Introduction to Geomagnetically Trapped Radiation

Martin Walt

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Chapter

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8

Diffusion in the L coordinate or radial diffusion

Particle diffusion through random increments in the L coordinate is frequently termed radial diffusion because the process changes the radial distances of trapped particles from the Earth. This type of diffusion is crucial in forming the radiation belts as it provides a mechanism for transporting particles from the outer boundary of the magnetosphere into the inner belt. It also leads to the redistribution of particles injected or accelerated during magnetic storms and substorms. While radial diffusion may be overshadowed at times by the massive injections which occur during large storms and substorms, its role in bringing particles inward, in accelerating trapped particles and in redistributing newly injected particles is of major importance.

Since the third invariant Φ is proportional to L^{-1}, radial diffusion must proceed by fluctuations in the third invariant. Variations in a trapped particle's third invariant require changes in the electric or magnetic fields that are more rapid than the particle drift frequency. Drift periods vary from tens of seconds to about a day (see Appendix B), hence, perturbations over a wide range of frequencies can alter the third invariant. Because the gyration and bounce periods are much shorter than the drift period, the first and second invariants are less likely to be affected by many of these field perturbations.

The paths of the mirror points of particles undergoing third invariant diffusion but with constant first and second invariant are shown in Figure 8.1. The trapped particles mirroring on the equatorial plane remain on that plane, as demanded by the need to keep $J = 0$. The mirror points of particles mirroring off the equatorial plane move along lines of almost constant latitude, the latitude increasing slightly with increasing L. As particles diffuse inward (outward) the momentum increases (decreases) in order to maintain a constant value for the magnetic moment.

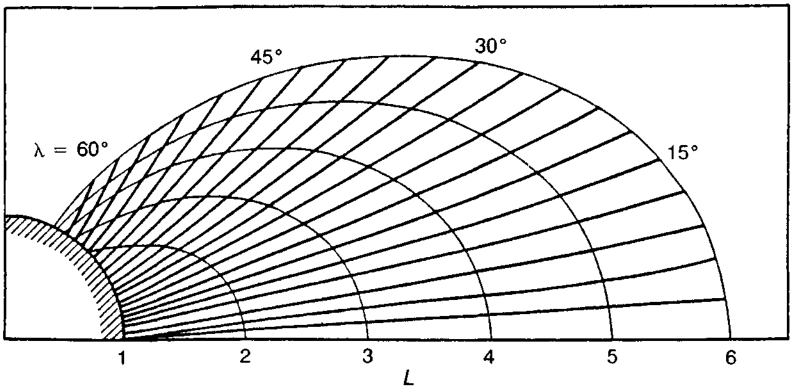

Figure 8.1. Diffusion paths of the mirroring points of trapped particles undergoing radial diffusion conserving the first and second adiabatic invariants.

In Chapter 6 a diffusion equation ((6.26)) for L shell diffusion was derived for a distribution function in the variables μ, J and L:

$$\frac{\partial f(\mu, J, L)}{\partial t} = \frac{\partial}{\partial L}\left[\frac{\langle(\Delta L)^2\rangle}{2}\frac{1}{L^2}\frac{\partial}{\partial L}(L^2 f)\right] \tag{8.1}$$

where $\langle(\Delta L)^2\rangle/2 = D_{LL}$ is the radial diffusion coefficient. In this chapter the diffusion coefficient will be evaluated for magnetic and electric perturbations, as both of these variations are common in the magnetosphere. To make the computation for D_{LL} easier, idealized models for the geometry of the perturbations will be assumed. While these simplifications are not completely justified, they illustrate the principals involved and are appropriate in view of present limited knowledge of the geometry of the magnetic and electric field variations.

It is expected that D_{LL} will depend on general statistical properties such as the power spectrum of a multitude of disturbances rather than on details of any single fluctuation. In each event particles will be moved inward or outward depending on their location at the time of the field changes. Summed over many events the cumulative motion of an individual particle may be inward or outward. However, the overall flow of particles will depend on the distribution in L of the particle populations as described in Chapter 6. If the coordinates μ, J and L are used, the flow will be inward wherever $\partial(L^2 f)/\partial L$ is positive.

The partial derivatives with respect to L in (8.1) are taken with μ and J constant. It is not possible to determine the overall direction of particle flow by plotting $j(E, \alpha_{eq} = (\pi/2))$ as a function of L. One must first convert $j(E, \alpha_{eq})$ into $f(\mu, J = 0, L)$ using equation (6.35) and selecting E

at each L to keep μ constant. A plot of $L^2 f(\mu, J, L)$ as a function of L with μ and J held constant will immediately reveal the direction of net particle flow as in Figure 6.3.

