Παραδείγματα:

▶ **Παράδειγμα 1.1.** Να υπολογισθούν τα ολοκληρώματα:

i. $\int\left(2x^4 - 5\sin x + 7\sqrt{x}\right)dx$ ii. $\int \sqrt{x}\left(1+\sqrt[3]{x}\right)^2 dx$

iii. $\int \frac{\left(\sqrt{x}-1\right)^2}{x}\,dx$

$$\boxed{\sqrt[k]{x^{\lambda}} = x^{\lambda/k}}$$

i) $2\int x^4 dx - 5\int \sin x\, dx + 7\int x^{1/2} dx =$

$= 2\frac{x^5}{5} + 5\cos x + 7\cdot\frac{x^{3/2}}{3/2} + c =$

$\sqrt{x^3} = \sqrt{x^2}\cdot\sqrt{x} = x\sqrt{x}$

$= \frac{2x^5}{5} + 5\cos x + \frac{14}{3}x\sqrt{x} + c$

ii. $\int \sqrt{x}\left(1+\sqrt[3]{x}\right)^2 dx = \int \sqrt{x}\left(1+2\sqrt[3]{x}+\sqrt[3]{x^2}\right)dx$

$= \int x^{1/2} + x^{1/2}\cdot 2x^{1/3} + x^{1/2}\cdot x^{2/3}\,dx$

$$\boxed{\sqrt[k]{x}\cdot\sqrt[\lambda]{x} = \sqrt[\lambda\cdot k]{x^{\lambda+k}}}$$

$x^{1/k}\, x^{1/\lambda} = x^{1/k+1/\lambda} = x^{\frac{\lambda+k}{k\cdot\lambda}}$

$= \sqrt[k\cdot\lambda]{x^{\lambda+k}}$

$= \int x^{1/2}dx + 2\int x^{5/6}dx + \int x^{7/6}dx$

$= \frac{x^{3/2}}{3/2} + 2\cdot\frac{x^{11/6}}{11/6} + \frac{x^{13/6}}{13/6} + c$

$\frac{\alpha/1}{\beta/\gamma} = \frac{\alpha\cdot\gamma}{\beta}$

$= \frac{2}{3}x^{3/2} + \frac{12}{11}x^{11/6} + \frac{6}{13}x^{13/6} + c$

iii. $\int \frac{\left(\sqrt{x}-1\right)^2}{x}\,dx = \int \frac{x - 2\sqrt{x} + 1}{x}\,dx$

$= \int 1\,dx - \int 2\boxed{\frac{x^{1/2}}{x}}dx + \int \frac{1}{x}\,dx =$ $= x^{-1/2}$

$= x - 2\frac{x^{-1/2+1}}{-\frac{1}{2}+1} + \ln|x| + c$

$= x - 4\cdot x^{1/2} + \ln|x| + c$

$\boxed{x^{-1/2}} = \frac{1}{x^{1/2}} = \frac{1}{\sqrt{x}}$

$$\frac{x^{1/2}}{x} = x^{\frac{1}{2}-1} = x^{-1/2}$$

Θεώρημα 2.1. *Αν οι f' και g' είναι συνεχείς, τότε:*

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

$$\int_a^b f(x)\,g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)\,g(x)dx$$

π.χ. $\int \boxed{x}\cdot\boxed{\ln x}\,dx =$

$\boxed{\int P(x)\cdot \ln x\, dx}$

πολυώνυμο

x^3+x+2

$\left(\frac{x^4}{4}+\frac{x^2}{2}+2x\right)'$

$$= \int \underbrace{\ln x}_{f(x)}\cdot \underbrace{\left(\frac{x^2}{2}\right)'}_{g'(x)} dx =$$

$$= \ln x\cdot\frac{x^2}{2} - \int (\ln x)'\cdot \frac{x^2}{2}dx$$

$$= \ln x\cdot\frac{x^2}{2} - \int \frac{x}{2}dx = \ln x\cdot\frac{x^2}{2} - \frac{1}{2}\frac{x^2}{2}+c$$

$$= \frac{\ln x\cdot x^2}{2} - \frac{x^2}{4}+c$$

π.χ. $\int \ln x\, dx = \int 1\cdot \ln x\, dx =$

$$= \int (x)'\cdot \ln x\, dx = x\cdot\ln x - \int x\cdot\frac{1}{x}dx$$

$$= x\ln x - x + c = x(\ln x - 1)+c$$

$\int \ln^2 x\, dx = \int 1\cdot \ln^2 x\, dx =$ παραγοντική ολοκλήρωση.

$= \dots$

▶ Παράδειγμα 2.2. Να υπολογισθούν τα ολοκληρώματα:

i. $\int xe^x\,dx$ **ii.** $\int xa^x\,dx$

i) $\int x\cdot e^x dx = \int x\,(e^x)'dx$

$\int P(x)\cdot \boxed{e^x}\,dx$ a^x

i) $\int x\cdot e^x dx = \int x (e^x)' dx$ $\quad \boxed{\int P(x)\cdot e \, dx}$ πολυώνυμο

$= x\cdot e^x - \int (x)' e^x dx$

$= x\cdot e^x - \int e^x dx =$

$= xe^x - e^x + c.$

π.χ. $\int x^2\cdot e^x dx = \int x^2\cdot (e^x)' dx =$

$= x^2\cdot e^x - \int 2x\cdot e^x dx = x^2\cdot e^x - 2\int x\cdot e^x dx$

$= x^2\cdot e^x - 2\left[x\cdot e^x - \int \underbrace{(x)'\cdot e^x dx}\right] =$

