ΕΘΝΙΚΟ ΚΑΠΟΔΙΣΤΡΙΑΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

ΤΜΗΜΑ ΟΙΚΟΝΟΜΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

ΜΕΤΑΠΤΥΧΙΑΚΟ ΠΡΟΓΡΑΜΜΑ ΣΤΗΝ ΟΙΚΟΝΟΜΙΚΗ ΕΠΙΣΤΗΜΗ

DYNAMICAL MATHEMATICS

(Continuous Time)

ΠΡΟΧΕΙΡΕΣ ΣΗΜΕΙΩΣΕΙΣ

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Contents

Chapter 1

Ordinary Differential Equations (ODE)

1.1 General

- Economic Dynamics is concerned with the movement of economic variables over time.
- There two ways of treating time in economics: discrete-time analysis and continuous-time analysis.
- Differential equations belong to continuous-time analysis, difference equations to discrete-time analysis.

1.2 Categories

Ordinary Differential Equations (ODE)

Partial Differential Equations (PDE)

1.3 Examples

$$y'(x) = 5y^2(x) + x$$

$$\frac{d^2f}{dx^2} + \left(\frac{df}{dx}\right)^2 = 1$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial x^2}$$

1.4 Syllabus

1. ODE of seperable variables
2. First Order Linear ODE
3. Bernoulli ODE
4. Linear ODE of higher order
5. Systems of Linear ODE
6. Stability Theory - Phase Portraits
7. Optimal Control

1.5 Separable ODE

$$\boxed{y'(x) = A(x) \cdot B(y(x))}$$

Examples

$y'(x) = 5x^2 \cdot y(x)$

$y'(x) = 5x^2 + y(x)$, this is NOT a separable ODE

Solution of the above equation is called any function which satisfies it.

Any additional relation of the form $y(x_0) = y_0$, satisfied by the solution, is called **initial condition**

1.1 Exercise: *Solve the equation:* $y'(x) = 5x^2 y(x)$, $y(0) = 1$.

$$y'(x) = 5x^2 y(x) \Rightarrow \frac{dy}{dx} = 5x^2 y(x) \ \Rightarrow \frac{dy}{y} = 5x^2 dx$$

$$\Rightarrow \int \frac{dy}{y} = \int 5x^2 dx \ \Rightarrow \ln|y(x)| = \frac{5x^3}{3} + C \ \Rightarrow \boxed{y(x) = Ce^{\frac{5x^3}{3}}}$$

Remark: This is a general solution. Actually it is an infinite set of solutions.

To find a specific solution we must use the initial condition.

$$y(0) = 1 \Rightarrow 1 = y(0) = Ce^{\frac{5\cdot 0^3}{3}} \Rightarrow 1 = C \Rightarrow \boxed{y(x) = e^{\dfrac{5x^3}{3}}}$$

1.2 Exercise: *Find a demand function with constant elasticity.*

Solution:

$$\epsilon_D = \frac{dD}{dp} \cdot \frac{p}{D} = k \Rightarrow \frac{dD}{D} = k\frac{dp}{p} \Rightarrow \int \frac{dD}{D} = k \int \frac{dp}{p}$$

$$\Rightarrow \ln D + C_1 = k \ln p + C_2 \Rightarrow \boxed{D(p) = Cp^k}$$

1.3 Exercise: *The rate of change of a population is proportional to the current value of the population. Find the population. (Malthus)*

Solution:

$\frac{dN}{dt} = kN \Rightarrow \frac{dN}{N} = kdt \Rightarrow \int \frac{dN}{N} = k \int dt \Rightarrow$

$\ln N = kt + C \Rightarrow N(t) = e^{kt+C} \Rightarrow N(t) = Ce^{kt}$

To calculate the constant C, we use the fact that initially the population is equal to N_0.

$$N(0) = N_0 \Rightarrow N_0 = Ce^{k\cdot 0} \Rightarrow C = N_0 \Rightarrow \boxed{N(t) = N_0 e^{kt}}$$

