure 7.2(a, b). Figure 7.3 compares calculated and observed values of the decay time, the time required for the flux to be reduced by the factor $1/e$. In general, agreement is excellent below $L = 1.3$, but above that limit electrons are removed much more rapidly than the theory permits. This result indicates that some other process, such as scattering by electromagnetic waves, is responsible for the observed diffusion in pitch angle.



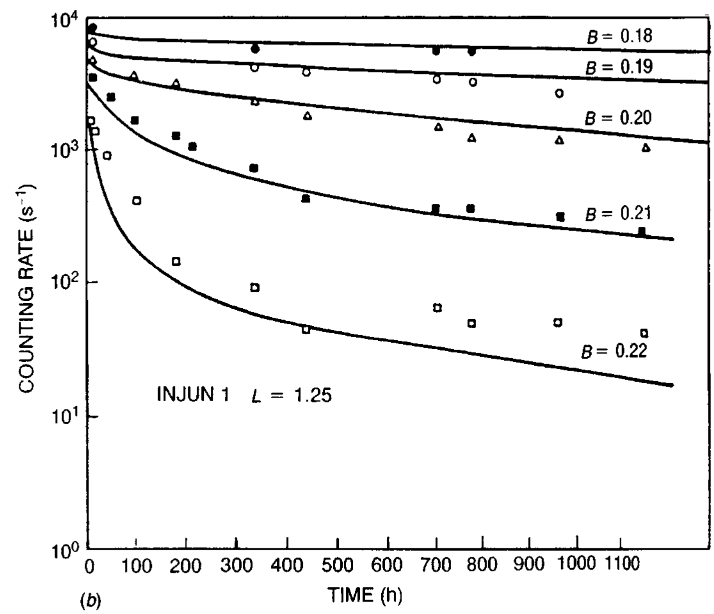


Figure 7.2(*a*) Loss of Starfish electrons at $L = 1.185$ by scattering with the ambient atmosphere. Symbols show the experimental values of omnidirectional flux of electrons (> 1 MeV), and lines are fluxes obtained from numerical integration of equation (7.12). (*b*) Same as Figure 7.2(*a*) but for $L = 1.25$. (From *J. Geophys. Res.* (1964) **69**, 397.)

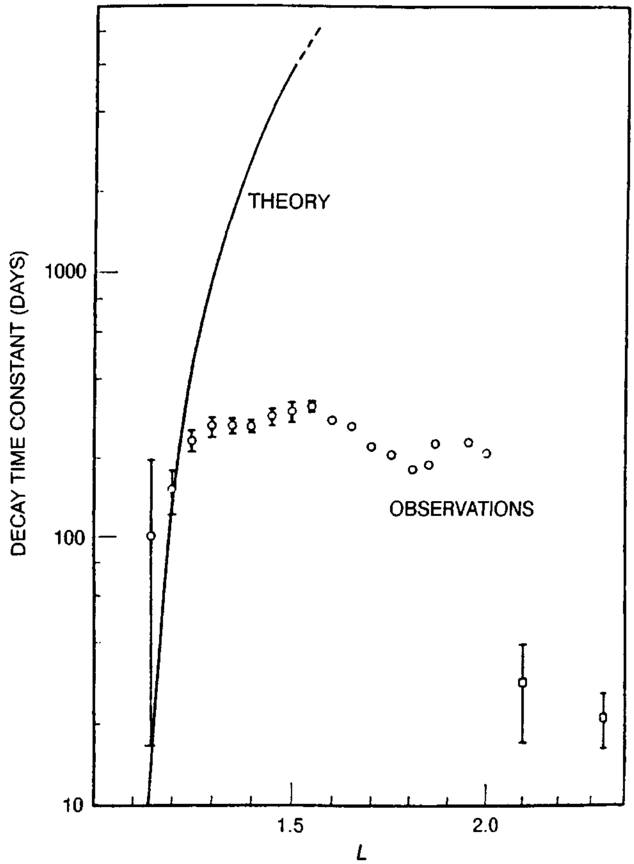


Figure 7.3. Comparison of trapped electron flux decay times from atmospheric scattering theory and from experiment. Theory fails above $L = 1.3$, indicating that other loss processes are present.

Diffusion in pitch angle by interactions with waves

It was pointed out in the preceding section that over most of the magnetosphere the observed losses from pitch-angle diffusion are much too large to be accounted for by collisions with the ambient atmosphere. The pitch angles must therefore be altered by electromagnetic fields which change the first adiabatic invariant. Since the gyration frequency is of the order of 5 kHz–1 megahertz for electrons and 3–300 Hz for protons, electromagnetic field variations at these or higher frequencies are required to alter the first adiabatic invariant and thereby change the pitch angle.

Many types of plasma waves occur in the magnetosphere. From the standpoint of trapped radiation, circularly polarized whistler and ion cyclotron waves appear to be the most important for their effects on trapped electrons and ions respectively. Since electron interactions with the whistler mode waves have received the most attention and are the easiest to calculate, they will be described here. The case for protons and ion cyclotron waves is similar except that the velocity of the protons is comparable to the phase velocity of the waves, and some of the approximations introduced for the electron case are not valid.

