MICROECONOMICS

Principles and Analysis

CONSUMER OPTIMISATION

WHAT WE'RE GOING TO DO:

- We'll solve the consumer's optimisation problem…
- …using methods that we've already introduced.
- This enables us to re-cycle old techniques and results.
- A tip:
  - Run the presentation for firm optimisation…
  - look for the points of comparison…
  - and try to find as many reinterpretations as possible.

THE PROBLEM

- Maximise consumer's utility $U(\mathbf{x})$ — *U assumed to satisfy the standard "shape" axioms*
- Subject to feasibility constraint $\mathbf{x} \in X$ — *Assume consumption set X is the non-negative orthant.*
- and to the budget constraint $\sum_{i=1}^{n} p_i x_i \le y$ — *The version with fixed money income*

OVERVIEW…

Two fundamental views of consumer optimisation

Consumer: Optimisation

- Primal and Dual problems
- Lessons from the Firm
- Primal and Dual again

AN OBVIOUS APPROACH?

- We now have the elements of a standard constrained optimisation problem:
  - the constraints on the consumer.
  - the objective function.
- The next steps might seem obvious:
  - set up a standard Lagrangean.
  - solve it.
  - interpret the solution.
- But the obvious approach is not always the most insightful.
- We're going to try something a little sneakier…

THINK LATERALLY…

- In microeconomics an optimisation problem can often be represented in more than one form.
- Which form you use depends on the information you want to get from the solution.
- This applies here.
- The same consumer optimisation problem can be seen in two different ways.
- I've used the labels "primal" and "dual" that have become standard in the literature.

A FIVE-POINT PLAN

- Set out the basic consumer optimisation problem.
- Show that the solution is equivalent to another problem.
- Show that this equivalent problem is identical to that of the firm.
- Write down the solution.
- Go back to the problem we first thought of...

The primal problem

The dual problem

The primal problem again

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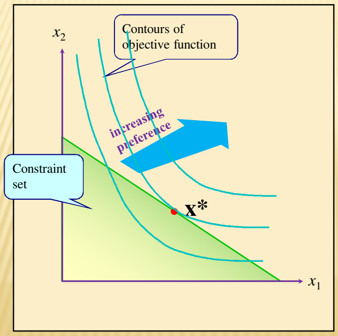
THE PRIMAL PROBLEM



- The consumer aims to maximise utility...
- Subject to budget constraint
- Defines the primal problem.
- Solution to primal problem

max $U(\mathbf{x})$ subject to

$$\sum_{i=1}^{n} p_i x_i \le y$$

- *But there's another way at looking at this*

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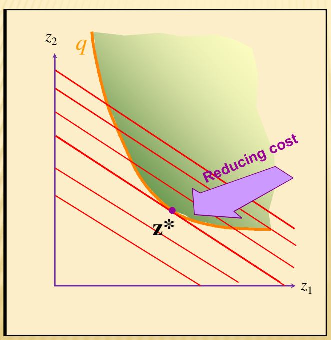
THE DUAL PROBLEM



- Alternatively the consumer could aim to minimise cost...
- Subject to utility constraint
- Defines the dual problem.
- Solution to the problem
- Cost minimisation by the firm

minimise

$$\sum_{i=1}^{n} p_i x_i$$

subject to $U(\mathbf{x}) \ge \upsilon$

- *But where have we seen the dual problem before?*

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TWO TYPES OF COST MINIMISATION

- The similarity between the two problems is not just a curiosity.
- We can use it to save ourselves work.
- All the results that we had for the firm's "stage 1" problem can be used.
- We just need to "translate" them intelligently
  - Swap over the symbols
  - Swap over the terminology
  - Relabel the theorems

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OVERVIEW...