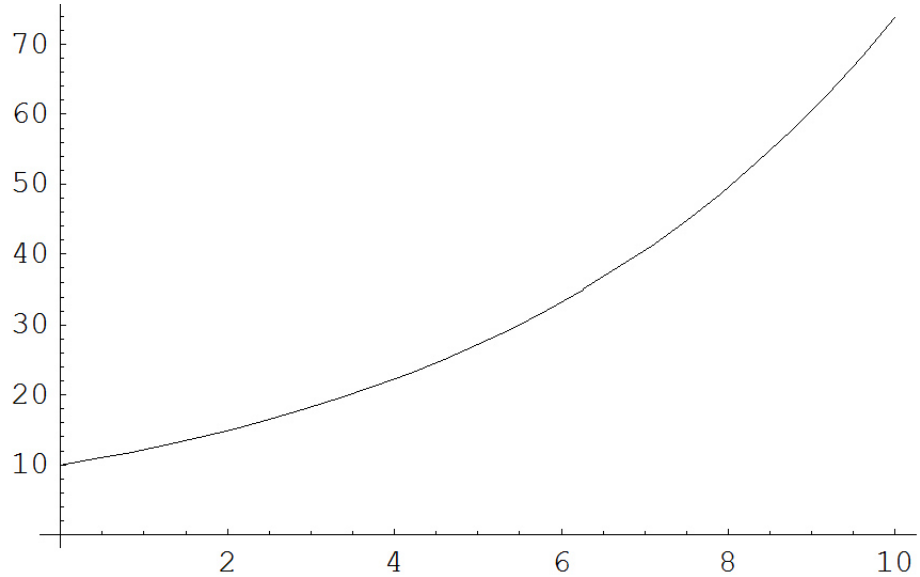


Figure 1.1: $k > 0$

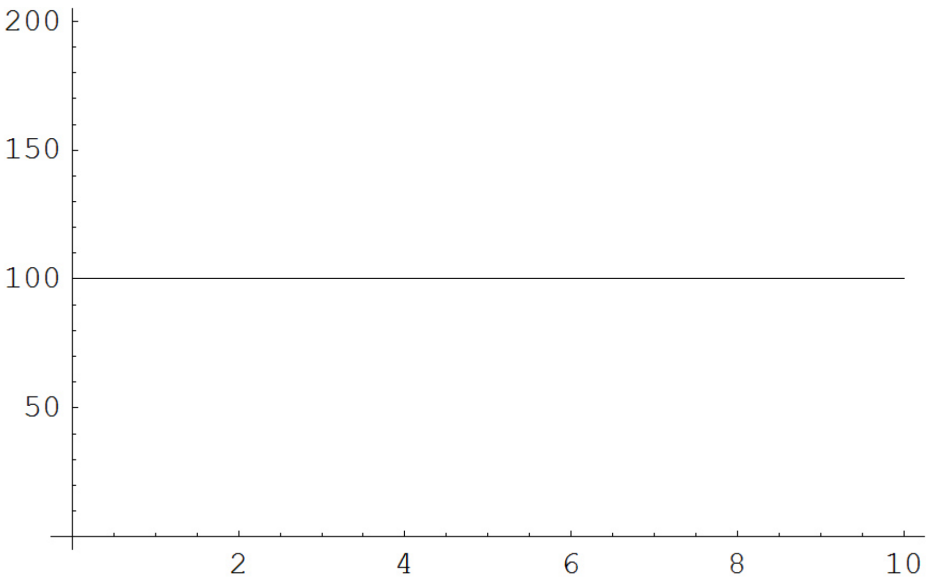


Figure 1.2: $k = 0$

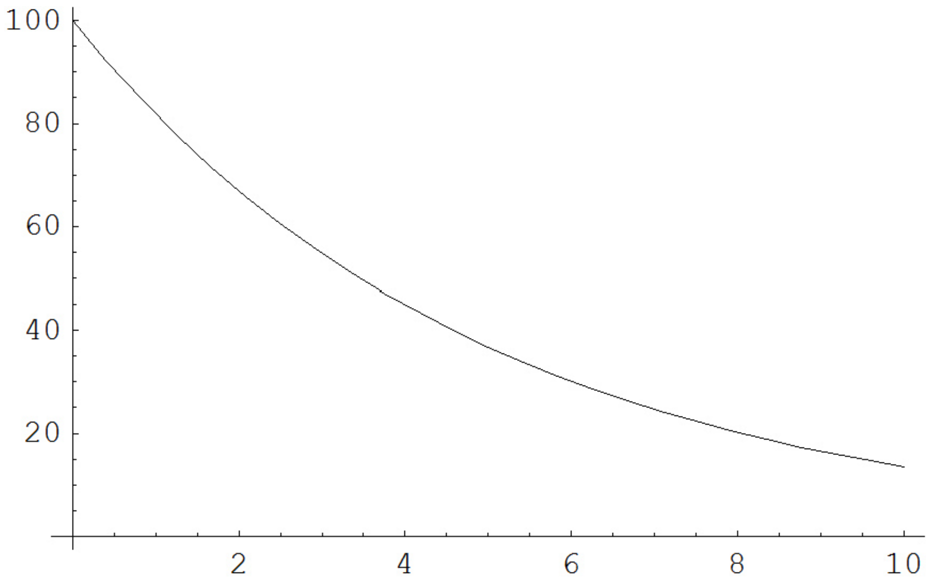


Figure 1.3: $k < 0$

1.4 Exercise: *The rate of change of a population is given by the equation:*

$$y'(t) = y(t)(a - by(t)) \quad , \quad a, b > 0$$

Find the population. (Verhulst 1841)

Remark: This is a generalization of Malthus equation. Now, the parameter k is not constant, but depending from the current value of the population.

Solution: The equation is of separable variables and it becomes:

$\dfrac{dy}{dt} = y(a - by) \Rightarrow \dfrac{dy}{y(a - by)} = dt \Rightarrow \displaystyle\int \frac{dy}{y(a - by)} = \int dt$ ar But,

$$\frac{1}{y(a - by)} = \frac{1/a}{y} + \frac{b/a}{a - by}$$

and thus:

$$\int \frac{dy}{y(a - by)} = \int \frac{1/a}{y} + \int \frac{b/a}{a - by} =$$

$$= \frac{1}{a} \ln|y| + \left(-\frac{1}{b}\right) \frac{b}{a} \ln|a - by| + C_1$$