A whistler or right-hand circularly polarized wave propagating parallel to the geomagnetic field will have **E** and **b** wave vectors perpendicular to the magnetic field. The situation is depicted in Figure 7.4, which indicates the sense of rotation of the vectors. For a wave of this type the phase velocity is given by a dispersion relation which relates the phase velocity to the frequency:

$$v_{\mathrm{ph}} = \frac{c[w(\Omega_{\mathrm{e}} - w)]^{1/2}}{\omega_{\mathrm{p}}} \tag{7.13}$$

where $\omega_{\mathrm{p}} = (e^2 N/\varepsilon_0 m_{\mathrm{e}})^{1/2}$ is the plasma frequency of the medium, ω is the wave frequency, Ω_{e} is the electron gyration frequency, N is the electron

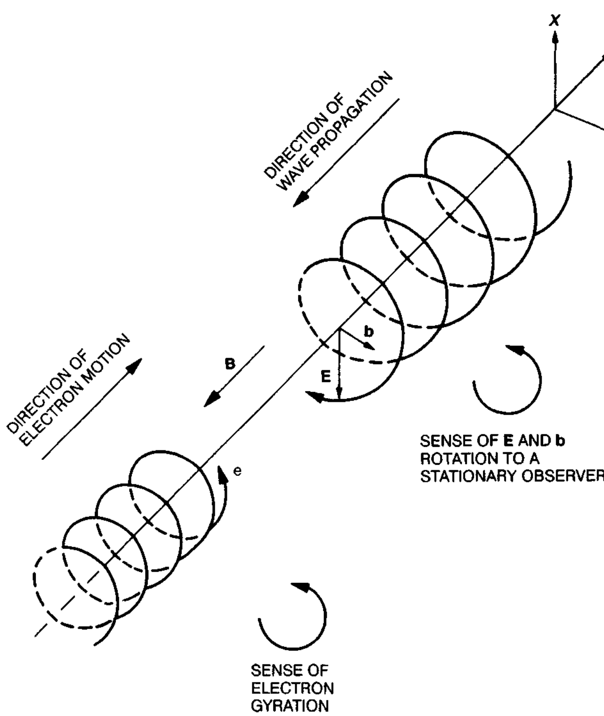


Figure 7.4. Interaction of a gyrating electron with a right-hand circular polarized wave propagating parallel to the magnetic field.

number density and m_e is the electron mass. For the phase velocity to be real, $\omega < \Omega_e$. The amplitudes of the **b** and **E** vectors are related by

$$\frac{|\mathbf{E}|}{|\mathbf{b}|} = v_{ph}$$

The phase velocity depends on the magnetic field intensity through Ω_e and on the ambient electron density through ω_p. Phase velocities of whistler waves are usually less than 0.1c. Therefore, for energetic electron interactions $v_{ph}/v \ll 1$.

In the stationary frame of reference the wave magnetic field is

$$\mathbf{b} = b[\hat{\mathbf{e}}_x \cos(\omega t + kz) - \hat{\mathbf{e}}_y \sin(\omega t + kz)] \tag{7.14}$$

for a wave moving in the negative z direction. For an electron whose guiding center moves in the positive z direction at velocity v_z:

$$z = v_z t + z_0 \tag{7.15}$$

The electron will therefore experience the Doppler shifted wave as

$$\mathbf{b} = b\{\hat{\mathbf{e}}_x \cos[(\omega + kv_z)t + kz_0] - \hat{\mathbf{e}}_y \sin[(\omega + kv_z)t + kz_0]\} \tag{7.16}$$

where the Doppler shifted frequency is

$$\omega_d = \omega + kv_z$$

The electron will see the **E** and **b** vectors rotate with angular frequency $\omega + kv_z$, and the phase of **b** is (see Figure 7.5).

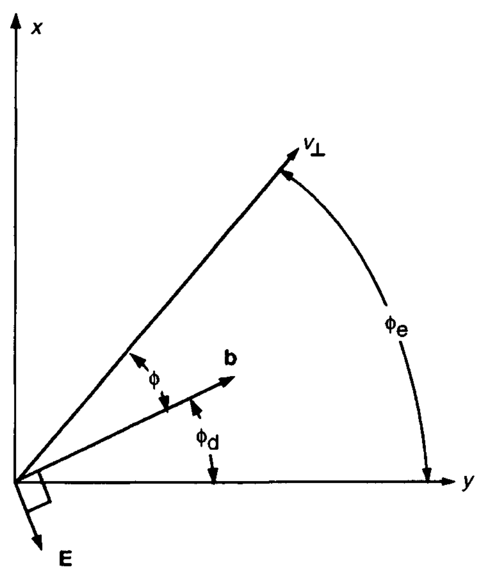


Figure 7.5. Definition of ϕ, the phase angle between the wave **b** vector and $\mathbf{v}_\perp$, the electron perpendicular velocity.

