MICROECONOMICS

*Principles and Analysis*

# CONSUMER OPTIMISATION

# WHAT WE'RE GOING TO DO:

- We’ll solve the consumer's optimisation problem…
- …using methods that we've already introduced.
- This enables us to re-cycle old techniques and results.
- A tip:
  - Run the presentation for firm optimisation…
  - look for the points of comparison…
  - and try to find as many reinterpretations as possible.

# THE PROBLEM

- Maximise consumer's utility

  $U(\mathrm{x})$

- Subject to feasibility constraint

  $\mathbf{x} \in X$

- and to the budget constraint

  $$\sum_{i=1}^{n} p_i x_i \leq y$$

*U assumed to satisfy the standard "shape" axioms*

*Assume consumption set X is the non-negative orthant.*

*The version with fixed money income*

# OVERVIEW…

*Two fundamental views of consumer optimisation*

- Consumer: Optimisation
  - Primal and Dual problems
  - Lessons from the Firm
  - Primal and Dual again

# AN OBVIOUS APPROACH?

- We now have the elements of a standard constrained optimisation problem:
  - the constraints on the consumer.
  - the objective function.
- The next steps might seem obvious:
  - set up a standard Lagrangean.
  - solve it.
  - interpret the solution.
- But the obvious approach is not always the most insightful.
- We're  going to try something a little sneakier…

# THINK LATERALLY…

- In microeconomics an optimisation problem can often be represented in more than one form.
- Which form you use depends on the information you want to get from the solution.
- This applies here.
- The same consumer optimisation problem can be seen in two different ways.
- I’ve used the labels “primal” and “dual” that have become standard in the literature.

# A FIVE-POINT PLAN

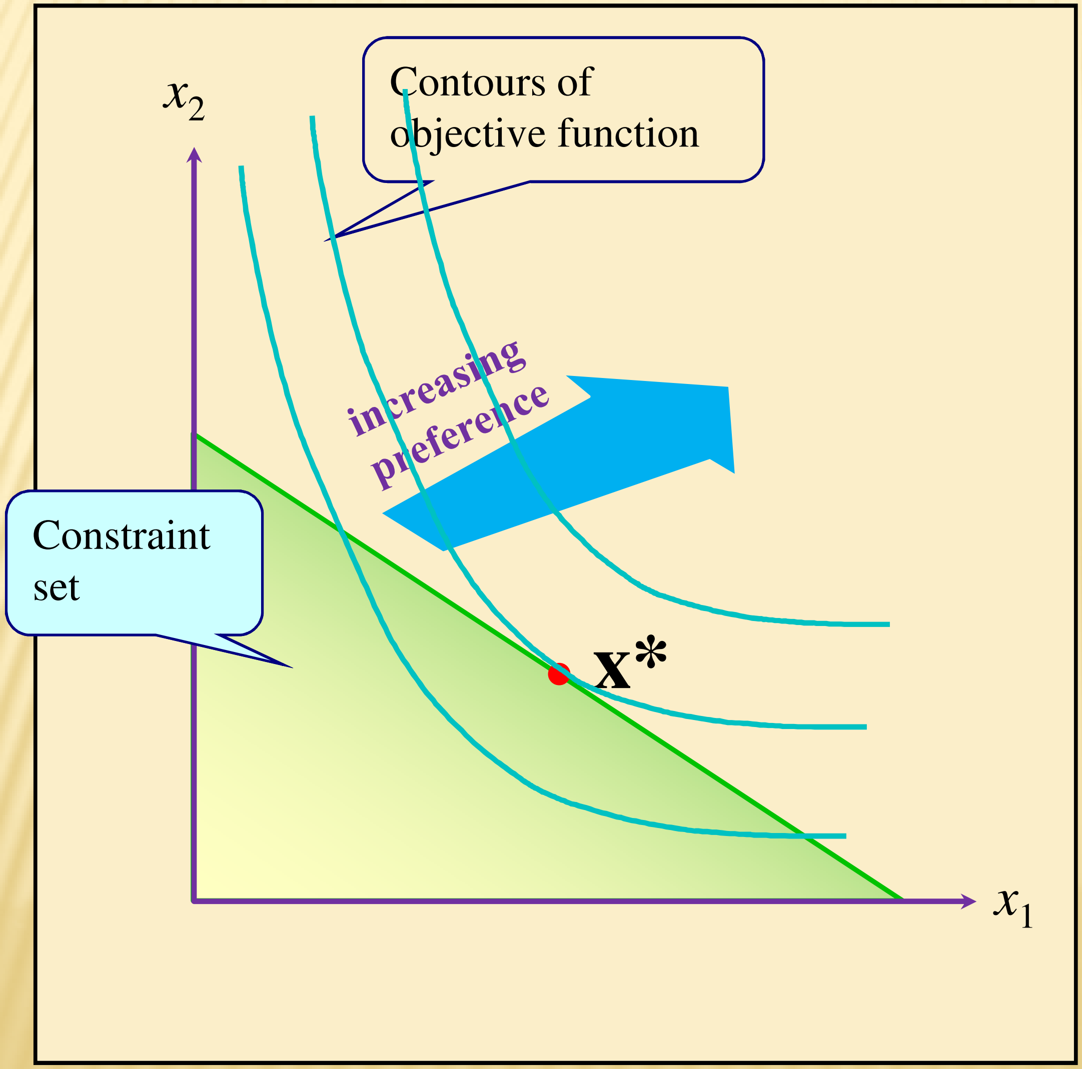
- Set out the basic consumer optimisation problem.
- Show that the solution is equivalent to another problem.
- Show that this equivalent problem is identical to that of the firm.
- Write down the solution.
- Go back to the problem we first thought of...