Reusing results on optimisation

- Consumer: Optimisation
  - Primal and Dual problems
  - Lessons from the Firm
  - Primal and Dual again

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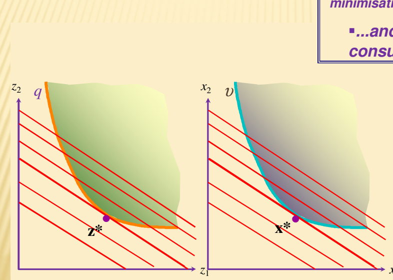
A LESSON FROM THE FIRM

- *Compare cost-minimisation for the firm...*
- *...and for the consumer*



- *The difference is only in notation*
- *So their solution functions and response functions must be the same*

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COST-MINIMISATION: STRICTLY QUASICONCAVE *U*

- Minimise

$$\sum_{i=1}^{n} p_i x_i \; + \; \lambda[\upsilon - U(\mathbf{x})]$$

Lagrange multiplier

- Because of strict quasiconcavity we have an interior solution.
- A set of $n+1$ First-Order Conditions

$$\lambda^* U_1(\mathbf{x}^*) = p_1$$
$$\lambda^* U_2(\mathbf{x}^*) = p_2$$
$$\dots \quad \dots \quad \dots$$
$$\lambda^* U_n(\mathbf{x}^*) = p_n$$

one for each good

$$\upsilon = U(\mathbf{x}^*)$$

utility constraint

- *Use the objective function*
- *...and output constraint*
- *...to build the Lagrangean*
- *Differentiate w.r.t. $x_1, \dots, x_n$ and set equal to 0.*
- *... and w.r.t λ*
- *Denote cost minimising values with a *.*

IF ICS CAN TOUCH THE AXES...

- Minimise

$$\sum_{i=1}^{n} p_i x_i \; + \; \lambda[\upsilon - U(\mathbf{x})]$$

- Now there is the possibility of corner solutions.
- A set of $n+1$ First-Order Conditions

$$\lambda^* U_1(\mathbf{x}^*) \le p_1$$
$$\lambda^* U_2(\mathbf{x}^*) \le p_2$$
$$\dots \quad \dots \quad \dots$$
$$\lambda^* U_n(\mathbf{x}^*) \le p_n$$

$$\upsilon = U(\mathbf{x}^*)$$

Can get "<" if optimal value of this good is 0

FROM THE FOC

- If both goods *i* and *j* are purchased and MRS is defined then...

$$\frac{U_i(\mathbf{x}^*)}{U_j(\mathbf{x}^*)} = \frac{p_i}{p_j}$$

- MRS = price ratio
  - "implicit" price = market price
- If good *i* could be zero then...

$$\frac{U_i(\mathbf{x}^*)}{U_j(\mathbf{x}^*)} \le \frac{p_i}{p_j}$$

- $\text{MRS}_{ji} \le$ price ratio
  - "implicit" price ≤ market price

THE SOLUTION...

- Solving the FOC, you get a cost-minimising value for each good...

$$x_i^* = H^i(\mathbf{p}, \upsilon)$$

- ...for the Lagrange multiplier

$$\lambda^* = \lambda^*(\mathbf{p}, \upsilon)$$

- ...and for the minimised value of cost itself.
- The *consumer's cost function* or *expenditure function* is defined as

$$C(\mathbf{p}, \upsilon) := \min_{\{U(\mathbf{x}) \ge \upsilon\}} \Sigma\, p_i x_i$$

vector of goods prices

Specified utility level

THE COST FUNCTION HAS THE SAME PROPERTIES AS FOR THE FIRM

- Non-decreasing in every price. Increasing in at least one price
- Increasing in utility υ.
- Concave in **p**
- Homogeneous of degree 1 in all prices **p**.
- Shephard's lemma.

OTHER RESULTS FOLLOW

- Shephard's Lemma gives demand as a function of prices and utility

$$H^i(\mathbf{p}, \upsilon) = C_i(\mathbf{p}, \upsilon)$$

H is the "*compensated*" or conditional demand function.

- Properties of the solution function determine behaviour of response functions.

Downward-sloping with respect to its own price, etc...

- "Short-run" results can be used to model side constraints

For example rationing.