$$\phi_d = \int \omega_d \, dt = \int (\omega + kv_z)\, dt \tag{7.17}$$

The perpendicular velocity vector of the electron, $\mathbf{v}_\perp$, gyrates about the field line with phase

$$\phi_e = \int \Omega_e \, dt + \phi_0 = \Omega_e t + \phi_0$$

Thus, the angular difference between $\mathbf{v}_\perp$ and $\mathbf{b}$ is

$$\phi = \phi_e - \phi_d = (\Omega_e - \omega - kv_z)t + \phi_0 \tag{7.18}$$

where ϕ_0 is the initial phase difference between ϕ_e and ϕ_d at time $t = 0$.

The electric and magnetic components of the wave exert forces on the electron which, in the non-relativistic case, are, for $q = -e$,

$$\dot{\mathbf{v}} = -\frac{e}{m}[\mathbf{E} + \mathbf{v} \times \mathbf{b}] \tag{7.19}$$

The components of $\dot{\mathbf{v}}$ perpendicular and parallel to z are

$$\dot{v}_z = -\frac{e}{m} b v_\perp \sin\phi \tag{7.20}$$

$$\dot{v}_\perp = \frac{e}{m}\{E \sin\phi + b v_z \sin\phi\} = \frac{e}{m} b(v_{ph} + v_z)\sin\phi \tag{7.21}$$

where use has been made of $|\mathbf{E}|/|\mathbf{b}| = v_{ph}$. The change in v_z and $v_\perp$ will cause changes in the electron kinetic energy and pitch angle. The rate of energy change is given by

$$\begin{aligned}\frac{d}{dt}\left(\frac{m}{2}(v_z^2 + v_\perp^2)\right) &= m(v_z \dot{v}_z + v_\perp \dot{v}_\perp) \\ &= -v_z e b v_\perp \sin\phi + v_\perp e b (v_{ph} + v_z)\sin\phi \\ &= e b v_{ph} v_\perp \sin\phi \end{aligned} \tag{7.22}$$

Only the electric field term from equation (7.21) contributes to (7.22), as is expected since deflection by the magnetic field will not change the particle energy.

The pitch angle α changes at a rate

$$\begin{aligned}\dot{\alpha} &= \frac{d}{dt}\left[\tan^{-1}\left(\frac{v_\perp}{v_z}\right)\right] \\ &= \frac{v_z \dot{v}_\perp - \dot{v}_z v_\perp}{v_z^2 + v_\perp^2}\end{aligned} \tag{7.23}$$

Using (7.20) and (7.21), equation (7.23) becomes

$$\dot{\alpha} = \frac{e}{m} b \left(1 + \frac{v_{ph}\cos\alpha}{v}\right)\sin\phi \tag{7.24}$$

In general, ϕ is a rapidly changing function of time given by equation (7.18). In this case $\dot{\alpha}$ will display a rapid sinusoidal variation, and the accumulated deflection will not be large. However, if the parallel velocity v_z of the particle has the resonant value

$$v_z = \frac{\Omega_e - \omega}{k} \tag{7.25}$$

then ϕ from (7.18) and thus $\dot{\alpha}$ from (7.24) will be constant, and appreciable changes in α can accumulate. From equation (7.22), changes in energy will also take place if ϕ is constant over an appreciable time interval. The electron is said to be in resonance with the wave. The resonant frequency of the wave is given by $\omega = \Omega - kv_z$.

The sign of $\dot{\alpha}$ and $d(\frac{1}{2}mv^2)/dt$ is determined by the phase angle ϕ. If $0 < \phi < \pi$ both α and the energy increase. For these values of ϕ, $\mathbf{v}_\perp \cdot \mathbf{E} < 0$ (see Figure 7.5), and the electron would be expected to gain energy at the expense of energy in the wave. If $\mathbf{v}_\perp \cdot \mathbf{E} > 0$ ($\pi < \phi < 2\pi$), the electron will lose energy and decrease its pitch angle. In a flux of electrons uniformly distributed in the azimuthal angle, ϕ_e of Figure 7.5, some electrons will be deflected to smaller α and some to larger α during encounters with waves of finite length. The overall effect of many such encounters will be a diffusion in pitch angle.

A single frequency wave of infinite length will impart net deflections only for particles in exact resonance. Waves even slightly off-resonance will successively increase and decrease α as the phase angle ϕ rotates through 2π. If, however, the wave is of finite length, off-resonant frequencies can change α permanently. In general, the shorter the duration of the wave, the further off-resonance the wave can be and still contribute to the deflection.