Radial diffusion induced by magnetic fluctuations

Third invariant diffusion can be driven by asymmetric fluctuations in the geomagnetic field. A simplified illustration of the effect of such a perturbation is given in Figure 8.2, which represents the equatorial plane and the response of a narrow band of equatorially trapped particles to a global magnetic disturbance. The postulated disturbance is illustrated schematically at the bottom of the diagram and consists of a sudden compression of the magnetosphere by an increase in solar wind pressure. After this compressive impulse the solar wind pressure gradually decreases, allowing the magnetosphere to relax to its original configuration. The compression will be greatest on the sunward side of the Earth (right-hand side of Figure 8.2). During this initial compression, the trapped particles are carried inward to the dashed line, this transport taking place before the particles have an opportunity to drift appreciably in longitude. Particles on the sunward side of the Earth are moved by the largest amount and are left in more intense magnetic fields. The dashed line showing the post-compression particle positions is not a line of constant B, nor is it the path of drifting particles. It is the instantaneous location of the particles which

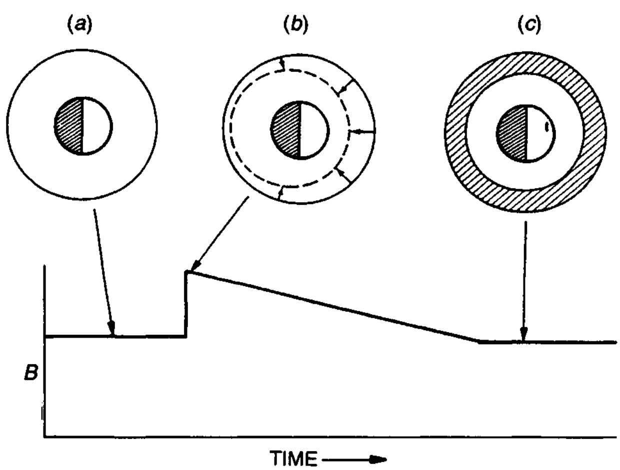

Figure 8.2(a–c) Effect of an asymmetric, sudden compression and slow relaxation of the geomagnetic field on a narrow band of equatorially trapped particles. After the recovery period the particles fill the shaded band.

were previously drifting in the narrow band of Figure 8.2(*a*). The sudden compression will change the values of the third invariant but not the values of μ and J. Particles on the sunward side will suffer the largest change in Φ and the largest increase in energy. Following this compression, the geomagnetic field relaxes slowly, keeping all adiabatic invariants constant. Meanwhile, each particle drifts about the Earth at constant μ, J and Φ and, after a number of orbits, the natural dispersion in drift velocity results in the broad band of particles depicted in Figure (8.2(*c*)). During this field relaxation all particles are moved outward as the magnetic field recovers. The overall effect of the sudden compression and slow relaxation is to move some particles inward (those initially on the sunward side of the Earth) and to transport some particles outward (those initially on the nightside). Many events of this type, each small in overall effect, will lead to a diffusion in the L coordinates of the particles.

The motion of the trapped particles under magnetic perturbation depends both on the magnetic field change and on the induced electric field. The induced electric field cannot be calculated directly from the magnetic field change as only $\nabla \times \mathbf{E}$ is given by $\partial \mathbf{B}/\partial t$. However, if one assumes that the cold plasma in the magnetosphere is a good electrical conductor in the direction parallel to the magnetic field and that the Earth itself, or the ionosphere, is a perfect conductor, then the induced electric field is completely specified. Where these conditions apply, the apparent velocity $\mathbf{v}_\mathrm{f}$ of the field line and the induced electric field are related by

$$\mathbf{v}_\mathrm{f} = (\mathbf{E} \times \mathbf{B})/B^2 \tag{8.2}$$

The instantaneous position of an individual magnetic field line is obtained by tracing its position through space, beginning with its fixed position at the conducting Earth or ionosphere. The position of the field line at subsequent time intervals permits a computation of $\mathbf{v}_\mathrm{f}$ and, by equation (8.2), an evaluation of $\mathbf{E}$. Since equation (8.2) also describes the electric drift velocity of ions and electrons, these particles will be carried with the moving field line. This simultaneous motion of particles and magnetic field is called the frozen field condition. While the frozen field condition obtains over much of the magnetosphere, it is not universally valid. In the outer magnetosphere where the cold plasma density is low, electric fields parallel to $\mathbf{B}$ occur, and the frozen field assumption is invalid. However, in the present idealized calculation, this assumption will be followed with the understanding that in the actual magnetosphere the induced electric field may be quite different.

The general approach used to find D_{LL} is to construct an idealized model of the field disturbance and to compute the radial displacement of a trapped particle experiencing the disturbance. This displacement is then squared and averaged over all possible initial longitudes of the particle and over many disturbances occurring at random times. In this way a diffusion coefficient is obtained as a function of the statistical characteristics of the disturbances. The full derivation of D_{LL} will be performed for equatorially trapped particles and the work extended later to other pitch angles.

Equations for the guiding center drift velocity were derived in Chapter 2 (equation (2.32)). Because we are considering only equatorially trapped particles, the curvature drift term is zero. In an electric field perpendicular to an inhomogeneous magnetic field $\mathbf{B}$ the perpendicular drift velocity is

$$\mathbf{v}_\perp = -\frac{1}{qB^2}\mathbf{B} \times (q\mathbf{E} - \mu\nabla B) \tag{8.3}$$