COMPARING FIRM AND CONSUMER

- Cost-minimisation by the firm...
- ...and expenditure-minimisation by the consumer
- ...are effectively identical problems.
- So the solution and response functions are the same:

	Firm	***Consumer***
Problem:	$\min_{\mathbf{z}} \sum_{i=1}^{m} w_i z_i + \lambda[q - \phi(\mathbf{z})]$	$\min_{\mathbf{x}} \sum_{i=1}^{n} p_i x_i + \lambda[\upsilon - U(\mathbf{x})]$
Solution function:	$C(\mathbf{w}, q)$	$C(\mathbf{p}, \upsilon)$
Response function:	$z_i^* = H^i(\mathbf{w}, q)$	$x_i^* = H^i(\mathbf{p}, \upsilon)$

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OVERVIEW...

Exploiting the two approaches

- Consumer: Optimisation
  - Primal and Dual problems
  - Lessons from the Firm
  - Primal and Dual again

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THE PRIMAL AND THE DUAL...

- There's an attractive symmetry about the two approaches to the problem
- In both cases the *p*s are given and you choose the *x*s. But…
- …constraint in the primal becomes objective in the dual…
- …and vice versa.

$$\sum_{i=1}^{n} p_i x_i + \lambda[\upsilon - U(\mathbf{x})]$$

$$U(\mathbf{x}) + \mu\left[\, y - \sum_{i=1}^{n} p_i x_i \right]$$

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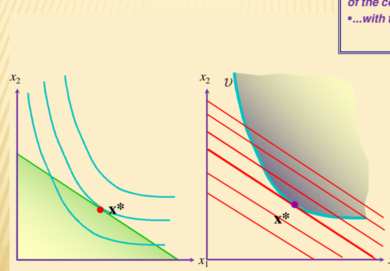
A NEAT CONNECTION

- *Compare the primal problem of the consumer...*
- *...with the dual problem*



- *The two are equivalent*
- *So we can link up their solution functions and response functions*

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UTILITY MAXIMISATION

- Maximise

$$U(\mathbf{x}) + \mu\left[\, y \geq \sum_{i=1}^{n} p_i x_i \right]$$

(Lagrange multiplier)

- If U is strictly quasiconcave we have an interior solution.
- A set of $n+1$ First-Order Conditions

$$U_1(\mathbf{x}^*) = \mu^* p_1$$
$$U_2(\mathbf{x}^*) = \mu^* p_2$$
$$\ldots \quad \ldots \quad \ldots$$
$$U_n(\mathbf{x}^*) = \mu^* p_n$$
$$y = \sum_{i=1}^{n} p_i x_i^*$$

(If U not strictly quasiconcave then replace "=" by "≤")

(budget constraint)

- *Use the objective function*
- *...and budget constraint*
- *...to build the Lagrangean*
- *Differentiate w.r.t. $x_1, \ldots, x_n$ and set equal to 0.*
- *... and w.r.t μ*
- *Denote utility maximising values with a *.*

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FROM THE FOC

- If both goods *i* and *j* are purchased and MRS is defined then...

$$\frac{U_i(\mathbf{x}^*)}{U_j(\mathbf{x}^*)} = \frac{p_i}{p_j}$$

(same as before)

- MRS = price ratio

"implicit" price = market price

- If good *i* could be zero then...

$$\frac{U_i(\mathbf{x}^*)}{U_j(\mathbf{x}^*)} \leq \frac{p_i}{p_j}$$

- $\text{MRS}_{ji} \leq$ price ratio

"implicit" price ≤ market price

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THE SOLUTION…

- Solving the FOC, you get a utility-maximising value for each good…

  $x_i^* = D^i(\mathbf{p}, y)$

- …for the Lagrange multiplier

  $\mu^* = \mu^*(\mathbf{p}, y)$

- …and for the maximised value of utility itself.
- The *indirect utility function* is defined as

$$V(\mathbf{p}, y) := \max_{\{\Sigma p_i x_i \le y\}} U(\mathbf{x})$$

vector of goods prices

money income

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A USEFUL CONNECTION

- The indirect utility function maps prices and budget into maximal utility

  $\upsilon = V(\mathbf{p}, y)$

- The cost function maps prices and utility into minimal budget

  $y = C(\mathbf{p}, \upsilon)$

- Therefore we have:

  $\upsilon = V(\mathbf{p}, C(\mathbf{p}, \upsilon))$

  $y = C(\mathbf{p}, V(\mathbf{p}, y))$

The indirect utility function works like an "inverse" to the cost function

The two solution functions have to be consistent with each other. Two sides of the same coin

Odd-looking identities like these can be useful

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THE INDIRECT UTILITY FUNCTION HAS SOME FAMILIAR PROPERTIES…

(All of these can be established using the known properties of the cost function)

- Non-increasing in every price. Decreasing in at least one price
- Increasing in income y.
- quasi-convex in prices **p**
- Homogeneous of degree zero in (**p**, y)
- Roy's Identity

But what's this…?

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ROY'S IDENTITY

$\upsilon = V(\mathbf{p}, y) = V(\mathbf{p}, C(\mathbf{p}, \upsilon))$

"function-of-a-function" rule

$0 = V_i(\mathbf{p}, C(\mathbf{p}, \upsilon)) + V_y(\mathbf{p}, C(\mathbf{p}, \upsilon))\, C_i(\mathbf{p}, \upsilon)$

$0 = V_i(\mathbf{p}, y) \qquad + V_y(\mathbf{p}, y) \qquad x_i^*$

$$x_i^* = -\frac{V_i(\mathbf{p}, y)}{V_y(\mathbf{p}, y)}$$

Marginal disutility of price i

Marginal utility of money income

$x_i^* = -V_i(\mathbf{p}, y)/V_y(\mathbf{p}, y) = D^i(\mathbf{p}, y)$

Ordinary demand function

- ***Use the definition of the optimum***
- ***Differentiate w.r.t. p_i.***
- ***Use Shephard's Lemma***
- ***Rearrange to get…***
- ***So we also have…***

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UTILITY AND EXPENDITURE

- Utility maximisation
- …and expenditure-minimisation by the consumer
- …are effectively two aspects of the same problem.
- So their solution and response functions are closely connected:

	Primal	***Dual***
Problem:	$\max_{\mathbf{x}} U(\mathbf{x}) + \mu\left[y - \sum_{i=1}^{n} p_i x_i\right]$	$\min_{\mathbf{x}} \sum_{i=1}^{n} p_i x_i + \lambda[\upsilon - U(\mathbf{x})]$
Solution function:	$V(\mathbf{p}, y)$	$C(\mathbf{p}, \upsilon)$
Response function:	$x_i^* = D^i(\mathbf{p}, y)$	$x_i^* = H^i(\mathbf{p}, \upsilon)$

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SUMMARY

A lot of the basic results of the consumer theory can be found without too much hard work.

We need two "tricks":

A simple relabelling exercise:

- cost minimisation is reinterpreted from output targets to utility targets.

The primal-dual insight:

- utility maximisation subject to budget is equivalent to cost minimisation subject to utility.

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1. COST MINIMISATION: TWO APPLICATIONS

- THE FIRM
- min *cost of inputs*
- subject to *output target*
- Solution is of the form $C(\mathbf{w},q)$

- THE CONSUMER
- min *budget*
- subject to *utility* target
- Solution is of the form $C(\mathbf{p},\upsilon)$

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2. CONSUMER: EQUIVALENT APPROACHES

- PRIMAL
- max utility
- subject to budget constraint
- Solution is a function of $(\mathbf{p},y)$

- DUAL
- min *budget*
- subject to *utility constraint*
- Solution is a function of $(\mathbf{p},\upsilon)$

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BASIC FUNCTIONAL RELATIONS

- $C(\mathbf{p},\upsilon)$ **cost (expenditure)**
- $H^i(\mathbf{p},\upsilon)$ **Compensated demand for good *i***
- $V(\mathbf{p}, y)$ **indirect utility**
- $D^i(\mathbf{p}, y)$ **ordinary demand for input *i***

Utility

H is also known as "Hicksian" demand.

money income

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WHAT NEXT?

- Examine the response of consumer demand to changes in prices and incomes.
- Household supply of goods to the market.
- Develop the concept of consumer welfare

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