An estimate of the width $\Delta\omega$ of the wave band contributing to $\dot{\alpha}$ for a wave of duration Δt is obtained as follows. If the change in ϕ is limited to π in time Δt:

$$\Delta\phi = (\Omega_e - \omega - kv_z)\Delta t = \pi \tag{7.26}$$

Consider the first factor on the right as a function of ω and expand it in a Taylor series about the resonant frequency $\omega = \Omega_e - kv_z$. The derivative of k with respect to ω is the reciprocal of the wave group velocity. Thus,

$$\Delta\phi = \left\{(\Delta\phi)_{\omega=\Omega_e - kv_z} + \frac{\partial}{\partial\omega}(\Delta\phi)\cdot\Delta\omega + \ldots\right\} = \pi$$

$$\sim -\left(1 + \frac{v_z}{v_g}\right)\Delta\omega\Delta t = \pi \tag{7.27}$$

and

$$|\Delta t| \approx \frac{\pi}{\left(1 + \frac{v_z}{v_g}\right)\Delta\omega} \tag{7.28}$$

The diffusion coefficient can be estimated for a series of waves of duration Δt interacting with the particle. With brackets denoting change per unit time:

$$\begin{aligned} D_{\alpha\alpha} &= \frac{\langle(\Delta\alpha)^2\rangle}{2} \approx \frac{1}{2}\left\langle\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2(\Delta t)^2\right\rangle \\ &\approx \frac{1}{2}\left(\frac{e}{m}\right)^2\frac{b^2}{\Delta\omega}\left(1 + \frac{v_{\mathrm{ph}}\cos\alpha}{v}\right)^2(\sin^2\phi)_{\mathrm{ave}}\frac{\pi}{\left(1 + \frac{v_z}{v_g}\right)} \\ &\approx \frac{\pi}{4}\left(\frac{e}{m}\right)^2\frac{b^2}{\Delta\omega}\left(1 + \frac{v_{\mathrm{ph}}\cos\alpha}{v}\right)^2\frac{1}{\left(1 + \frac{v_z}{v_g}\right)} \end{aligned} \tag{7.29}$$

since for particles uniformly distributed in ϕ_0, $(\sin^2\phi)_{\mathrm{ave}} = \frac{1}{2}$. The factor $b^2/\Delta\omega$ is interpreted as the power spectral density of the waves at the resonant frequency. In situations where $v_{\mathrm{ph}} \ll v$ and $v_g \ll v_z$, $D_{\alpha\alpha}$ can be approximated as

$$D_{\alpha\alpha} \approx \frac{\pi}{4}\left(\frac{e}{m}\right)^2\left(\frac{b^2}{\Delta\omega}\right)\frac{v_g}{v_z} \tag{7.30}$$

Other approximations, also based on heuristic arguments, give slightly different results.

A more quantitative expression for the pitch-angle diffusion coefficient can be derived by expressing the wave in general form and averaging over the stochastic variations of the wave field. The magnetic field as experienced by the electron is

$$\left.\begin{aligned} \mathbf{b} &= b_x(t)\hat{\mathbf{e}}_x + b_y(t)\hat{\mathbf{e}}_y \\ \mathbf{v}_\perp &= v_\perp\cos(\Omega t + \eta)\hat{\mathbf{e}}_x + v_\perp\sin(\Omega t + \eta)\hat{\mathbf{e}}_y \end{aligned}\right\} \tag{7.31}$$

Returning to equation (7.24) and recognizing that $\sin\phi = |\mathbf{v}_\perp \times \mathbf{b}|/v_\perp b$,

$$\begin{aligned} \dot{\alpha} &= \frac{e}{m}\left(1 - \frac{v_{\mathrm{ph}}\cos\alpha}{v}\right)\left[\frac{v_x b_y - v_y b_x}{v_\perp}\right] \\ &= K[\cos(\Omega t + \eta)\cdot b_y(t) - \sin(\Omega t + \eta)b_x(t)] \end{aligned} \tag{7.32}$$

where K is equal to the first two factors of equation (7.32).

To find $\Delta\alpha$ equation (7.32) is integrated over a finite time interval containing several gyrations. In fact, the integration time will be long compared to the coherence time of the wave. Under these conditions the two terms on the right-hand side will contribute equally to $\Delta\alpha$ and only one need be calculated:

$$(\Delta\alpha)^2 = 4K^2\int_0^t \cos(\Omega\xi' + \eta)b(\xi')\,\mathrm{d}\xi'\int_0^t \cos(\Omega\xi'' + \eta)b(\xi'')\,\mathrm{d}\xi'' \qquad (7.33)$$

Expanding the cosine terms, averaging over the electron initial phase angle η, and rearranging the integrands gives

$$(\Delta\alpha)^2 = 4K^2\int_0^t \mathrm{d}\xi'\int_0^t \mathrm{d}\xi'' b(\xi')b(\xi'')\tfrac{1}{2}\cos\Omega(\xi'' - \xi') \qquad (7.34)$$

Let $\xi'' - \xi' = \tau$,

$$(\Delta\alpha)^2 = 2K^2\int_0^t \mathrm{d}\xi'\int_{-\xi'}^{t-\xi'} \mathrm{d}\tau b(\xi')b(\xi' + \tau)\cos\Omega\tau \qquad (7.35)$$