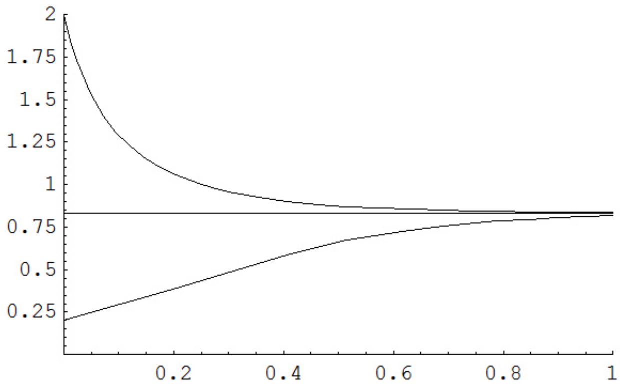


Figure 1.4: Verhulst

and

$$\int dt = t + C_2$$

and after some manipulations we get:

$$\boxed{y(t) = \frac{a}{b + Ke^{-at}}}$$

Finally, using the fact that $y(0) = y_0$ we get:

$$\boxed{y(t) = \frac{a}{b + \left(\frac{a}{y_0} - b\right)e^{-at}}}$$

Remark: $\lim_{t \to +\infty} y(t) = \frac{a}{b}$

This is an asymptotical stable equilibrium point

1.6 First Order Linear Differential Equations

$$\boxed{a(x)y'(x) + b(x)y(x) = \gamma(x)}$$

Examples: $5y' + 6y = 8x^2$ $2xy'(x) + 6x^2y(x) = 8\ln x$ $2x' + 9tx = 8t^3$

SOLUTION

- We solve the "homogeneous" equation: $a(x)y'(x) + b(x)y(x) = 0$. We name the solution $y_h(x)$
- We find a particular solution $y_p(x)$ of the equation: $a(x)y'(x) + b(x)y(x) = \gamma(x)$
- The general solution is: $\boxed{y(x) = y_h(x) + y_p(x)}$

1.5 Exercise: *Solve the ODE:* $y'(x) + 5y(x) = 6$.

Solution: First, we solve the homogeneous equation.

$$y'(x) + 5y(x) = 0 \Leftrightarrow \frac{dy}{dx} = -5y \Rightarrow \int \frac{dy}{y} = -5 \int dx$$

$$\Rightarrow \ln|y| = -5x + C \Rightarrow y_h(x) = ce^{-5x}$$

We find now a particular solution. To achieve that we consider, for the time being, that c is a function of x. Substituting ce^{-5x} into the original equation we get:

$$(ce^{-5x})' + 5ce^{-5x} = 6 \Rightarrow c'e^{-5x} + c(-5e^{-5x}) + 5ce^{-5x} = 6$$

$$c' = 6e^{5x} \Rightarrow c = \int 6e^{5x} = \frac{6}{5}e^{5x}$$

So, the particular solution is: $y_p(x) = \frac{6}{5}e^{5x}e^{-5x} = \frac{6}{5}$

Finally, the general solution is:

$$\boxed{y(x) = ce^{-5x} + \frac{6}{5}}$$

1.6 Exercise: *Solve the ODE:* $xy'(x) + 4y(x) = x^5$.

Solution: First, we solve the homogeneous equation:

$$xy' + 4y = 0 \Rightarrow x\frac{dy}{dx} = -4y \Rightarrow$$

$$\frac{dy}{-4y} = \frac{dx}{x} \Rightarrow \int \frac{dy}{y} = -4 \int \frac{dx}{x}$$

$$\Rightarrow \ln|y| = -4\ln|x| + K = \ln(x)^{-4} + \ln C \Rightarrow \ln|y| = \ln \frac{C}{x^4}$$

$$\Rightarrow \boxed{y_h(x) = \frac{C}{x^4}}$$

For the particular solution we have:

$$x\left(\frac{C}{x^4}\right)' + \frac{C}{x^4} \cdot 4 = x^5 \Rightarrow x \cdot \frac{C'x^4 - C \cdot 4x^3}{x^8} + 4 \cdot \frac{C}{x^4} = x^5$$

$$\Rightarrow C' = x^8 \Rightarrow C = \int x^8 dx = \frac{x^9}{9}$$

$$\Rightarrow y_p = \frac{x^9/9}{x^4} = \frac{x^5}{9}$$

So, the general solution is: $\boxed{y(x) = \frac{C}{x^4} + \frac{x^5}{9}}$

1.7 Exercise: *We consider that the capital flow, $K(t)$, follows the next equation:*

$$\frac{dK(t)}{dt} = I(t) - aK(t)$$

where $I(t)$ is the investment flow and $0 < a < 1$ a use coefficient. If $I(t) = I_0$, find $K(t)$ and $K(\infty)$.

Solution:

$$\tfrac{dK}{dt} + aK(t) = I_0$$

Homogeneous part: $\frac{dK}{dt} + aK(t) = 0$

$$\Rightarrow \int \frac{dK}{K} = -a \int dt \Rightarrow K_h(t) = Ce^{-at}$$

Particular Solution: We treat C as a function: $C = C(t)$

$$(Ce^{-at})' + aCe^{-at} = I_0 \Rightarrow C' = I_0 e^{at} \Rightarrow C = \int I_0 e^{at} dt = \frac{I_0}{a} e^{at}$$

and thus $y_p = \frac{I_0}{a} e^{at} e^{-at} = \frac{I_0}{a}$ The general solution is $\boxed{K(t) = ce^{-at} + \frac{I_0}{a}}$

