Computational Geometry

Polytope representations and duality

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Contents

Vertex and Facet Representation of Polytopes
Minkowski-Weyl

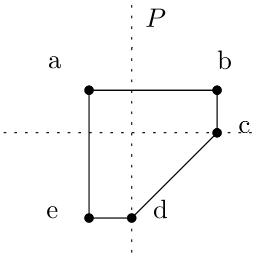
- A polytope can be represented in two dual ways:
  1. **Vertex Representation (V-representation)**:
     - Describes a polytope as the convex hull of a finite set of vertices:
     $$P = \text{conv}\{v_1, v_2, \ldots, v_n\}. \tag{1}$$
  2. **Facet Representation (H-representation)**:
     - Describes a polytope as the intersection of a finite number of half-spaces:
     $$P = \{x \in \mathbb{R}^d \mid Ax \leq b\}. \tag{2}$$
- Convex hull problem: V-rep to H-rep
- Vertex enumeration problem: H-rep to V-rep
- Equivalent via duality

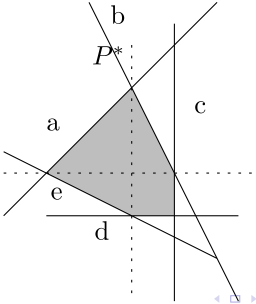
Polar Duality of Polytopes

- The **polar dual** of a polytope P is defined as:

$$P^* = \{y \in \mathbb{R}^d \mid x \cdot y \leq 1, \forall x \in P\}. \tag{3}$$

- **Key Properties:**
  - The dual of a *convex polytope* is also a convex polytope.
  - The **vertices** of P correspond to the **facets** of P^* and vice versa.
  - If P is given in *V-representation* (vertices), P^* is given in *H-representation* (half-spaces), and vice versa.





Numerical example I

The polytope P has the following vertices:

$$(-1,1), \quad (2,1), \quad (-1,-2), \quad (0,-2), \quad (2,0)$$

Defining inequalities:

$$\begin{aligned} 0 &: \quad x_1 \geq -1 \\ 1 &: \quad x_2 \geq -2 \\ 2 &: \quad -x_1 + x_2 \geq -2 \\ 3 &: \quad -x_1 \geq -2 \\ 4 &: \quad -x_2 \geq -1 \end{aligned}$$

The polar dual P^* has inequalities corresponding to the vertices of P, and vice versa.

Numerical example II

The dual polytope P^* has the following vertices:

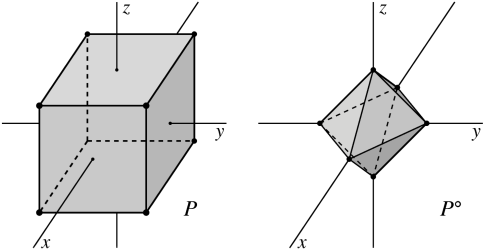
$$(0,1), \quad (-1,0), \quad (0,-1/2), \quad (1/2,-1/2), \quad (1/2,0)$$

Defining inequalities of P^* (corresponding to the vertices of P):

$$\begin{aligned}
0 &: \quad -x_1 + x_2 \geq -1 \\
1 &: \quad 2x_1 + x_2 \geq -1 \\
2 &: \quad -x_1 - 2x_2 \geq -1 \\
3 &: \quad -2x_2 \geq -1 \\
4 &: \quad 2x_1 \geq -1
\end{aligned}$$

Examples of Polar Polytopes

1. The cube and the octahedron(cross-polytope) are polar duals.



2. What is the dual of the i) triangular prism, ii) simplex?

Fourier-Motzkin (FM) elimination

Topics:

- From linear equalities (Gauss elim.) to linear inequalities (FM elim)
- Equivalence of representations (V-rep $=$ H-rep)
- FM as a convex hull algorithm
- FM as an LP solver

Equivalence of Polytope Representations

A V-polytope is defined as:

$$x \in \mathbb{R}^d, \quad x = \sum_{i=1}^{n} \lambda_i v_i + \sum_{i=1}^{n'} t_i r_i, \quad t_i \geq 0, \quad \lambda_i \geq 0, \quad \sum_{i=1}^{n} \lambda_i = 1$$

where $\{v_1, v_2, \ldots, v_n\}$ are the vertices and $\{r_1, r_2, \ldots, r_{n'}\}$ are the rays of the cone.

A H-polytope is defined by a set of halfspaces:

$$x \in \mathbb{R}^d \;:\; \{Ax \leq z\}, \; A \in \mathbb{R}^{m \times d}, z \in \mathbb{R}^m$$

where A is a matrix of coefficients and z is a vector of bounds.

Fourier-Motzkin for convex hull computations

What is the representation of the following polytope?

$$(x, \lambda, t) \in \mathbb{R}^{d+n+n'}, x = \sum_{i=1}^{n} \lambda_i v_i + \sum_{i=1}^{n'} t_i r_i, \quad t_i \geq 0, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1$$

Project in x coordinates, by eliminating λ, t.

What is the representation of the new "eliminated" polytope?

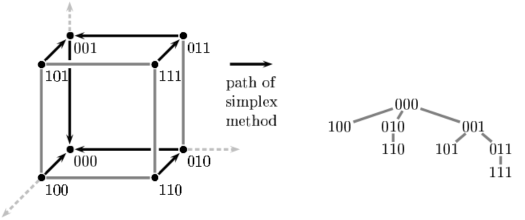
Reverse Search Method

[Avis, Fukuda]

1. Find an initial vertex using linear programming.
2. Define an adjacency structure where vertices represent polyhedral vertices.
3. Use **reverse search** to traverse the vertex graph efficiently.
4. Maintain only local information to reduce memory usage.

Complexity

- Time: $O(VE)$ where V = vertices, E = edges.
- Space: $O(d)$ (stores only one vertex at a time).



(a) The "simplex tree" induced by the objective $(-\sum x_i)$.

(b) The corresponding reverse search tree.

Output sensitive convex hull algorithms

A convex hull algorithm is total-polynomial it runs in time bounded by a polynomial in n, d, m (# input points, dimension, # output facets).

- Generic posotion: There are output sensitive algorithms (e.g. gift wrapping)
- Generic posotion: Reverse search algorithm is both output-senstive and compact (its space complexity is polynomial in the input size only)
- There is no known total-polynomial algorithm for the general (degenerate) case

Software

- cdd, ppl (double description method/dual Fourier-Motzkin)
- lrs (reverse search method)
- CGAL, polymake's bb (triangulation / beneath and beyond based)
- qhull (quickhull algorithm)

- Avis, et al - "How good are convex hull algorithms." presents how various algorithms fail to be polynomial, through constructions of "nasty" polytopes.
- Assarf et al - "Computing convex hulls and counting integer points with polymake" experimentally compare various implemetations of main CH algorithms

Convex hulls in biology

- ecmtool: fast and memory-efficient enumeration of elementary conversion modes Bianca Buchner, Tom J Clement, Daan H de Groot, Jürgen Zanghellini, 2023 `https://doi.org/10.1093/bioinformatics/btad095`
- Enumerated 4 212 839 045 vertices of a polytope with 304 variables and 338 facets within 2.6 weeks (using 60 threads)
- Use a paraller version of lrs, mplrs