Let the magnetic field be composed of a dipole field $\mathbf{B}_\mathrm{d}$ and a disturbance field $\mathbf{b}$ where $\mathbf{b} \ll \mathbf{B}_\mathrm{d}$. The fact that the disturbance is much smaller than the dipole field will allow the disturbance field to be considered a perturbation on the usual gradient $\mathbf{B}$ and electric field drifts. In this case it is assumed that the magnetic changes are caused by magnetospheric boundary currents, which then produce field changes interior to the boundary. Equation (3.23) is therefore suitable to express the disturbance field in terms of spherical harmonics:

$$\mathbf{b} = -\nabla\Psi = -\nabla \mathrm{R_E} \sum_{n=1}^{\infty}\left(\frac{r}{\mathrm{R_E}}\right)^n \sum_{m=0}^{n}(\bar{g}_n^m \cos m\phi + \bar{h}_n^m \sin m\phi)P_n^m(\cos\theta) \tag{8.4}$$

For small perturbations near the Earth, only the leading terms with $n < 3$ are important. A further simplification results from aligning the dipole perpendicular to the solar wind and labeling the meridian containing the Sun $\phi = 0$. Thus $\bar{h}_n^m = 0$ for all n, m and $\bar{g}_n^m = 0$ when $n + m$ is even. The only terms remaining are $\bar{g}_1^0$ and $\bar{g}_2^1$ and the disturbance field becomes

$$\begin{aligned}\mathbf{b}(t) &= -\nabla\left[\bar{g}_1^0 r\cos\theta + \sqrt{3}\frac{r^2}{\mathrm{R_E}}\bar{g}_2^1 \sin 2\theta\cos\phi\right] \\ &= [-S(t)\cos\theta - A(t)r\sin 2\theta\cos\phi]\hat{\mathbf{e}}_r \\ &\quad + [S(t)\sin\theta - A(t)r\cos 2\theta\cos\phi]\hat{\mathbf{e}}_\theta \\ &\quad + A(t)r\cos\theta\sin\phi\hat{\mathbf{e}}_\phi \end{aligned} \tag{8.5}$$

By expressing $\mathbf{b}$ in the form of (8.5) the Maxwell equation $\nabla \cdot \mathbf{B} = 0$ is automatically satisfied. The time-dependent coefficients $S(t)$ and $A(t)$ are parameters representing the parts of the disturbance field which are symmetric and asymmetric respectively, about the polar axis.

The induction electric field will be computed from the motion of the field lines by

$$\mathbf{E} = -\mathbf{v}_{\mathrm{f}} \times \mathbf{B} \tag{8.6}$$

where the field line velocity $\mathbf{v}_{\mathrm{f}}$ is obtained by tracing the equatorial crossing of a field line whose feet are fixed at a lower altitude, taken for convenience at the origin of the dipole.

The field line equations in polar coordinates are given by

$$\frac{\mathrm{d}r}{B_r} = \frac{r\,\mathrm{d}\theta}{B_\theta}, \quad \frac{r \sin\theta\,\mathrm{d}\phi}{B_\phi} = \frac{r\,\mathrm{d}\theta}{B_\theta} \tag{8.7}$$

with $\mathbf{B} = \mathbf{B}_{\mathrm{d}} + \mathbf{b}$.

In the spirit of perturbation theory these equations can be integrated from the origin to the equator by replacing r and ϕ in the expressions for B_{r}, B_θ and B_ϕ by their dipole values

$$r = R_0 \sin^2\theta, \quad \phi = \phi_0 \tag{8.8}$$

This approximation sets the field $\mathbf{B}_{\mathrm{d}} + \mathbf{b}$ at each position of the undistorted path equal to the distorted magnetic field values. With the additional assumption that $A(t)$ and $S(t)$ are much smaller than B the equations can be integrated analytically from $\theta = 0$ to $\pi/2$ to give r and ϕ in the equatorial plane in terms of $A(t)$ and $S(t)$ and the constants R_0 and ϕ_0. These constants are the radial distance and longitude of the equatorial crossing of the undistorted field line. The changes in r and ϕ as a function of $A(t)$ and $S(t)$ can then be interpreted as motion of the field line, and the resulting electric field can be calculated from equation (8.6). The electric field in the equatorial plane obtained in this manner is

$$\mathbf{E} = \frac{1}{7} r^2 \frac{\mathrm{d}A}{\mathrm{d}t} \sin\phi\,\hat{\mathbf{e}}_r + r\left(\frac{1}{2}\frac{\mathrm{d}S}{\mathrm{d}t} + \frac{8}{21} r \frac{\mathrm{d}A}{\mathrm{d}t}\cos\phi\right)\hat{\mathbf{e}}_\phi \tag{8.9}$$

The magnetic symmetry assumed for $\mathbf{b}$ gives $b_r = b_\phi = E_\theta = 0$ in the equatorial plane. The radial component of the drift velocity of equatorial particles from equation (8.3) reduces to

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \left(-\frac{E_\phi}{B} + \frac{\mu}{qBr}\frac{\partial b_\theta}{\partial \phi}\right) \tag{8.10}$$

Substituting values for E_ϕ and b_θ from equations (8.9) and (8.5) and using the dipole value for B in (8.10) results in

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -r\left(\frac{1}{2B_{\mathrm{d}}}\frac{\mathrm{d}S}{\mathrm{d}t} + \frac{8}{21}\frac{r}{B_{\mathrm{d}}}\frac{\mathrm{d}A}{\mathrm{d}t}\cos\phi\right) - \frac{\mu}{qB_{\mathrm{d}}} A \sin\phi \tag{8.11}$$