By using the initial condition $K(0) = K_0$ we get: $K_0 = K(0) = ce^0 + \frac{I_0}{a} \Rightarrow$ $c = K_0 - \frac{I_0}{a}$

And hence the final solution is: $\boxed{K(t) = \left(K_0 - \frac{I_0}{a}\right) e^{-at} + \frac{I_0}{a}}$

<u>Limit Behavior</u>

$$K(\infty) = \left(K_0 - \frac{I_0}{a}\right) e^{-a\cdot\infty} + \frac{I_0}{a} = 0 + \frac{I_0}{a} = \frac{I_0}{a}$$

1.8 Exercise: *The price of a good depends from the time, $p(t)$. The demand function is $D(t) = a + bp(t), b < 0$. The supply function is $S(t) = \gamma + \delta p(t)$, $\delta > 0$. Considering that the rate of change of the price is proportional of the current available quantity of the good, find $p(t)$ and $p(\infty)$.*

Solution: The rate of change of the price is $\frac{dp(t)}{dt} = \lambda(D - S)$, $\lambda > 0$.

By substituting we get the differential equation: $\boxed{\frac{dp(t)}{dt} - \lambda(b - \delta)p(t) = \lambda(a - \gamma)}$

<u>Homogeneous part:</u> $\frac{dp(t)}{dt} - \lambda(b - \delta)p(t) = 0$

$\Rightarrow \frac{dp(t)}{p} = \lambda(b - \delta)dt \Rightarrow \int \frac{dp(t)}{p} = \lambda(b - \delta) \int dt \Rightarrow$

$\Rightarrow \ln p(t) = \lambda(b - \delta)t + K \Rightarrow p_h(t) = C \cdot e^{-\lambda(\delta - b)t}$

<u>Particular Solution:</u> We consider, for the time being, that C is a function of t.

$$\left(C \cdot e^{-\lambda(\delta - b)t}\right)' - \lambda(b - \delta)C \cdot e^{-\lambda(\delta - b)t} = \lambda(a - \gamma)$$

$\Rightarrow C' = \lambda(a - \gamma) \cdot e^{-\lambda(\delta - b)t} \Rightarrow C = \frac{a - \gamma}{\delta - b} e^{-\lambda(\delta - b)t}$

$\Rightarrow \boxed{p_P(t) = \frac{a - \gamma}{\delta - b}}$

$\Rightarrow p(t) = C \cdot e^{-\lambda(\delta - b)t} + \frac{a - \gamma}{\delta - b}$

Initial condition: $p(0) = p_0$

$$p_0 = C + \frac{a - \gamma}{\delta - b} \quad \Rightarrow \quad C = p_0 - \frac{a - \gamma}{\delta - b}$$

$$p(t) = \left[p_0 - \frac{a - \gamma}{\delta - b}\right] e^{-\lambda(\delta - b)t} + \frac{a - \gamma}{\delta - b}$$

$$p(\infty) = \frac{a-\gamma}{\delta - b}$$

1.7 Bernoulli's Differential Equation

$$\boxed{a(x)y'(x) + b(x)y(x) = \gamma(x)y^{\rho}(x)}$$

Solution:

$$\boxed{z(x) = y^{1-\rho}(x)}$$

1.9 Exercise: *Solve the equation:* $y'(x) + y(x) = y^2(x)$

Solution: We set $z(x) = y^{1-2}(x) \Rightarrow z(x) = y^{-1}(x) = \dfrac{1}{y(x)} \Rightarrow \boxed{y(x) = \frac{1}{z(x)}}$.

$$\left(\frac{1}{z(x)}\right)' + \frac{1}{z(x)} = \frac{1}{z^2(x)} \Rightarrow \frac{1' \cdot z - z' \cdot 1}{z^2} + \frac{1}{z} = \frac{1}{z^2} \Rightarrow \boxed{-z' + z = 1}$$

which is a solvable linear differential equation of first order

$$\boxed{z(x) = 1 + C \cdot e^x} \Rightarrow y(x) = \frac{1}{1 + C \cdot e^x}$$

1.10 The Solow Model *We consider the production function* $Q = K^a L^b$, $a + b = 1$ *and the rates of change:* $\dfrac{dK}{dt} = \dot{K} = sQ$, $\dfrac{dL}{dt} = \dot{L} = nL$, $1 > s, n > 0$. *Study the time evolution of the quantity* $k = K/L$.

Solution: First we form a differential equation.

$$\dot{k} = \left(\frac{\dot{K}}{L}\right) = \frac{\dot{K}L - K\dot{L}}{L^2} = \frac{sQL - KnL}{L^2} = \frac{sK^aL^{1-a}L - KnL}{L^2} = \frac{sK^aL^{1-a}L}{L^2} - \frac{nKL}{L^2}$$

$$= s\frac{K^a}{L^a} - n\frac{K}{L} \Rightarrow \boxed{\dot{k} + nk = sk^a}$$

This is a Bernoulli's equation. We set $\boxed{z = k^{1-a}}$ and we get $\boxed{\dot{z} + (1-a)nz = s(1-a)}$

The final solution is:

$$k = \sqrt[1-a]{\left[k_0^{1-a} - \frac{s}{n}\right] e^{-(1-a)nt} + \frac{s}{n}} \quad , \quad k(\infty) = \sqrt[1-a]{\frac{s}{n}}$$

Chapter 2

Linear Differential Equations of Higher Order with Constant Coefficients

2.1 General

$$\boxed{a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = g(x)}$$

Example: $5y'''' + 6y''' - 7y'' - 8y' + 9y = 5x^2 + 6$

Solution

$$y(x) = y_h(x) + y_p(x)$$

2.2 The Homogeneous Solution

We form the **characteristic** algebraic equation:

$$\boxed{a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda_1 + a_0 = 0}$$

and we solve it.