The primal problem

The dual problem

The primal problem again

# THE PRIMAL PROBLEM



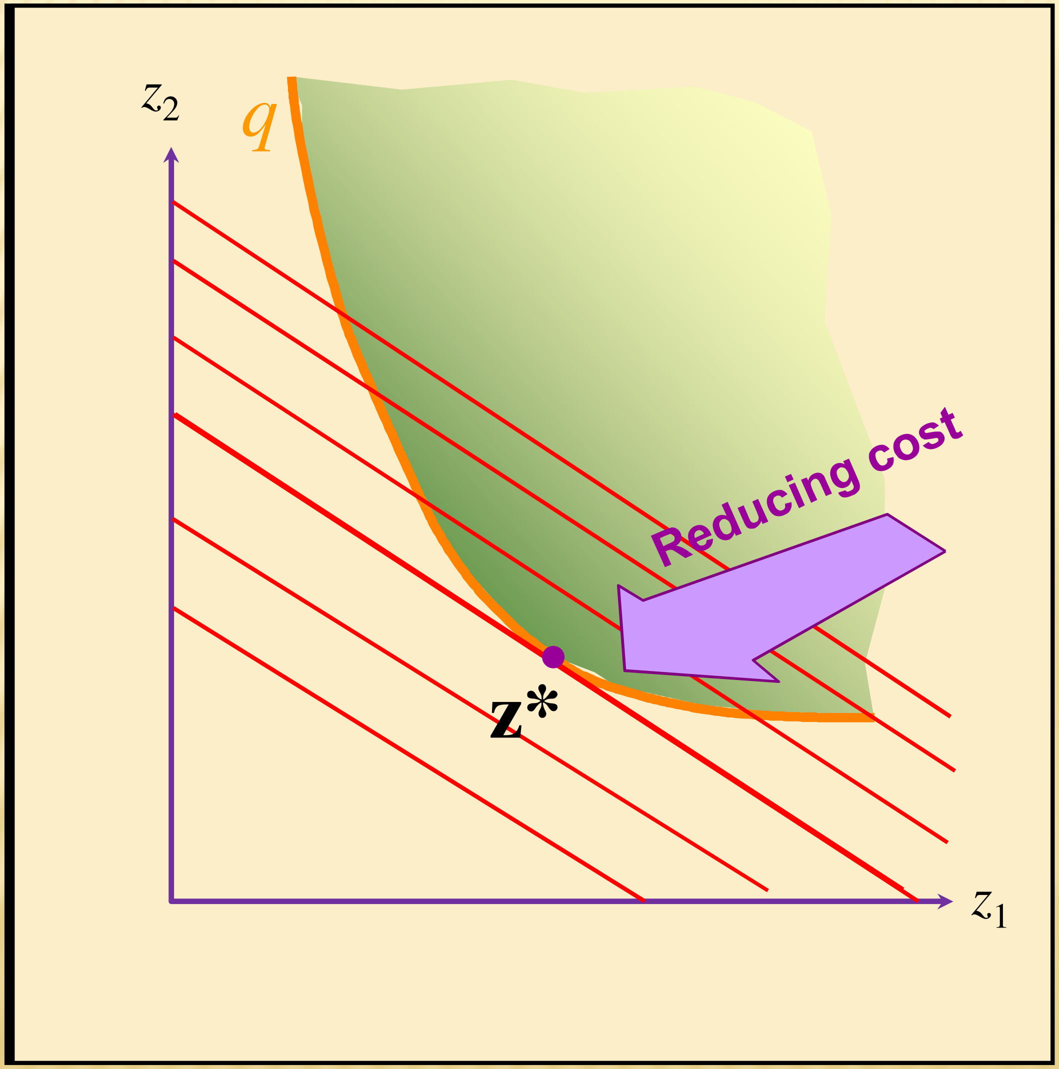

- *The consumer aims to maximise utility...*
- *Subject to budget constraint*
- *Defines the primal problem.*
- *Solution to primal problem*

max $U(\mathbf{x})$ subject to

$$\sum_{i=1}^{n} p_i x_i \leq y$$

- *But there's another way at looking at this*

# THE DUAL PROBLEM




- *Alternatively the consumer could aim to minimise cost...*
- *Subject to utility constraint*
- *Defines the dual problem.*
- *Solution to the problem*