In equation (8.11) the time-dependent quantities on the right-hand side are the coefficients A and S and the particle coordinates r and ϕ. In keeping with the usual policy of perturbation theory we will use the unperturbed values of these coordinates for the particle position. Therefore, on the right-hand side of equation (8.11) set $r = r_0$ and $\phi = \Omega_D t + \eta$ where Ω_D is the angular drift frequency of the particle and η is the particle longitude at $t = 0$. The magnetic moment can be replaced by its value in terms of the angular drift velocity $\mu = \Omega_D r_0^2 q/3$. Integrating equation (8.11) over time from zero to t gives the radial displacement at time t:

$$\int_{r_0}^{r(t)} \mathrm{d}r = r(t) - r_0 = -\frac{5}{7}\frac{r_0^2\Omega_D}{B_d}\int_0^t A(\xi)\sin(\Omega_D\xi + \eta)\,\mathrm{d}\xi$$
$$- \frac{r_0}{2B_d}[S(t) - S(0)] - \frac{8}{21}\frac{r_0^2}{B_d}\{A(t)\cos(\Omega_D t + \eta) - A(0)\cos\eta\} \tag{8.12}$$

With the exception of the first term on the right-hand side, all terms are bounded and of order b/B_d. On the other hand, the integral term can grow without limit as t increases, provided $A(\xi)$ has frequencies in the neighborhood of Ω_D. This term is therefore the important one for radial displacements.

Only the asymmetric part of the disturbance field survives in computing radial displacements. This result is as expected since symmetric compressions and relaxations will return particles to their original radial positions. Also, electric and magnetic drifts contribute almost equally to the coefficient 5/7 of the dominant term. Therefore, the assumptions regarding the induced electric field are quite important to the result.

The diffusion coefficient is constructed from the average value of the square of the radial displacement. The technique is similar to the one used to derive pitch-angle diffusion coefficients in Chapter 7. The square of (8.12), keeping only the dominant term, can be manipulated to give

$$[r(t) - r_0]^2 = \left(\frac{5}{7}\right)^2\left(\frac{r_0^2\Omega_D}{B_d}\right)^2\int_0^t \mathrm{d}\,\xi' A(\xi')\sin(\Omega_D\xi' + \eta)$$
$$\times \int_0^t \mathrm{d}\,\xi'' A(\xi'')\sin(\Omega_D\xi'' + \eta)$$
$$= \left(\frac{5}{7}\right)^2\left(\frac{r_0^2\Omega_D}{B_d}\right)^2\int_0^t \mathrm{d}\,\xi'\int_0^t \mathrm{d}\,\xi'' A(\xi')A(\xi'')$$
$$\times \sin(\Omega_D\xi' + \eta)\sin(\Omega_D\xi'' + \eta) \tag{8.13}$$

Equation (8.13) can be modified to bring out the physical content. Expand the sine terms using the trigonometric sums of angles formula and multiply the two factors to give

$$\begin{aligned}\sin(\Omega_D\xi' + \eta)\sin(\Omega_D\xi'' + \eta) = {} & \sin\Omega_D\xi' \sin\Omega_D\xi'' \cos^2\eta \\ & + \sin\Omega_D\xi' \cos\Omega_D\xi'' \cos\eta \sin\eta \\ & + \cos\Omega_D\xi' \sin\Omega_D\xi'' \sin\eta\cos\eta \\ & + \cos\Omega_D\xi' \cos\Omega_D\xi'' \sin^2\eta \end{aligned} \tag{8.14}$$

The quantity needed for the diffusion coefficient is $(r(t) - r_0)^2/t$ averaged over initial particle positions in longitude and averaged over a representative sample of the magnetic fluctuations. Averaging (8.14) over η eliminates the two terms containing $\sin\eta\cos\eta$ and replaces the $\sin^2\eta$ and $\cos^2\eta$ factors by $\frac{1}{2}$. The result is

$$[r(t) - r_0]^2 = \frac{1}{2}\left(\frac{5}{7}\right)^2\left(\frac{r_0^2\Omega_D}{B_d}\right)^2 \int_0^t d\,\xi' \int_0^t d\,\xi'' A(\xi')A(\xi'')\cos\Omega_D(\xi'' - \xi') \tag{8.15}$$

In averaging over η it was assumed that the particles were evenly distributed over η or drift phase. Since the magnetic disturbance is asymmetric, the particle distribution after the compression will not be uniform. Any subsequent disturbance would then act on a non-uniform phase distribution, and the $\sin\eta\cos\eta$ terms in (8.14) would not be zero. However, because the angular drift velocity depends on particle energy and pitch angle, in time the dispersion in Ω_D will restore the uniform distribution in η. This assumption of an efficient phase mixing is usually made in derivations of D_{LL}.