Now exchange the order of integration and modify the limits as needed:

$$(\Delta\alpha)^2 = 2K^2\left\{\int_{-t}^{0} \mathrm{d}\tau\cos\Omega\tau\int_{-\tau}^{t} b(\xi')b(\xi' + \tau)\,\mathrm{d}\xi' + \int_0^t \mathrm{d}\tau\cos\Omega\tau\int_0^{t-\tau} b(\xi')b(\xi' + \tau)\,\mathrm{d}\xi'\right\} \qquad (7.36)$$

The integral $1/t\int_0^t b(\xi')b(\xi' + \tau)\,\mathrm{d}\xi'$ is the auto-correlation function of a component of the wave magnetic field. It is usually written as $\langle b(\xi')b(\xi' + \tau)\rangle$. These correlation integrals are functions only of the 'lag' τ and are small for lags larger than the correlation length or coherence time of the wave.

Equation (7.36) then becomes, replacing τ by $-\tau$ in the first term,

$$(\Delta\alpha)^2 = 2K^2\left\{\int_0^t \mathrm{d}\tau\cos\Omega\tau(t - \tau)\langle b(\xi')b(\xi' - \tau)\rangle + \int_0^t \mathrm{d}\tau\cos\Omega\tau(t - \tau)\langle b(\xi')b(\xi' - \tau)\rangle\right\} \qquad (7.37)$$

Because $\langle b(\xi')b(\xi' + \tau)\rangle = \langle b(\xi')b(\xi' - \tau)\rangle$ and the value is small unless τ is less than the coherence time, $(t - \tau) \approx t$. Also, because the integrand is zero at large τ, the upper limit of the integrals can be increased to infinity:

$$(\Delta\alpha)^2 = 4K^2 t\int_0^\infty \mathrm{d}\tau\cos\Omega\tau\langle b(\xi')b(\xi' + \tau)\rangle = K^2 t P'_{\mathrm{b}}(\Omega) \qquad (7.38)$$

where

$$P'_{\mathrm{b}}(\Omega) = 4\int_0^\infty \mathrm{d}\tau\langle b(\xi')b(\xi' + \tau)\rangle\cos\Omega\tau \qquad (7.39)$$

is the power spectral density at the gyration frequency for a component of the wave as measured in the moving frame of the particle guiding center.

This spectral density is related to the power spectral density in the rest frame at the Doppler shifted frequency by

$$P'_b(\Omega) = P_b(\Omega - kv_z)\frac{d(\Omega - kv_z)}{d\Omega} = \frac{P_b(\Omega - kv_z)}{[1 + v_z/v_g]} \tag{7.40}$$

The diffusion coefficient is then

$$\begin{aligned} D_{\alpha\alpha} &= \frac{(\Delta\alpha)^2}{2t} = \frac{1}{2}\left(\frac{e}{m}\right)^2\left(1 - \frac{v_{ph}\cos\alpha}{v}\right)^2 \frac{P_b(\Omega - kv_z)}{[1 + v_z/v_g]} \\ &\approx \frac{1}{2}\left(\frac{e}{m}\right)^2\left(\frac{v_g}{v_z}\right)P_b(\Omega - kv_z) \end{aligned} \tag{7.41}$$

for $v \gg v_{ph}$ and $v_z \gg v_g$.

Equation (7.41) is the local value of the diffusion coefficient describing the change in the local pitch angle. To calculate the behavior of trapped particles it is necessary to convert the pitch angle to some quantity which is constant during the bounce and then average over the bounce motion. If one chooses $\cos\alpha_e$ as the variable, the procedure to be followed is given in equations (7.6) and (7.7).

The energy loss described in equation (7.22) indicates that the electrons change energy as they diffuse in pitch angle. The magnitude of this energy change for a given $\Delta\alpha$ can be estimated from equations (7.22) and (7.24):

$$\begin{aligned} \frac{\Delta E}{E} &= \frac{1}{E}\frac{dE}{d\alpha}\Delta\alpha = \frac{1}{E}\frac{dE}{dt}\frac{dt}{d\alpha}\Delta\alpha \\ &= \frac{2mv_{ph}v_\perp v}{mv^2(v + v_{ph}\cos\alpha)}\Delta\alpha \\ &\sim \frac{2v_{ph}v_\perp}{v^2} \end{aligned} \tag{7.42}$$

for $v_{ph} \ll v$ and for $\Delta\alpha = 1$. If $v_{ph} \ll v$, the fractional change in energy will be small. Therefore, for electrons it is usually permissible to ignore the energy change during pitch-angle diffusion and use a diffusion equation with some function of the pitch angle as the independent variable. With independent variables L, E and x the only diffusion term is the one containing D_{xx} and the equation to be used is (6.33).