- *Cost minimisation by the firm*

minimise

$$\sum_{i=1}^{n} p_i x_i$$

subject to $U(\mathbf{x}) \geq \upsilon$

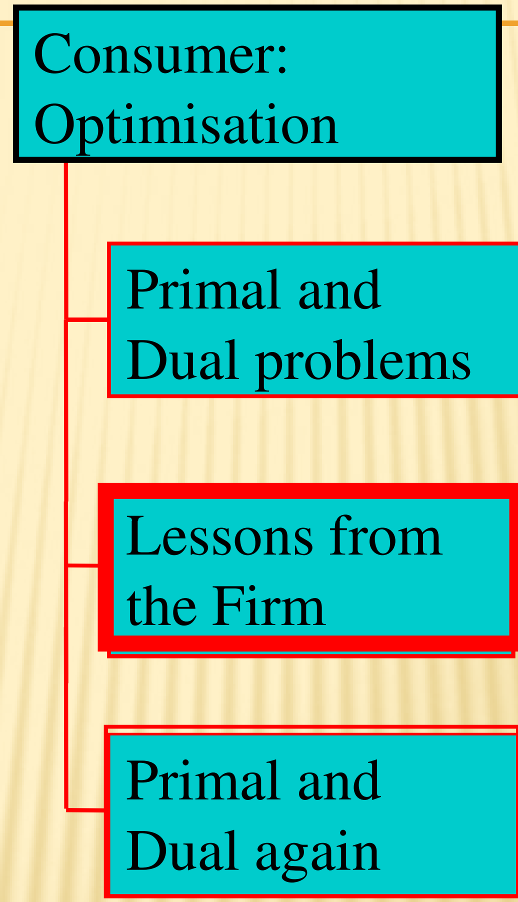
- *But where have we seen the dual problem before?*

# TWO TYPES OF COST MINIMISATION

- The similarity between the two problems is not just a curiosity.
- We can use it to save ourselves work.
- All the results that we had for the firm's "stage 1" problem can be used.
- We just need to "translate" them intelligently
  - Swap over the symbols
  - Swap over the terminology
  - Relabel the theorems

