Άσκηση: Να λυθεί η Δ.Ε.

$$y' = -\frac{x+y-2}{x-y+4}$$

Λύση: Θέτουμε $\boxed{\begin{array}{l} x+y-2 = u+v \\ x-y+4 = u-v \end{array}} \Rightarrow$

$\Rightarrow \begin{array}{l} x = u-1 \Rightarrow dx = du \\ y = v+3 \Rightarrow dy = dv \end{array} \Rightarrow$

$\Rightarrow \frac{dy}{dx} = -\frac{x+y-2}{x-y+4} \Leftrightarrow \frac{dv}{du} = -\frac{u+v}{u-v}$, ομογενής

Θέτουμε $\boxed{v = t\cdot u} \Rightarrow \frac{dv}{du} = \frac{dt}{du}u + t \Rightarrow$

$\Rightarrow$ (Ε) $\Leftrightarrow \frac{dt}{du}u + t = -\frac{1+t}{1-t} \Rightarrow$

$\Rightarrow \int \frac{1-t}{t^2-2t-1}\,dt = \int \frac{du}{u} \Rightarrow -\frac{1}{2}\ln|t^2-2t-1| = \ln u + c$

$\Rightarrow \boxed{t^2-2t-1 = (ku)^{-2}} \Rightarrow \dots$

Γενικά: $\boxed{y' = \frac{a_1x+b_1y+\gamma_1}{a_2x+b_2y+\gamma_2}}$ τότε κάνουμε

$\boxed{\begin{array}{l} a_1x+b_1y+\gamma_1 = u+v \\ a_2x+b_2y+\gamma_2 = u-v \end{array}}$

Ομογενείς Διαφορικές Εξισώσεις

$$y' = \frac{P(x,y)}{Q(x,y)} \qquad (E)$$

όπου P, Q ομογενείς συναρτήσεις, βαθμού k

δηλ: $P(\lambda x, \lambda y) = \lambda^k P(x,y)$

$Q(\lambda x, \lambda y) = \lambda^k Q(x,y)$

Τρόπος Λύσεως: Θέτουμε

$$\boxed{w(x) = \frac{y(x)}{x}} \quad (\sigma) \Longrightarrow \boxed{y(x) = w(x)\cdot x} \Longrightarrow$$

$$\Longrightarrow y'(x) = w'(x)\cdot x + w(x) \overset{(E)}{\Longrightarrow}$$

$$\Longrightarrow w'x + w = \frac{P(x,wx)}{Q(x,wx)} = \frac{x^k P(1,w)}{x^k Q(1,w)}$$

$$\Longrightarrow \boxed{w'x + w = \frac{P(1,w)}{Q(1,w)}}$$ χωριζομένων μεταβλητών.

Άσκηση: Να λυθεί η Δ.Ε.

$$y' = \frac{x^2+3y^2}{2xy} \quad (E)$$

Λύση: Έστω $\dfrac{P(x,y)}{Q(x,y)} = \dfrac{x^2+3y^2}{2xy} \Rightarrow$

$$\Rightarrow \frac{P(\lambda x,\lambda y)}{Q(\lambda x,\lambda y)} = \frac{x^2\lambda^2+3\lambda^2y^2}{2\lambda x\lambda y} = \frac{\lambda^2(x^2+3y^2)}{\lambda^2 2xy}$$

$\Rightarrow$ είναι ομογενής Δ.Ε.

Θέτουμε $\boxed{y = wx} \Rightarrow \boxed{y' = w'x + w} \Rightarrow$

$$(E) \Leftrightarrow w'x + w = \frac{x^2 + 3w^2x^2}{2xwx} \Rightarrow$$

$$\Rightarrow w'x + w = \frac{1+3w^2}{2w} \Leftrightarrow \boxed{x\frac{dw}{dx} = \frac{1+w^2}{2w}}$$

$$\Rightarrow \int \frac{2w}{w^2+1}\,dw = \int \frac{dx}{x} \Rightarrow \ln(w^2+1) = \ln x + c$$

$$\Rightarrow w^2+1 = kx \Rightarrow \boxed{\frac{y^2}{x^2} + 1 = kx}$$