A more graphic description of the diffusion in pitch angle can be obtained by transforming to a frame of reference moving with the phase velocity of the wave. Since $\mathbf{v}_{ph} = (\mathbf{E} \times \mathbf{b})/b^2$, the electric field of the wave will be zero in that frame:

$$\mathbf{E}' = \mathbf{E} + \mathbf{v}_{ph} \times (\mathbf{B}_0 + \mathbf{b}) = \mathbf{E} + \mathbf{v}_{ph} \times \mathbf{b} = 0 \tag{7.43}$$

and the electron energy will be conserved. This condition expressed in terms of $\mathbf{v}_\perp$ and $\mathbf{v}_\parallel = \mathbf{v}_z$ is

$$\tfrac{1}{2}\mathrm{d}[v_\perp^2 + (v_\parallel - v_{ph})^2] = v_\perp\,\mathrm{d}v_\perp + (v_\parallel - v_{ph})\,\mathrm{d}v_\parallel = 0 \tag{7.44}$$

This differential equation in velocity $(v_\perp, v_\parallel)$ space describes an element of a circle whose center is located at $-v_{ph}$ on the $v_\parallel$ axis (see Figure 7.6). This diffusion path differs from the constant energy path centered at the origin, although the difference will be small if $v_{ph} \ll v$. As the particle moves along the diffusion path of equation (7.44) the parallel velocity will change and the resonant frequency will also change. The motion along the line will take place in a number of small steps, each increment being in a random direction and resulting from interaction with a wave.

From equation (7.22) it is clear that an individual electron will either lose or gain energy depending on the azimuthal phase angle ϕ. However, the overall flow of the diffusing electrons depends on the distribution of the particles in velocity space. Figure 7.7 illustrates a hypothetical distribution function $F(v_\parallel, v_\perp)$, where $F(v_\parallel, v_\perp)2\pi v_\perp\,\mathrm{d}v_\perp\,\mathrm{d}v_\parallel$ is the number of particles within the differential element of velocity space. In this diagram more particles are at large pitch angles, and the net diffusion is towards smaller α and lower energies.

The general form of the diffusion equation for wave–particle interactions can be derived by assuming that the current of particles flowing along the diffusion path of equation (7.44) is proportional to the slope of F along that path. The change in F along an increment $\mathrm{d}\mathbf{v}$ is

$$\mathrm{d}\mathbf{v} \cdot \nabla F = \frac{\partial F}{\partial v_\perp}\mathrm{d}v_\perp + \frac{\partial F}{\partial v_\parallel}\mathrm{d}v_\parallel \tag{7.45}$$

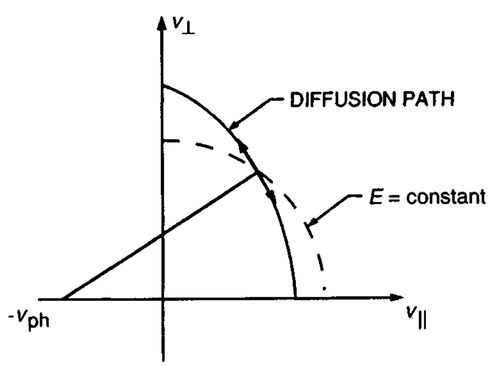


Figure 7.6. Diffusion path of a particle interacting with waves compared to the constant energy path. As the pitch angle decreases, the particle energy decreases.

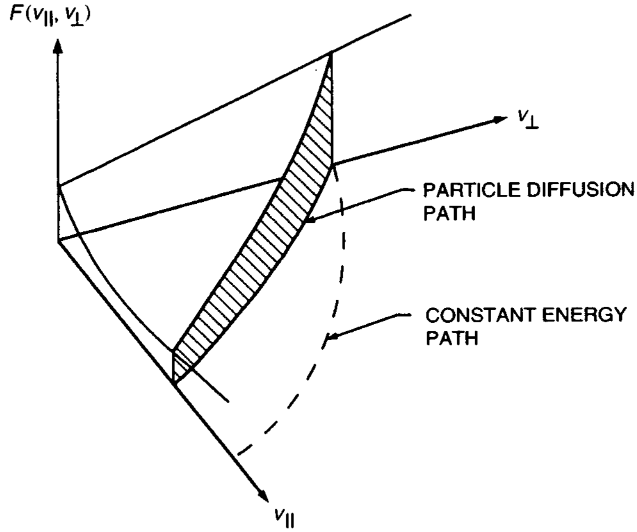


Figure 7.7. Particle distribution in velocity space showing the diffusion path and the constant energy path.

But $dv_\perp$ and $dv_\parallel$ are related by (7.44), which, combined with (7.45), yields

$$\text{current} \propto -D\left(v_\perp \frac{\partial F}{\partial v_\parallel} - (v_\parallel - v_{\text{ph}})\frac{\partial F}{\partial v_\perp}\right) = -D\mathcal{D}_v F \qquad (7.46)$$

The proportionality constant D is recognized as the diffusion coefficient, and the differential operator $\mathcal{D}_v$ computes the gradient of F along the diffusion path. The rate of change in F is equal to the divergence of the current and is given by

$$\frac{\partial F}{\partial t} = \frac{1}{\mathcal{J}}\mathcal{D}_v[D\mathcal{J}\mathcal{D}_v F] \qquad (7.47)$$

where $\mathcal{J}$ is the Jacobian ($\mathcal{J} = v_\perp$), relating a volume element in velocity space $dv_x\, dv_y\, dv_z$ to $v_\perp\, d_\phi\, dv_\perp\, dv_\parallel$. More rigorous methods have confirmed the validity of (7.47).