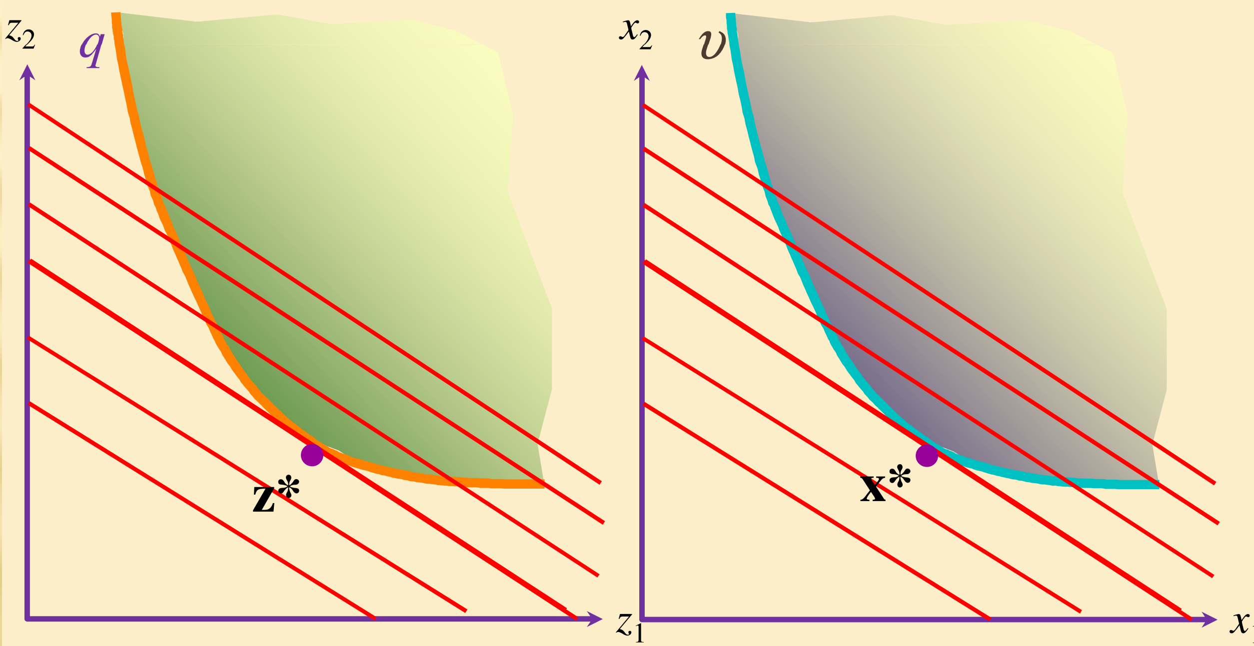
# OVERVIEW...

*Reusing results on optimisation*



# A LESSON FROM THE FIRM

- Compare cost-minimisation for the firm...
  - ...and for the consumer




- The difference is only in notation
- So their solution functions and response functions must be the same

# COST-MINIMISATION: STRICTLY QUASICONCAVE *U*

- Minimise

$$\sum_{i=1}^{n} p_i x_i \quad + \; \lambda[\upsilon \leq U(\mathbf{x})]$$

Lagrange multiplier

- Because of strict quasiconcavity we have an interior solution.
- A set of $n$+1 First-Order Conditions

$$\left.\begin{array}{l} \lambda^* U_1(\mathbf{x}^*) \;= p_1 \\ \lambda^* U_2(\mathbf{x}^*) \;= p_2 \\ \dots \quad \dots \quad \dots \\ \lambda^* U_n(\mathbf{x}^*) \;= p_n \end{array}\right\}$$

one for each good

$$\upsilon = U(\mathbf{x}^*)$$

utility constraint

- *Use the objective function*
- *...and output constraint*
- *...to build the Lagrangean*
- *Differentiate w.r.t. $x_1, \dots, x_n$ and set equal to 0.*
- *... and w.r.t λ*
- *Denote cost minimising values with a * .*