Case I: If the roots are different real numbers, $r_1, r_2, \ldots, r_n$, then:

$$y_h(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

Case II: If the roots are equal real numbers, $r_1 = r_2 = \cdots = r_n = r$, then:

$$y_h(x) = c_1 e^{rx} + c_2 x e^{rx} + c_3 x^2 e^{rx} + \cdots + c_n x^{n-1} e^{rx}$$

Case III: If two roots are complex numbers, $r_1 = a + bi, r_2 = a - bi$, then:

$$y_h(x) = \cdots + e^{ax}(c_1 \sin(bx) + c_2 \cos(bx)) + \cdots$$

Case IV: If we have many roots of different kinds then $y_h(x)$ consists from different sums of the the above type.

Examples

roots: 1,2,-3,-4,-7, $\sqrt{2}$

$$y_h(x) = c_1 e^{x} + c_2 e^{2x} + c_3 e^{-3x} + c_4 e^{-4x} + c_5 e^{-7x} + c_6 e^{\sqrt{2}x}$$

roots: 1,1,1,1,1

$$y_h(x) = c_1 e^{x} + c_2 x e^{x} + c_3 x^2 e^{x} + c_4 x^3 e^{x} + c_5 x^4 e^{x}$$

roots: 2,3,4,4,6+2i,6-2i

$$y_h(x) = c_1 e^{2x} + c_2 e^{3x} + c_3 e^{4x} + c_4 x e^{4x} + e^{6x}(c_5 \sin(2x) + c_6 \cos(2x))$$

2.1 Exercise: *The risk, accordingly to Arrow-Pratt is given by the relation $r(x) = -u''(x)/u'(x)$, where $u(x)$ a utility function. Find $u(x)$ which gives $r(x) = 1$.*

Solution:

$$-\frac{u''}{u'} = 1 \Rightarrow u'' = -u' \Rightarrow u'' + u' = 0$$

The characteristic equation is: $\lambda^2 + \lambda = 0$ with roots $\lambda = 0$ and $\lambda = -1$.

So, the solution is:

$$\boxed{u(x) = c_1 e^{0x} + c_2 e^{-x} = c_1 + c_2 e^{-x}}$$

2.3 The Particular Solution

There are two methods

- The Lagrange's method
- The method of the undetermined coefficients

THE METHOD OF THE UNDETERMINED COEFFICIENTS

We follow the next table:

$g(x)$	Particular solution
An n-degree polynomial $p_n(x)$	$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$
$e^{ax}p_n(x)$	$e^{ax}(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0)$
$e^{ax}p_n(x)\sin(bx)$ or $e^{ax}p_n(x)\cos(bx)$	$e^{ax}(A_n x^n + \cdots + A_0)\sin(bx)+$ $+e^{ax}(B_n x^n + \cdots + B_0)\cos(bx)$

2.2 Exercise: *Solve the ODE:* $y''''(x) = 5x$.

Solution: The homogeneous solution

$$\lambda^4 = 0 \Rightarrow \lambda = 0, 0, 0, 0 \Rightarrow$$

$$\Rightarrow y_h(x) = c_1 e^{0x} + c_2 x e^{0x} + c_3 x^2 e^{0x} + c_4 x^3 e^{0x} \Rightarrow$$

$$\Rightarrow y_h(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

The particular solution

We set $y_p(x) = Ax + B$

$\Rightarrow (Ax + B)'''' = 5x \Rightarrow 0 = 5x$ a contradiction

We set $y_p(x) = Ax^2 + Bx + C$

$\Rightarrow (Ax^2 + Bx + C)'''' = 5x \Rightarrow 0 = 5x$ a contradiction

Finally, we set $y_p(x) = Ax^5 + Bx^4 + Cx^3 + Dx^2 + E$

$\Rightarrow (Ax^5 + Bx^4 + \cdots)'''' = 5x \Rightarrow 120Ax + B = 5x \Rightarrow A = 1/24, B = 0$

$y_p(x) = \frac{1}{24}x^5 + 0x^4 + Cx^3 + Dx^2 + E$ and finally:

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{1}{24}x^5$$

2.3 Exercise: *Solve the ODE:* $y''(x) + 9y(x) = \sin 7x$.

Solution: The homogeneous solution

$$\lambda^2 + 9 = 0 \Rightarrow \lambda_1 = 3i, \lambda_2 = -3i \Rightarrow$$

$$\Rightarrow y_h(x) = e^{0\cdot x}(c_1 \sin 3x + c_2 \cos 3x)$$

$$\Rightarrow y_h(x) = c_1 \sin 3x + c_2 \cos 3x$$

The particular solution

We set $y_p(x) = A_0 \sin 7x + B_0 \cos 7x$

$\Rightarrow y_p'(x) = A_0 \cdot 7 \cdot \cos 7x - B_0 \cdot 7 \cdot \sin 7x$

$\Rightarrow y_p''(x) = -A_0 \cdot 49 \cdot \sin 7x - B_0 \cdot 49 \cdot \cos 7x$