Now change the inner variable of integration to $\zeta = \xi'' - \xi'$ where ζ varies from $-\xi'$ to $t - \xi'$ giving

$$[r(t) - r_0]^2 = \frac{1}{2}\left(\frac{5}{7}\right)^2\left(\frac{r_0^2\Omega_D}{B_d}\right)^2 \int_0^t d\,\xi' \int_{\xi'}^{t-\xi'} d\,\zeta A(\xi')A(\xi' + \zeta)\cos\Omega_D\zeta \tag{8.16}$$

$A(\xi')$ is assumed to fluctuate randomly with zero mean. Over a sufficiently long period of time integrals such as

$$\frac{1}{t}\int_0^t A(\xi')\,A(\xi' + \zeta)\,d\xi' = \frac{1}{t}\int_0^t A(\xi')A(\xi' - \zeta)\,d\xi' \tag{8.17}$$

will be equal and will be independent of the time interval chosen. They depend only on the 'lag', ζ, which is the difference in the arguments of the two factors in the integrand. Now, reverse the order of integration in the double integral of (8.16) and use (8.17) to simplify the result:

$$[r(t) - r_0]^2 = \left(\frac{5}{7}\right)^2\left(\frac{r_0^2\Omega_D}{B_d}\right)^2 \int_0^t d\,\zeta \cos\Omega_D\zeta \int_0^t d\,\xi'\,A(\xi')\,A(\xi' + \zeta) \tag{8.18}$$

The inner integral is t times the auto-correlation function of $A(\xi')$, which is written $\langle A(\xi')A(\xi'+\zeta)\rangle$. It is a function of ζ, not of ξ', and its value will be large when ζ is small so that $A(\xi')$ and $A(\xi'+\zeta)$ are nearly equal. For large ζ, $A(\xi')$ and $A(\xi'+\zeta)$ are uncorrelated and are as likely as not to have different signs. For ζ greater than this correlation length, the autocorrelation function will be zero. As long as the time interval t is larger than the correlation period, the integration over ζ can be extended to infinity giving

$$[r(t)-r_0]^2 = \left(\frac{5}{7}\right)^2\left(\frac{r_0^2\Omega_D}{B_d}\right)^2 t\int_0^\infty d\,\zeta\langle A(\xi')A(\xi'+\zeta)\rangle\cos\Omega_D\zeta \quad (8.19)$$

The diffusion coefficient in terms of $(\Delta r)^2$ is for magnetic field fluctuations

$$D_{LL}^M = \frac{\langle(\Delta L)^2\rangle}{2} = \frac{[r(t)-r_0]^2}{R_E^2 2t} = \frac{1}{8}\left(\frac{5}{7}\right)^2\left(\frac{r_0^2\Omega_D}{R_E B_d}\right)^2 P_A(\Omega_D) \quad (8.20)$$

where

$$P_A(\Omega_D) = 4\int_0^\infty d\,\zeta\langle A(\xi')A(\xi'+\zeta)\rangle\cos\Omega_D\xi \quad (8.21)$$

is the power spectral density of the field variation evaluated at the drift frequency. Thus, the radial diffusion coefficient will be large when the magnetic fluctuations occur at frequencies near the particle drift frequency.

The diffusion coefficient can be expressed in more familiar terms by setting $r_0 = LR_E$, $\Omega_D = 2\pi\nu_{\text{drift}}$, and $B_d = B_0/L^3$. With these substitutions

$$D_{LL}^M = \frac{\pi^2}{2}\left(\frac{5}{7}\right)^2\frac{R_E^2L^{10}}{B_0^2}\nu_{\text{drift}}^2 P_A(\nu_{\text{drift}}) \quad (8.22)$$

The variables in equation (8.22) are the L value and drift frequency which is a function of L and μ. For non-relativistic particles $\nu_{\text{drift}} \propto \mu/L^2$ so that D_{LL}^M is influenced by the ν dependence of $P_A(\nu)$. In the special case where $P_A(\nu)$ varies as ν^{-2}, D_{LL}^M will have no ν_{drift} dependence, and particles of all energies will diffuse at the same rates. If the power spectrum varies as ν^{-n}, D_{LL}^M will be proportional to $L^{6+2n}\mu^{2-n}$. Since the magnitude of D_{LL}^M depends directly on $P_A(\nu_{\text{drift}})$ and the L variation depends on the spectral content, it is to be expected that observed values of D_{LL}^M and their L dependence will change with global magnetic activity.

A similar calculation for off-equatorial particles is more complex but follows the same principals. The curvature drift term must be included in equation (8.3) and the projected change in r at the equator must be averaged over the complete bounce motion, weighting the contribution at each field line segment by the time the particle spends in that segment. The result of this averaging is the mirroring latitude correction factor,

$\Gamma(\alpha_{eq})$, shown in Figure 8.3. This factor is the ratio of the diffusion coefficient at pitch angle α_{eq} to the diffusion coefficient at $\alpha_{eq} = \pi/2$. The magnetic perturbations are most effective in diffusing particles with large equatorial pitch angles so that diffusion proceeds most rapidly for particles confined to the equatorial plane.