# IF ICS CAN TOUCH THE AXES...

- Minimise

$$\sum_{i=1}^{n} p_i x_i \quad + \ \lambda[\upsilon - U(\mathbf{x})]$$

- Now there is the possibility of corner solutions.
- A set of $n+1$ First-Order Conditions

$$\left.\begin{aligned} \lambda^* U_1(\mathbf{x}^*) &\leq p_1 \\ \lambda^* U_2(\mathbf{x}^*) &\leq p_2 \\ \dots \quad \dots & \quad \dots \\ \lambda^* U_n(\mathbf{x}^*) &\leq p_n \end{aligned}\right\}$$

$$\upsilon = U(\mathbf{x}^*)$$

Can get "<" if optimal value of this good is 0

# FROM THE FOC

- If both goods $i$ and $j$ are purchased and MRS is defined then...

$$\frac{U_i(\mathbf{x}^*)}{U_j(\mathbf{x}^*)} = \frac{p_i}{p_j}$$

- MRS = price ratio
  - "implicit" price = market price

- If good $i$ could be zero then...

$$\frac{U_i(\mathbf{x}^*)}{U_j(\mathbf{x}^*)} \leq \frac{p_i}{p_j}$$

- $\text{MRS}_{ji} \leq$ price ratio
  - "implicit" price ≤ market price

# THE SOLUTION…

- Solving the FOC, you get a cost-minimising value for each good…

$$x_i^* = H^i(\mathrm{p}, \upsilon)$$

- …for the Lagrange multiplier

$$\lambda^* = \lambda^*(\mathrm{p}, \upsilon)$$

- …and for the minimised value of cost itself.
- The *consumer's cost function* or *expenditure function* is defined as

$$C(\mathrm{p}, \upsilon) := \min_{\{U(\mathrm{x}) \geq \upsilon\}} \Sigma\, p_i\, x_i$$

vector of goods prices

Specified utility level

# THE COST FUNCTION HAS THE SAME PROPERTIES AS FOR THE FIRM

- Non-decreasing in every price. Increasing in at least one price
- Increasing in utility $\upsilon$.
- Concave in p
- Homogeneous of degree 1 in all prices **p**.
- Shephard's lemma.

# OTHER RESULTS FOLLOW

- Shephard's Lemma gives demand as a function of prices and utility

$$H^i(\mathbf{p}, \upsilon) = C_i(\mathbf{p}, \upsilon)$$

*H* is the "*compensated*" or conditional demand function.

- Properties of the solution function determine behaviour of response functions.

Downward-sloping with respect to its own price, etc…

- "Short-run" results can be used to model side constraints

For example rationing.

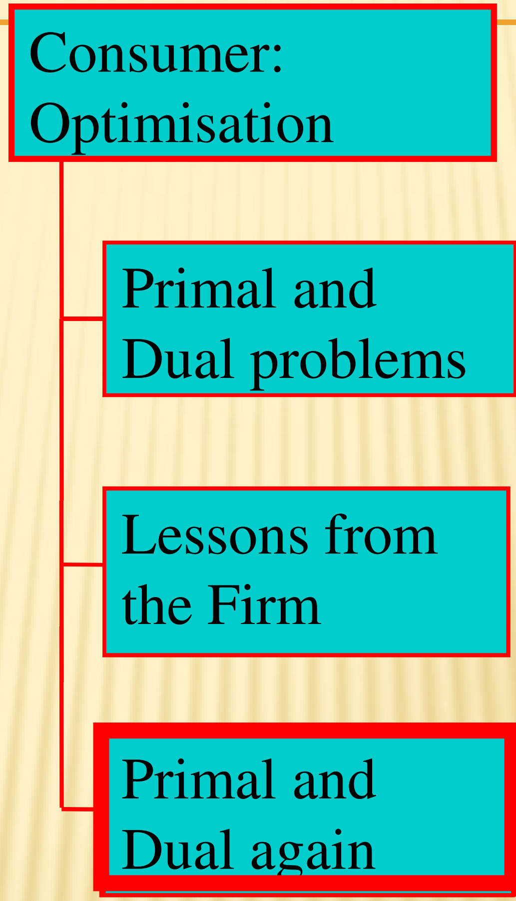
# COMPARING FIRM AND CONSUMER

- Cost-minimisation by the firm...
- ...and expenditure-minimisation by the consumer
- ...are effectively identical problems.
- So the solution and response functions are the same:

| | ***Firm*** | ***Consumer*** |
| --- | --- | --- |
| Problem: | $\min_{\mathbf{z}} \sum_{i=1}^{m} w_i z_i + \lambda[q - \phi(\mathbf{z})]$ | $\min_{\mathbf{x}} \sum_{i=1}^{n} p_i x_i + \lambda[\upsilon - U(\mathbf{x})]$ |
| Solution function: | $C(\mathbf{w}, q)$ | $C(\mathbf{p}, \upsilon)$ |
| Response function: | $z_i^* = H^i(\mathbf{w}, q)$ | $x_i^* = H^i(\mathbf{p}, \upsilon)$ |

# OVERVIEW…

*Exploiting the two approaches*



# THE PRIMAL AND THE DUAL…

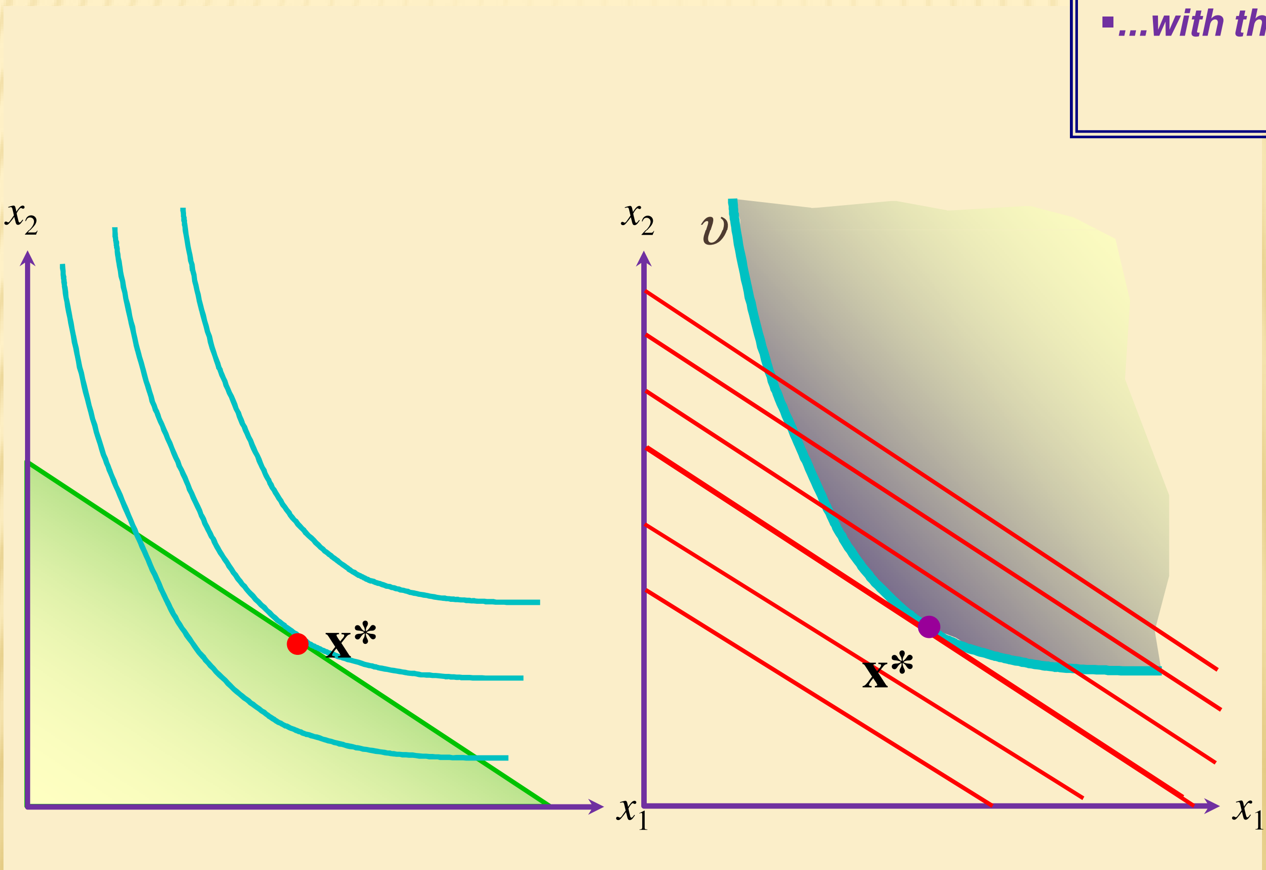
- There's an attractive symmetry about the two approaches to the problem
- In both cases the *p*s are given and you choose the *x*s. But…
- …constraint in the primal becomes objective in the dual…
- …and vice versa.