By substitution we get:

$$-49A_0 \sin 7x - 49B_0 \cos 7x + 9A_0 \sin 7x + 9B_0 \cos 7x = \sin 7x$$

$$\left.\begin{array}{r} -49A_0 + 9A_0 = 1 \\ -49B_0 + 9B_0 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{c} A_0 = -\frac{1}{40} \\ B_0 = 0 \end{array}\right\} \Rightarrow y_p(x) = -\frac{1}{40}\sin 7x$$

and finally:

$$\boxed{y(x) = c_1 \sin 3x + c_2 \cos 3x - \frac{1}{40}\sin 7x}$$

Chapter 3

Calculus of Variations

3.1 The Euler Equation

$$\max \quad or \quad \min V[y(t)] = \int_0^T F[t, y(t), y'(t)]dt$$

$$\text{when} \quad y(0) = A \quad , \quad y(T) = Z$$

$A, T, Z \in \mathbf{R}$ given

Theorem. (The First Order Conditions) The stationary "points", $y^*(t)$ satisfy the *Euler* equation:

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right) = 0}$$

Theorem. (The Second Order Conditions) If the matrix:

$$\boxed{\begin{pmatrix} F_{y'y'} & F_{y'y} \\ F_{yy'} & F_{yy} \end{pmatrix}_{y(t)=y^*(t)}}$$

is positive semidefinite then $y^*(t)$ is a local minimum. If it is negative semidefinite then $y^*(t)$ is a local maximum.

3.1 Exercise: *Solve the problem:*

$$\min \quad \int_0^2 [12ty(t) + (y'(t))^2]dt$$

when $y(0) = 0$ *and* $y(2) = 8$.

Solution: First Order Conditions.
We set $F(t, y, y') = 12ty + (y')^2$.
The Euler equation is:

$$\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right) = 012t - \frac{d}{dt}(2y') = 0 \Rightarrow 12t - 2y'' = 0 \Rightarrow$$

$\Rightarrow \quad \boxed{y''(t) = 6t}$

Solution of the ODE.

The homogeneous part, $\quad y'' = 0 \Rightarrow \hat{\pi}^2 = 0 \Rightarrow \hat{\pi}_1 = \hat{\pi}_2 = 0$
$\Rightarrow \quad y_h(t) = c_1 e^{0t} + c_2 t e^{0t} = c_1 + c_2 t$
The particular solution, $y_p(t) = At^3 \Rightarrow A = 1 \Rightarrow y_p(t) = t^3$

$\Rightarrow y^*(t) = c_1 + c_2 t + t^3, \quad y(0) = 0, y(2) = 8 \Rightarrow \boxed{y^*(t) = t^3}$

Second Order Conditions.

$$\begin{pmatrix} F_{y'y'} & F_{y'y} \\ F_{yy'} & F_{yy} \end{pmatrix}_{y(t)=y^*(t)} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

the eigenvalues of this matrix are 2 and 0 and thus the matrix is positive semidefinite. Hence, the final solution is

$$\boxed{y^*(t) = t^3}$$

$y(t)$	$V(y(t))$
$4t$	160
$8\frac{\sin t}{\sin 2}$	246.63
t^3	134.4
$\sqrt{\sqrt{2060.8t + 0.04} - 0.2}$	$+\infty$

3.2 Exercise: *Solve the problem:*

$$\min \quad or \quad \max \int_1^e (3x' - tx'^2)dt$$

when $x(1) = 0$ *and* $x(e) = 1$.

Solution: First Order Conditions.
We set $F(t, x, x') = 3x' - tx'^2$.
The Euler equation is:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right) = 0 \Rightarrow \frac{d}{dt}(3 - 2tx') = 0 \Rightarrow 3 - 2tx' = c_0$$

Solution of the ODE.

Seperable variables, $x' = \frac{c_1}{t}$ and integrating we get:
$\Rightarrow \quad x(t) = c_1 \ln t + c_2$

The constants of integration satisfy the pair of equations:

$$0 = c_1 \ln 1 + c_2 \quad , \quad 1 = c_1 \ln e + c_2$$

$$\Rightarrow \quad c_1 = 1 \quad , c_2 = 0$$

$$\boxed{x^*(t) = \ln t}$$

Second Order Conditions.

$$\begin{pmatrix} F_{x'x'} & F_{x'x} \\ F_{xx'} & F_{xx} \end{pmatrix}_{x(t)=x^*(t)} = \begin{pmatrix} -2t & 0 \\ 0 & 0 \end{pmatrix}$$

The eigenvalues of this matrix are $-2t$ and 0 and since $1 < t < e$ the matrix is negative semidefinite. Hence, we have maximum.

3.3 Exercise: *A firm has received an order for* B *units of product to be delivered by time* T.

- *The production rate is equal to the rate of change of inventory.*
- *The unit production cost is proportional to the production rate.*

- *The unit cost of holding inventory is constant.*

We are seeking a production schedule for filling this order at minimum cost.

Solution: By $x(t)$ we denote the inventory.

Total Cost=Production Cost + Holding Cost

Production rate=$x'(t)$

Unit production cost=$c_1 x'(t)$,
Unit holding cost=$c_2 x(t)$
$c_1, c_2 \geq 0$
The total cost at any moment t=$c_1[x'(t)]^2 + c_2 x(t)$

$$\text{Total Cost} = \int_0^T [c_1[x'(t)]^2 + c_2 x(t)]dt$$

Furthermore, $x(0) = 0$ and $x(T) = B$.