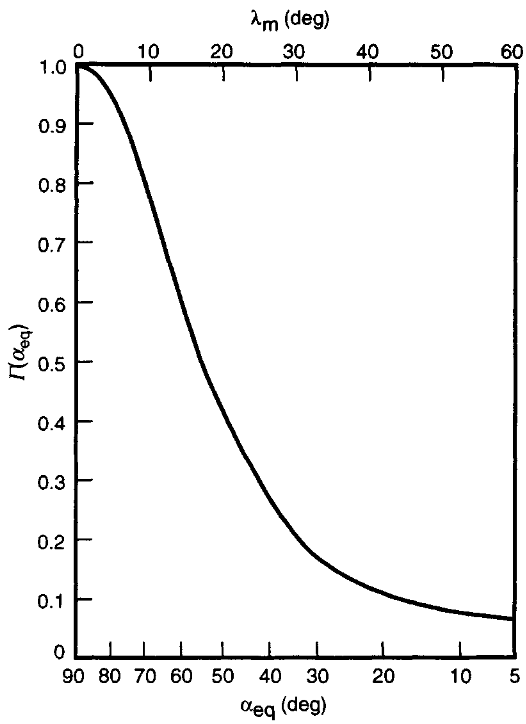

Figure 8.3. Latitude-dependent factor of the radial diffusion coefficient for magnetic fluctuations. The curve shows the more rapid diffusion of equatorially trapped particles.

Radial diffusion induced by electric potential fields

Large-scale electric potential fields are imposed on the magnetosphere by the solar wind and by plasma circulation within the magnetosphere. However, the magnitude and geometry of these fields is uncertain at present, and estimates of diffusion from this mechanism are somewhat speculative. Nevertheless, it is important to estimate the magnitude and character of diffusion from electric potential fields in order to assess the importance of this mechanism.

The calculation of electric field diffusion proceeds in much the same way as diffusion by magnetic perturbations. Again, the development will be restricted initially to equatorial particles with the results for off-equatorial particles considered at the end. The applied electric field will be assumed perpendicular to $\mathbf{B}_d$ at all positions in the magnetosphere. The starting point for the calculation is the radial component of the $\mathbf{E} \times \mathbf{B}/B^2$ drift velocity from equation (8.3)

$$\frac{dr}{dt} = -\frac{E_\phi}{B_d} \tag{8.23}$$

The time variations are assumed to be stochastic in the same sense as the magnetic perturbations in the preceding section. Only the ϕ component of $\mathbf{E}$ is involved in radial displacements and its equatorial values can be represented by a Fourier expansion in longitude ϕ:

$$E_\phi(r_0, \phi, t) = \sum_{n=1}^{N} E_{\phi n}(r_0, t)\cos[n\phi + \gamma_n(r_0, t)] \tag{8.24}$$

The number of terms in the sum of equation (8.24) will depend on the complexity or spatial structure of the electric field. For example a uniform dawn-to-dusk electric field would contain only the $n = 1$ term. A simplification will be to let the phase constants γ_n be independent of t. This assumption fixes the longitude of the nodes of the electric field components, a reasonable assumption if the electric field has its origin in the solar wind or in the magnetospheric tail.

The radial displacement of a particle whose initial coordinates are r_0 and $\phi = \eta$ is obtained by integrating (8.23) from zero to t replacing ϕ by $\Omega_D t + \eta$:

$$r(t) - r_0 = -\frac{1}{B_d}\int_0^t \sum E_{\phi n}(r_0, \xi)\cos[n\Omega_D\xi + n\eta + \gamma_n(r_0)]\,d\xi \tag{8.25}$$

With this equation the expression for $\langle(\Delta r)^2\rangle$ is obtained in the same manner as for magnetic perturbations. Equation (8.25) is squared and averaged over η. The averaging over initial longitude η will eliminate all terms except the power spectrum expressions. Also, only the fluctuating part of the electric field has an influence on the motion. This result is expected. A steady electric field will distort the azimuthal drift path, but the orbit will remain closed and no net displacement will occur. Squaring equation (8.25), averaging over longitude, and rearranging the integrals as in equation (8.12)–(8.22) results in the diffusion coefficient

$$D^{\mathrm{E}}_{LL} = \left\langle \frac{(\Delta L)^2}{2} \right\rangle = \frac{1}{2R_{\mathrm{E}}^2 B_d^2} \sum_{n=1}^{N} \int_0^t \langle \tilde{E}_{\phi n}(r_0, \xi') \tilde{E}_{\phi n}(r_0, \xi' + \zeta) \rangle \cos \Omega_{\mathrm{D}} \zeta \, \mathrm{d}\zeta \tag{8.26}$$

where the fluctuating part of the electric field is denoted by coefficients $\tilde{E}_{\phi n}$.

For particles mirroring off the equatorial plane the diffusion coefficient can be derived by starting with equation (8.23) with L replacing r as the radial coordinate. If the electric field is always perpendicular to $\mathbf{B}_{\mathrm{d}}$, the instantaneous change in L for a particle at latitude λ is given by

$$\mathrm{R_E} \frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{E_\phi(L, \lambda)}{B(L, \lambda)} \cdot \frac{\sqrt{(1 + 3\sin^2 \lambda)}}{\cos^3 \lambda} \tag{8.27}$$