$= x^2\cdot e^x - 2x\cdot e^x + 2e^x + c = (x^2 - 2x + 2)e^x + c$

ii) $\int x\cdot \alpha^x dx = \int x\cdot \left(\frac{\alpha^x}{\ln\alpha}\right)' dx$

$(\alpha^x)' = \alpha^x\cdot \ln\alpha$

$\int \alpha^x dx = \frac{\alpha^x}{\ln\alpha} + c$

$= x\cdot\frac{\alpha^x}{\ln\alpha} - \int (x)'\cdot\frac{\alpha^x}{\ln\alpha} dx =$

$= \frac{x\cdot\alpha^x}{\ln\alpha} - \frac{1}{\ln\alpha}\int \alpha^x dx = \frac{x\cdot\alpha^x}{\ln\alpha} - \frac{1}{\ln\alpha}\cdot\frac{\alpha^x}{\ln\alpha} + c$

$= \frac{x\alpha^x}{\ln\alpha} - \frac{\alpha^x}{\ln^2\alpha} + c$

▶ **Παράδειγμα 2.3.** Να υπολογισθεί το ολοκλήρωμα:

i. $\int e^{-x} \sin x \; dx$

$I = \int e^{-x}\sin x dx = \int \left(\frac{e^{-x}}{-1}\right)' \cdot \sin x dx$

$\boxed{\int e^{\alpha x+\beta}\cdot \sin x dx \text{ ή } \int e^{\alpha x+\beta}\cdot \cos x dx}$

$$I = \int e^{-x}\sin x\,dx = \int \left(\frac{e^{-x}}{-1}\right)' \cdot \sin x\,dx$$

$$\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + c$$

$$= -e^{-x}\cdot\sin x - \int -e^{-x}\cdot\cos x\,dx =$$

$$= -e^{-x}\cdot\sin x + \int e^{-x}\cdot\cos x\,dx =$$

$$= -e^{-x}\cdot\sin x + \int (-e^{-x})'\cdot\cos x\,dx =$$

$$= -e^{-x}\sin x - e^{-x}\cdot\cos x - \int -e^{-x}\cdot(-\sin x)\,dx$$

$$= -e^{-x}\cdot\sin x - e^{-x}\cdot\cos x - \int e^{-x}\cdot\sin x\,dx \Rightarrow$$

$$I = -e^{-x}\cdot\sin x - e^{-x}\cos x - I \Rightarrow$$

$$2I = -e^{-x}\cdot\sin x - e^{-x}\cos x \Rightarrow$$

$$\boxed{I = \frac{-e^{-x}(\sin x + \cos x)}{2} + c,\quad c\in\mathbb{R}.}$$

▶ **Παράδειγμα 2.4.** Να υπολογισθούν τα ολοκληρώματα

i. $\int x\cos x\ dx$ **ii.** $\int x\sin x\ dx$

i) $\int x\cdot\cos x\,dx = \int x\cdot(\sin x)'\,dx$

$= x\cdot\sin x - \int \sin x\,dx =$

$= x\cdot\sin x + \cos x + c,\ c\in\mathbb{R}.$

$\int p(x)\cdot\sin x\,dx$ ή
$\int p(x)\cdot\cos x\,d$
(παρόμοιο με το $\int p(x)\cdot e^{x}dx$)

▶ **Παράδειγμα 3.1.** Να υπολογισθούν τα ολοκληρώματα:

i. $\int \boxed{3x^2}(x^3+5)\boxed{dx}$ **ii.** $\int (x^3+2x+6)(3x^2+2)\,dx$

Θέτω $u = x^3+5$

$\frac{du}{dx} = (x^3+5)'$

dx

$f'(x) = \frac{df}{dx}$ Σύμβολο του Leibniz

$\frac{df}{...} = f'(x)$

$$\frac{du}{dx} = (x^3+5)'$$

$$\frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{du}{3x^2}$$

$$\left.\frac{df}{dx}\right|_{x=1} = f'(1)$$

$f(x,y)$

$$\boxed{\int 3x^2(x^3+5)\,dx} =$$

$$\int \cancel{3x^2} \cdot u \cdot \frac{du}{\cancel{3x^2}} =$$

$$\int u\,du = \frac{u^2}{2} + c$$

$$= \frac{(x^3+5)^2}{2} + c,\ c \in \mathbb{R}$$

$$f(x,y) = x^2 + y^2$$

$$\frac{\partial f(x,y)}{\partial x} = 2x$$

$$\frac{\partial f(x,y)}{\partial y} = 2y$$

$$I = \int (x^3 + 2x + 6)^9 (3x^2 + 2)\,dx$$

Θέτω $u = x^3 + 2x + 6$

$$\frac{du}{dx} = 3x^2 + 2 \Rightarrow \boxed{dx = \frac{du}{3x^2+2}}$$

$$I = \int u^9 \cdot \cancel{(3x^2+2)} \cdot \frac{du}{\cancel{3x^2+2}} = \int u^9\,du = \frac{u^{10}}{10} + c$$

$$= \frac{(x^3+2x+6)^{10}}{10} + c, \quad c \in \mathbb{R}.$$

π.χ.

$$I = \int_1^e \frac{\ln x}{x}\,dx \left(= \int_1^e \ln x \cdot (\ln x)'\,dx \right)$$

Θέτω $\boxed{u = \ln x}$

$$\frac{du}{dx} = \frac{1}{x} \Rightarrow \boxed{dx = x \cdot du}$$

Αν $x = 1 \to u = \ln 1 = 0$

Αν $x = e \to u = \ln e = 1$

$$I = \int_0^1 \frac{u}{\cancel{x}} \cdot \cancel{x} \cdot du = \int_0^1 u\,du = \left.\frac{u^2}{2}\right|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$