So, we have the problem:

$$\boxed{\begin{array}{ll} \min & \displaystyle\int_0^T [c_1[x'(t)]^2 + c_2 x(t)]dt \\ s.t. & x(0) = 0 \quad , \quad x(T) = B \end{array}}$$

First Order Conditions.
We set $F(t, x, x') = c_1(x')^2 + c_2 x$.
The Euler equation is:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right) = 0$$

$$\Rightarrow c_2 - \frac{d}{dt}(2c_1 x') = 0 \Rightarrow$$

$$\Rightarrow \qquad \boxed{2c_1 x'' = c_2}$$

$$\Rightarrow \qquad x(t) = \frac{c_2}{4c_1}t^2 + k_1 t + k_2$$

Using the initial conditions: $x(0) = 0, x(T) = B$, we get:

$$\boxed{x^*(t) = \frac{c_2}{4c_1}t(t-T) + \frac{B}{T}t, \quad 0 \le t \le T}$$

Second Order Conditions.

$$\begin{pmatrix} F_{x'x'} & F_{x'x} \\ F_{xx'} & F_{xx} \end{pmatrix}_{x(t)=x^*(t)} = \begin{pmatrix} 2c_1 & 0 \\ 0 & 0 \end{pmatrix}$$

The eigenvalues of this matrix are $2c_1$ and 0 and since $c_1 \ge 0$ the matrix is positive semidefinite.

Hence, we have minimum.

3.2 The Multidimensional Case

$\vec{x}(t)$ a vector of n state variables

The Euler Equation:

$$\boxed{\nabla F - \frac{d}{dt}\nabla_{\vec{x}'}F = 0}$$

3.4 Exercise: *Find the stationary point for:*

$$\int_0^{\frac{\pi}{2}} [(x')^2 + (y')^2 + 2xy]dt$$

$$x(0) = 0, \quad x(\pi/2) = 1$$

$$y(0) = 0, \quad y(\pi/2) = -1$$

Solution: Successively we have

$$\nabla F = (2y, 2x), \quad \nabla_{\vec{\dot{x}}} F = (2x', 2y') \Rightarrow$$

$$\Rightarrow \frac{d}{dt}\nabla_{\vec{\dot{x}}} F = (2x'', 2y'') \Rightarrow$$

$$\nabla F - \frac{d}{dt}\nabla_{\vec{\dot{x}}} F = 0 \Leftrightarrow 2y - 2x'' = 0, \quad 2x - 2y'' = 0$$

$$\Rightarrow x'''' - x = 0 \Rightarrow s^4 - 1 = 0 \Rightarrow s = 1, -1, i, -i$$

$$x(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$

$$y(t) = x''(t) = c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t$$

The boundary conditions will give: $c_1 = c_2 = c_3 = 0$, $c_4 = 1$, $\Rightarrow$

$$x^*(t) = \sin t \quad , \quad y^*(t) = -\sin t$$

3.5 Exercise: *(Ramsey, F.P., 1928. A mathematical theory of saving, Economic Journal, pp 543-549. (A simplified version)*

Consider an one-sector economy in which a single homogeneous good is produced by two factors of production, labor and capital.

$L(t)$ and $K(t)$ be the states of labor and capital over time.

The next relations hold:

- *Consumption,* $C(t) = Q(L(t), K(t)) - S(t) = Q(L, K) - S$
- *Savings,* $S(t) = \dfrac{dK}{dt}$
- *Utility function,* $U = U(C(t))$
- *Disutility of work,* $V = V(L(t))$

Study the problem.

Total Utility:

$$J = \int_{t_0}^{t_1} [U(C(t)) - V(L(t))]dt =$$

$$= \int_{t_0}^{t_1} [U(Q(L,K) - \dot{K}) - V(L)]dt = \int_{t_0}^{t_1} I(L,K,\dot{K})dt$$

Euler's equations:

$$\nabla I - \frac{d}{dt}\nabla_{\dot{\vec{x}}} I = 0 \Rightarrow \begin{pmatrix} \partial I/\partial L \\ \partial I/\partial K \end{pmatrix} - \frac{d}{dt}\begin{pmatrix} \partial I/\partial \dot{L} \\ \partial I/\partial \dot{K} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\frac{\partial I}{\partial L} = 0} \text{ and } \boxed{\frac{\partial I}{\partial K} = \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{K}}\right)}$$

Calculations:

$$\frac{\partial I}{\partial L} = \frac{\partial U}{\partial C}\frac{\partial C}{\partial L} - \frac{\partial V}{\partial L} = \frac{\partial U}{\partial C}\frac{\partial Q}{\partial L} - \frac{\partial V}{\partial L}$$

$$\frac{\partial I}{\partial K} = \frac{\partial U}{\partial C}\frac{\partial C}{\partial K} = \frac{\partial U}{\partial C}\frac{\partial Q}{\partial K}$$

$$\frac{\partial I}{\partial \dot{K}} = \frac{\partial U}{\partial C}\frac{\partial C}{\partial \dot{K}} = \frac{\partial U}{\partial C}(-1)$$

$$\Rightarrow \frac{\partial U}{\partial C}\frac{\partial Q}{\partial L} - \frac{\partial V}{\partial L} = 0 \quad and \quad \frac{\partial U}{\partial C}\frac{\partial Q}{\partial K} = \frac{d}{dt}\left(\frac{\partial U}{\partial C}(-1)\right)$$