$$\sum_{i=1}^{n} p_i x_i + \lambda[\upsilon - U(\mathbf{x})]$$

$$U(\mathbf{x}) + \mu\left[ y - \sum_{i=1}^{n} p_i x_i \right]$$

# A NEAT CONNECTION

- *Compare the primal problem of the consumer...*
- *...with the dual problem*




- *The two are equivalent*
- *So we can link up their solution functions and response functions*

# UTILITY MAXIMISATION

- Maximise

$$U(\mathbf{x}) + \mu\left[\, y \geq \sum_{i=1}^{n} p_i x_i \right]$$

Lagrange multiplier

- If $U$ is strictly quasiconcave we have an interior solution.
- A set of $n$+1 First-Order Conditions

$$U_1(\mathbf{x}^*) = \mu^* p_1$$
$$U_2(\mathbf{x}^*) = \mu^* p_2$$
$$\cdots \quad \cdots \quad \cdots$$
$$U_n(\mathbf{x}^*) = \mu^* p_n$$

If $U$ not strictly quasiconcave then replace "=" by "≤"

$$y = \sum_{i=1}^{n} p_i x_i^*$$

budget constraint

- *Use the objective function*
- *...and budget constraint*
- *...to build the Lagrangean*
- *Differentiate w.r.t. $x_1, \ldots, x_n$ and set equal to 0.*
- *... and w.r.t $\mu$*
- *Denote utility maximising values with a * .*

# FROM THE FOC

- If both goods $i$ and $j$ are purchased and MRS is defined then...

$$\frac{U_i(\mathbf{x}^*)}{U_j(\mathbf{x}^*)} = \frac{p_i}{p_j}$$

- (same as before)

- MRS = price ratio

- "implicit" price = market price

- If good $i$ could be zero then...

$$\frac{U_i(\mathbf{x}^*)}{U_j(\mathbf{x}^*)} \leq \frac{p_i}{p_j}$$

- $\text{MRS}_{ji} \leq$ price ratio

- "implicit" price ≤ market price

# THE SOLUTION…

- Solving the FOC, you get a utility-maximising value for each good…

$$\mathbf{x}_i^* = D^i(\mathbf{p}, y)$$

- …for the Lagrange multiplier

$$\mu^* = \mu^*(\mathbf{p}, y)$$

- …and for the maximised value of utility itself.
- The *indirect utility function* is defined as

$$V(\mathbf{p}, y) := \max_{\{\Sigma p_i x_i \le y\}} U(\mathbf{x})$$

vector of goods prices

money income

# A USEFUL CONNECTION

- The indirect utility function maps prices and budget into maximal utility

$$\upsilon = V(\mathbf{p}, y)$$

- The cost function maps prices and utility into minimal budget

$$y = C(\mathbf{p}, \upsilon)$$

- Therefore we have:

$$\upsilon = V(\mathbf{p}, C(\mathbf{p}, \upsilon))$$
$$y = C(\mathbf{p}, V(\mathbf{p}, y))$$

The indirect utility function works like an "inverse" to the cost function

The two solution functions have to be consistent with each other. Two sides of the same coin

Odd-looking identities like these can be useful

# THE INDIRECT UTILITY FUNCTION HAS SOME FAMILIAR PROPERTIES…

***(All of these can be established using the known properties of the cost function)***

- Non-increasing in every price. Decreasing in at least one price
- Increasing in income *y*.
- quasi-convex in prices p
- Homogeneous of degree zero in (**p**, *y*)
- Roy's Identity

But what's this…?