The first factor is the electric field drift velocity at λ and the second is the ratio of the field line separation at the equator to the field line separation at λ. This factor is needed because at λ a smaller displacement perpendicular to $\mathbf{B}_{\mathrm{d}}$ is needed to traverse a given ΔL. The disturbance electric field and the dipole magnetic field map from λ to the equator as

$$E_\phi(L, \lambda) = E_\phi(L, 0)/\cos^3 \lambda \tag{8.28}$$

and

$$B(L, \lambda) = B(L, 0)\sqrt{(1 + 3\sin^2 \lambda)}/\cos^6 \lambda$$

Therefore the latitude factors in equation (8.27) cancel leaving

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{1}{\mathrm{R_E}} \frac{E_\phi(L, 0)}{B(L, 0)} \tag{8.29}$$

which is the same as equation (8.23) which was written for the equatorial plane. This surprising result indicates that radial diffusion by potential electric fields proceeds at the same rate for off-equatorial particles as it does for particles trapped on the equator if the field lines are equipotentials. Expressed in terms of the power spectra of the Fourier components of the electric field, the diffusion coefficient for electric fields from (8.26) is

$$D^{\mathrm{E}}_{\mathrm{LL}}(L, \nu_{\mathrm{drift}}) = \frac{L^6}{8\mathrm{R_E^2} B_0^2} \sum_{n=1}^{N} P_n(L, n\nu)_{\nu = \nu\mathrm{drift}} \tag{8.30}$$

where $P_n(L, n\nu)$ is the power spectral density of the nth harmonic of the electric field fluctuations evaluated at the same harmonic of the drift frequency. The need for harmonics stems from the fact that if the disturbance field has n nodes, it must vary at n times the particle drift frequency to maintain the resonance condition.

Because v_{drift} for constant μ depends on L, the overall variation of D^E_{LL} with L will depend on the frequency dependence of $P_n(L,\nu)$ as well as on the L^6 term. For example, if $P_n \propto L^0\nu^{-m}$ then $D^E_{LL} \propto L^{6+2m}/\mu^m$. As in the case of magnetically driven diffusion, the process is much more rapid at larger L values.

Observed and derived values of D_{LL}

Changes in the radial distribution of trapped particles have been interpreted as evidence for radial diffusion. Efforts to explain these observations by applying equation (8.1) have led to a number of experimental determinations of D_{LL}. Two general approaches are used. If the distribution in L is evolving with time, equation (8.1) is solved as an initial value problem. The observed initial distribution is specified and numerical integration of (8.1) predicts the distribution at latter times. D_{LL} is adjusted to cause the calculated distributions to match the observed ones. Values of D_{LL} have also been obtained with (8.1) by adding source and loss terms and solving for the equilibrium distribution. The boundary conditions needed are that $f(\mu, J, L)$ is equal to the experimental values at some outer boundary and falls to zero at the inner boundary $L = 1$. Again D_{LL} is varied to give a best fit. In this latter technique it is necessary to know the particle sources and losses. Except for the neutron decay source of protons, the internal sources can usually be ignored, but the loss rates from pitch-angle scattering are important.

In four instances narrow bands of electrons were injected into the magnetosphere by high-altitude nuclear weapon detonations. The subsequent spreading of these sharp initial distributions allows a straightforward extraction of D_{LL} from experiment. The values of D_{LL} are not sensitive to the assumed loss processes, but are, of course, characteristic of diffusion only during the time immediately following the injection.

Figure 8.4 is a compilation of theoretical (dashed lines) and experimental (solid lines) values of the radial diffusion coefficient. As was expected from the theoretical expressions derived earlier, D_{LL} increases with L, varying as L^6 to L^{10}. This general agreement confirms that radial diffusion processes occur as described. However, improved precision in the measurements is needed and it is necessary to understand how D_{LL} responds to changing magnetic activity. There are very large differences in the coefficients obtained by these methods, indicating that the experimental uncertainties are large or (more likely) that the observed diffusion coefficient is time dependent. In view of the many approximations used in

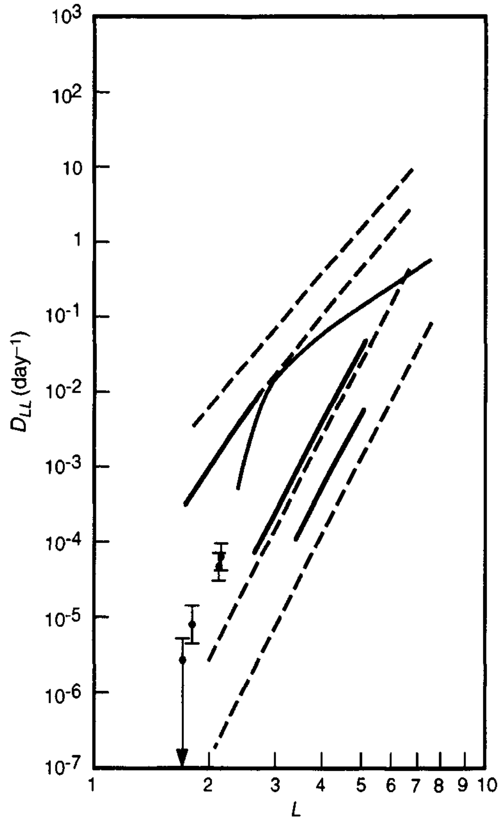

Figure 8.4. Experimental (solid lines) and theoretical (dashed lines) values of D_{LL}.

deriving theoretical values of D_{LL}, it is not surprising that theory and experiment differ somewhat. It is also expected that D_{LL} would reflect the intensity of magnetic disturbances, and these are known to vary greatly with time.