To reduce (7.47) to a more familiar form for a simplified case assume that $v_\parallel \gg v_{\text{ph}}$ so that only the pitch angle α changes during diffusion. The change from $(v_\perp, v_\parallel)$ to (v, α) coordinates is obtained by inserting

$$\left.\begin{aligned} \frac{\partial}{\partial v_\perp} &= \sin\alpha \frac{\partial}{\partial v} + \frac{\cos\alpha}{v}\frac{\partial}{\partial \alpha} \\ \frac{\partial}{\partial v_\parallel} &= \cos\alpha \frac{\partial}{\partial v} - \frac{\sin\alpha}{v}\frac{\partial}{\partial \alpha} \end{aligned}\right\} \qquad (7.48)$$

into the differential operator giving $\mathcal{D}_v = -\partial/\partial\alpha$. With these substitutions (7.47) becomes

$$\frac{\partial F}{\partial t} = \frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left[D_{\alpha\alpha}\sin\alpha\frac{\partial F}{\partial\alpha}\right] \tag{7.49}$$

Equation (7.49) is a local equation describing the pitch-angle diffusion at a point. For trapped particles the variable α must be replaced by some quantity which remains constant during the adiabatic bounce motion, and the equation must be averaged over a bounce cycle. As described in the preceding section, the equatorial pitch angle is a convenient variable for this purpose. By changing α to α_{eq} in equation (7.49), multiplying by $ds/v\cos\alpha$ and integrating over a complete bounce trajectory, the bounce averaged diffusion equation becomes

$$\frac{\partial F}{\partial t} = \frac{1}{\tau_b\sin 2\alpha_{eq}}\frac{\partial}{\partial\alpha_{eq}}\left[\bar{D}_{\alpha_{eq}\alpha_{eq}}\tau_b\sin 2\alpha_{eq}\frac{\partial F}{\partial\alpha_{eq}}\right] \tag{7.50}$$

where the averaged diffusion coefficient is

$$\bar{D}_{\alpha_{eq}\alpha_{eq}} = \frac{1}{\tau_b}\oint D_{\alpha_{eq}\alpha_{eq}}(s)\frac{ds}{v\cos\alpha} \tag{7.51}$$

This equation is entirely equivalent to (6.33). By changing the distribution function $f(x, E, L)$ in (6.33) to a velocity or phase space distribution function using (6.32), and by replacing the independent variable $x = \cos\alpha_{eq}$ by α_{eq}, equation (6.33) becomes (7.50).

Coupling of particle and wave energy

The diffusion in pitch angle by waves which are not produced by the particles themselves is sometimes termed parasitic diffusion. If the power spectrum of the waves is known, the change in the particle distribution can be calculated using equation (7.50) for the diffusion equation and (7.51) with (7.41) for the diffusion coefficient. This approach is satisfactory for estimating the diffusion rates from waves which are not produced by the particles themselves.

As described in the derivation of equation (7.22), particles can exchange energy with the waves, either gaining or losing energy depending on the phase angle between $\mathbf{v}_\perp$ and $\mathbf{b}$. The wave is augmented or reduced by the electric and magnetic fields produced by the particle currents. The wave response is characterized by a growth rate γ where

$$\frac{d}{dt}|b|^2 = 2\gamma|b|^2 \tag{7.52}$$

A positive γ denotes a growing wave. The form of γ is given by

$$\gamma(\omega) \approx -g(\omega)\int_0^\infty \mathrm{d}v_\perp v_\perp^2\left[v_\perp\frac{\partial F}{\partial v_\parallel} - (v_\parallel - v_{\mathrm{ph}})\frac{\partial F}{\partial v_\perp}\right]_{v_\parallel=\Omega_e-\omega/k} \qquad (7.53)$$

where $g(\omega)$ is a slowly varying function of ω. The integrand of (7.53) contains the differential operator of (7.46) and represents the slope of the distribution function along the diffusion path.

With reference to Figure 7.8 the integrand is $v_\perp^2$ times the derivative of F along the diffusion path. The integral over $v_\perp$ then includes the contributions of these derivatives at all $v_\parallel = (\Omega_e - \omega)/k$. The growth rate will be large if the slope of F along the diffusion path is large and a net flow of electrons occurs towards smaller α and lower energy.