# ROY'S IDENTITY

$$\upsilon = V(\mathbf{p}, y) = V(\mathbf{p}, C(\mathbf{p}, \upsilon))$$

"function-of-a-function" rule

$$0 = V_i(\mathbf{p}, C(\mathbf{p}, \upsilon)) + V_y(\mathbf{p}, C(\mathbf{p}, \upsilon))\ C_i(\mathbf{p}, \upsilon)$$

$$0 = V_i(\mathbf{p}, y) \qquad + V_y(\mathbf{p}, y) \qquad x_i^*$$

- *Use the definition of the optimum*
- *Differentiate w.r.t. $p_i$.*
- *Use Shephard's Lemma*
- *Rearrange to get…*
- *So we also have…*

$$x_i^* = -\frac{V_i(\mathbf{p}, y)}{V_y(\mathbf{p}, y)}$$

Marginal disutility of price $i$

Marginal utility of money income

$$x_i^* = -V_i(\mathbf{p}, y)/V_y(\mathbf{p}, y) = D^i(\mathbf{p}, y)$$

Ordinary demand function

# UTILITY AND EXPENDITURE

- Utility maximisation
- ...and expenditure-minimisation by the consumer
- ...are effectively two aspects of the same problem.
- So their solution and response functions are closely connected:

| | ***Primal*** | ***Dual*** |
|---|---|---|
| Problem: | $\max_{\mathbf{x}} U(\mathbf{x}) + \mu\left[y - \sum_{i=1}^{n} p_i x_i\right]$ | $\min_{\mathbf{x}} \sum_{i=1}^{n} p_i x_i + \lambda[\upsilon - U(\mathbf{x})]$ |
| Solution function: | $V(\mathbf{p}, y)$ | $C(\mathbf{p}, \upsilon)$ |
| Response function: | $x_i^* = D^i(\mathbf{p}, y)$ | $x_i^* = H^i(\mathbf{p}, \upsilon)$ |

# SUMMARY

- A lot of the basic results of the consumer theory can be found without too much hard work.
- We need two "tricks":

1. A simple relabelling exercise:
   - cost minimisation is reinterpreted from output targets to utility targets.
2. The primal-dual insight:
   - utility maximisation subject to budget is equivalent to cost minimisation subject to utility.

# 1. COST MINIMISATION: TWO APPLICATIONS

- THE FIRM
- min *cost of inputs*
- subject to *output target*
- Solution is of the form $C(\mathbf{w},q)$

- THE CONSUMER
- min *budget*
- subject to *utility* target
- Solution is of the form $C(\mathbf{p},\upsilon)$

# 2. CONSUMER: EQUIVALENT APPROACHES

- PRIMAL
- max utility
- subject to budget constraint
- Solution is a function of $(\mathbf{p}, y)$

- DUAL
- min *budget*
- subject to *utility constraint*
- Solution is a function of $(\mathbf{p}, v)$

# BASIC FUNCTIONAL RELATIONS

- $C(\mathbf{p}, \upsilon)$ **cost (expenditure)**
- $H^i(\mathbf{p}, \upsilon)$ **Compensated demand for good *i***
- $V(\mathbf{p}, y)$ **indirect utility**
- $D^i(\mathbf{p}, y)$ **ordinary demand for input *i***

Utility

*H* is also known as "Hicksian" demand.

money income

# WHAT NEXT?

- Examine the response of consumer demand to changes in prices and incomes.
- Household supply of goods to the market.
- Develop the concept of consumer welfare