From the standpoint of theory, the assumption of small disturbance fields is quite restrictive. The phase or longitude averaging is also suspect if disturbances occur so frequently that the distribution is unable to relax to a uniform distribution in longitude before the next impulse occurs. Estimates of the particle diffusion in larger field changes and arbitrary time variations is best done by simulation, tracking a number of particles through the time-dependent electric and magnetic fields and tabulating their behavior. Again, the applicability of such results to the magnetosphere is dependent on the accuracy of the assumed field variations.

The magnitude of D_{LL} implies that diffusive changes in the L distribution could take place in a day at $L \approx 5$ but would require many days to be noticeable at $L = 2$. The strong L dependence implies that particles diffusing inward from the outer boundary spend most of their time at low L values.

Dilution of phase space density

It is apparent from the results of Chapter 6 that the phase space density of particles becomes smaller as the particles diffuse along a radial path to smaller L. This decrease might seem to contradict Liouville's theorem discussed in Chapter 4 which predicts that the phase space density along a dynamic path is preserved. However, a closer examination of the details of the diffusion shows that no inconsistency occurs. The evolution of a band of particles of equal μ responding to an electric field disturbance is

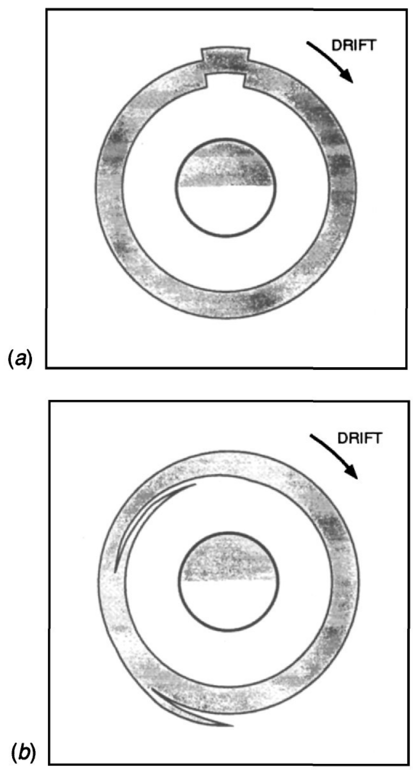

Figure 8.5. Phase space mixing after radial perturbation. Differential drift rates mix regions containing particles and voids, thereby diluting the phase space density.

illustrated in Figure 8.5. For simplicity, only equatorial particles ($J = 0$) are considered, and the disturbance is taken to be an azimuthal electric potential field in the midnight sector which acts momentarily and displaces particles in that sector outward. (The fringing field will displace particles inward at other longitudes, but this smaller motion will be ignored.) The distribution after this impulse is shown in Figure 8.5(a); all particles initially have the same μ and J which are preserved. Section (b) of the diagram illustrates the evolution of the distribution with time after the electric field is removed. The lower drift rate of particles at larger L produces a spiral in particle density, and, as time increases, the spiral becomes more tightly wound. the resulting distribution is then a fine-grained mixture of regions containing particles at densities given by Liouville's theorem and of voids containing no particles.

Eventually, the structure will become too detailed to observe, and the dispersion in drift rates for particles with slightly different energies or pitch angles will effectively mix the two regions. The overall effect is to produce a distribution in which the phase space density appears to decrease.

Problems

1. If two closely spaced field lines lie in the same meridian but are separated by Δr_{eq} at the equator and by Δr_λ at latitude λ, show that $\Delta r_{\text{eq}}/\Delta r_\lambda = \sqrt{(1 + 3\sin^2\lambda)}/\cos^3\lambda$.

2. An electric field is perpendicular to $\mathbf{B}$ and is in the ϕ direction throughout the magnetosphere. Show that if $E_\phi(L, \lambda)$ is its value at latitude λ, it will map to an intensity $E_\phi(L, \lambda = 0) = E_\phi(L, \lambda)\cos^3\lambda$ at the equator.

3. It is sometimes convenient to use a distribution function proportional to the phase space density of particles, but it is also desirable to work with the radial diffusion equation in the L coordinate. Show that the diffusion equation in these terms is

$$\partial F/\partial t = L^2 \partial/\partial L(D_{LL} L^{-2} \partial F/\partial L)$$

where the partial derivatives are performed with μ and J constant.

4. Assume that a flux of equatorially trapped protons has an exponential energy spectrum $j(E) = C\exp(-E/E_0)$ and is trapped at L_1. A large-scale electric field carries the group of protons to L_2 while conserving the first and second adiabatic invariants and the phase space density. Show that the new flux also has an exponential energy spectrum with an e-folding constant of $E_0(L_1/L_2)^3$.

5. Equation (8.11) expresses $\mathrm{d}r/\mathrm{d}t$ with the first term giving the contribution to $\mathrm{d}r/\mathrm{d}t$ from the induction electric field and the second term giving the contribution from the magnetic field changes. Starting with (8.11) derive (8.12). What fraction of the only important term (the first) in (8.12) comes from the induction electric field and what fraction is derived from magnetic field variations?

6. Show that the angular drift frequency of an equatorially trapped particle at r_0 is given by $\Omega_{\mathrm{D}} = 3\mu/qr_0^2$.