The coupling of the particle distribution with wave growth rates changes the dynamic behavior of the wave and particle systems. If the waves propagate parallel to **B**, the conservation of wave energy is expressed by

$$\frac{\partial b^2}{\partial t} + v_g\frac{\partial b^2}{\partial z} = 2(\gamma - \ell)b^2 \qquad (7.54)$$

where ℓ represents the internal loss rate of wave energy. Equations (7.41), (7.47), (7.53) and (7.54) relate the wave and particle behavior. An initial

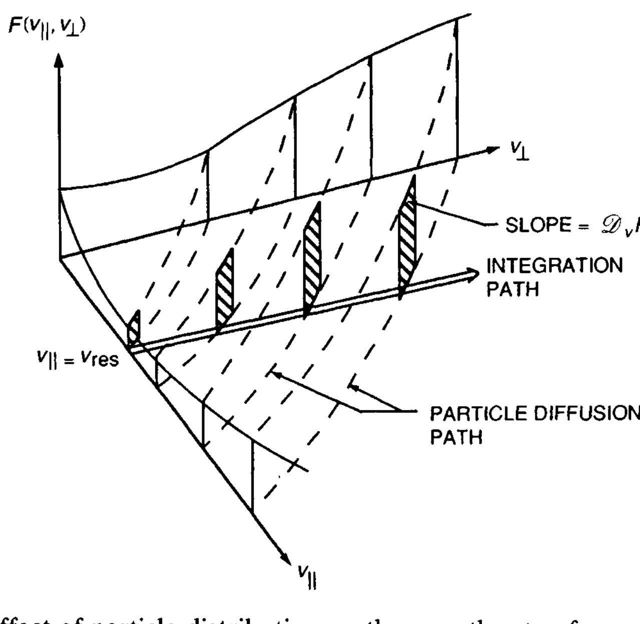


Figure 7.8. Effect of particle distribution on the growth rate of waves. The growth rate for a resonant $v_\parallel$ (equation (7.43)) contains the slope of the distribution along the diffusion path integrated over $v_\perp$.

distribution of particles with an empty loss cone will generally lead to wave growth. The growing waves will increase $D_{\alpha\alpha}$ by equation (7.41), leading to more rapid diffusion and loss of particles into the loss cone. This process should provide a regulation mechanism for trapped radiation. An increased source of electrons would lead to greater wave growth and increased loss rates, thereby limiting the particle flux.

While the concepts of a flux limiting mechanism are valid, the complex geometry and inhomogeneities in the magnetosphere make quantitative calculations ambiguous. In particular, the propagation of wave energy out of the particle interaction region reduces the effectiveness of the flux limiting mechanism. Idealized calculations usually assume that the waves travel only parallel to **B** and are reflected at each end of the field line with reflectivity $\mathcal{R}$. These assumptions lead to a steady-state situation in which

$$1/\mathcal{R} = \exp(\gamma \Delta s / v_{\text{ph}}) \tag{7.55}$$

where Δs is the length of the wave–particle interaction region. Equation (7.55) simply states that the loss of wave energy at each reflection is balanced by the growth during passage through Δs.

Discussion

This chapter has presented a basic description of pitch-angle scattering. However, in the interests of clarity this treatment was simplified, and many factors of importance to wave–particle interactions were ignored. In particular, only one type of wave was considered, a parallel propagating electromagnetic, whistler-mode wave. Other important approximations were that the wave had a broad frequency spectrum, the wave amplitude was small and the particles were uniformly distributed in gyrophase.

In fact, whistler-mode waves usually propagate at some angle to the magnetic field. This condition causes the wave to be elliptically polarized and extends the wave–particle interaction to harmonics of the particle gyrofrequency. The diffusion coefficient produced by such waves is substantially different from the one expressed in equation (7.41). The assumption that the waves are of small amplitude allowed the forces on the particle to be evaluted at positions given by the unperturbed motion of the particle through the wave. Larger wave amplitudes will alter the trajectories so that the particle motion must be computed throughout the encounter. In extreme cases, the particle can become trapped in the fields of the wave and the resonance time is thus greatly extended.

Problems

1. Using equation (7.12), but neglecting the energy loss term, show that an isotropic flux will remain isotropic regardless of the form of the atmospheric collision coefficient $\langle(\Delta x)^2\rangle$.

2. D_{xx} is the diffusion coefficient in terms of $x = \cos\alpha_{\mathrm{eq}}$, but we wish to use B_{m}, the magnetic field at the mirroring point, as the independent variable. Find the expression for $\langle(\Delta B_{\mathrm{m}})^2\rangle$ in terms of D_{xx}, B_{eq} and B_{m}.

3. Starting with equation (7.2) and retaining only terms of order $\ln 2/\eta_{\mathrm{min}}$, show that the first and second Fokker–Planck coefficients for atmospheric scattering (before bounce averaging) satisfy the relationship (6.20).

4. Derive the bounce averaged pitch-angle diffusion equation (7.50) from the local equation (7.49).

5. If the reflectivity of the ionosphere for electromagnetic wave energy is 0.2 and the equatorial interaction region at $L = 2$ is 10^3 km in length, what must be the growth rate in the interaction region to sustain a 5 kHz parallel propagating wave? Assume that the cold plasma is hydrogen with a density of $2 \times 10^9\ \mathrm{m}^{-3}$.