Solving, we get:

$$\boxed{\frac{\partial Q}{\partial L} = \frac{\frac{\partial V}{\partial L}}{\frac{\partial U}{\partial C}}} \quad , \quad \boxed{\frac{\partial Q}{\partial K} = -\frac{\frac{d}{dt}\left(\frac{\partial U}{\partial C}\right)}{\frac{\partial U}{\partial C}}}$$

Rate of wages= (marginal disutility of labor)/(marginal utility of consumption)

Rate of interest= (-the rate of change of the marginal utility of consumption)/(marginal utility of consumption)

Chapter 4

Systems of Linear Differential Equations

4.1 The Description

$$\dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t)$$

$$\dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t)$$
$$\dot{x}_3(t) = a_{31}x_1(t) + a_{32}x_2(t) + \cdots + a_{3n}x_n(t)$$
$$\vdots$$
$$\dot{x}_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t)$$

or shortly

$$\vec{x}'(t) = A\vec{x}(t)$$

A the coefficients matrix.

$\vec{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ the **state** vector.

Examples:

$$\begin{array}{l} x'(t) = 5x(t) - 7y(t) \\ y'(t) = 3x(t) + 2y(t) \end{array} \text{ or } \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} 5 & -7 \\ 3 & 2 \end{pmatrix}$$

4.2 The Solution Method

We find the eigenvalues of A: $r_1, r_2, \ldots, r_n$ and the corresponding eigenvectors: $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$.

We have the next cases:

Case I: If $r_1 \neq r_2 \neq \cdots \neq r_n \in \mathbf{R}$, then:

$$\boxed{\vec{x}(t) = c_1 e^{r_1 t}\vec{w}_1 + c_2 e^{r_2 t}\vec{w}_2 + \cdots + c_n e^{r_n t}\vec{w}_n}$$

Case II: If for some eigenvalues $\cdots r_1 = r_2 = \cdots = r_n = r \cdots$, then:

$$\boxed{\vec{x}(t) = \cdots + c_1 e^{rt}\vec{w}_1 + c_2 t e^{rt}\vec{w}_2 + \cdots + c_n t^{n-1} e^{rt}\vec{w}_n + \cdots}$$

Case III: Assume that A has a pair of complex eigenvalues $a \pm ib$ and corresponding eigenvalues $\vec{u} \pm i\vec{w}$. These eigenvalues correspond to the part of the solution....

$$\boxed{\vec{x}(t) = \cdots + e^{at}cos(bt)(c_1\vec{u} + c_2\vec{w}) - e^{at}sin(bt)(c_2\vec{u} + c_1\vec{w}) + \cdots}$$

Conclusions

- We have closed forms of solutions. (Complete Theory)
- Negative real parts of the eigenvalues imply absolute stability.
- The above theory does not work in practise.

4.1 Exercise: *Solve the system of ODE:* $x'(t) = -y(t)$, $y'(t) = x(t)$.

Solution: The matrix expression of the system is:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We have to find the eigenvalues and the eigenvectors of the matrix.

Eigenvalues. $|A - \lambda I| = 0 \Rightarrow$

$$\left|\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0 \Rightarrow$$

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda_1 = i, \quad \lambda_2 = -i$$

Eigenvectors , $\lambda = i$

$$(A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\begin{matrix} -ix - y = 0 \\ x - iy = 0 \end{matrix} \Rightarrow x = iy \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} iy \\ y \end{pmatrix} = y\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$\begin{pmatrix} i \\ 1 \end{pmatrix}$ is the so-called basic eigenvector. Furthermore,

$$\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{u} + i\vec{w}$$

So, Finally:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{0\cdot t}cos(1\cdot t)\left(c_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} - c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) -$$

$$-e^{0\cdot t}sin(1\cdot t)\left(c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

or

$$x(t) = k_1 sint + k_2 cost$$
$$y(t) = k_1^* sint + k_2^* cost$$

4.2 Exercise: *Solve, as a system, the ODE:*

$$ay''(x) + by'(x) + cy(x) = 0$$

Solution: We transform the ODE to a system of differential equations.

We define $\boxed{z = y'}$ and we have:

$$z' = y'' = -\frac{b}{a}y' - \frac{c}{a}y$$

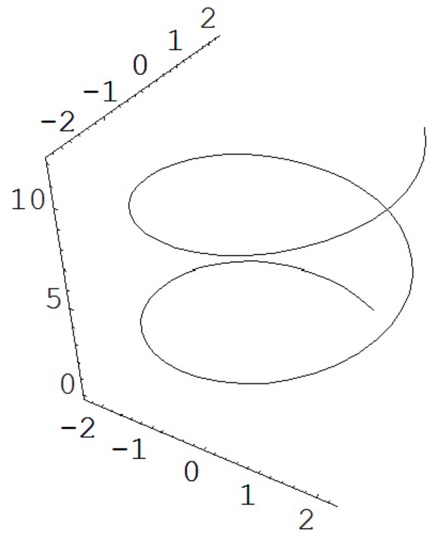


Figure 4.1: An orbit

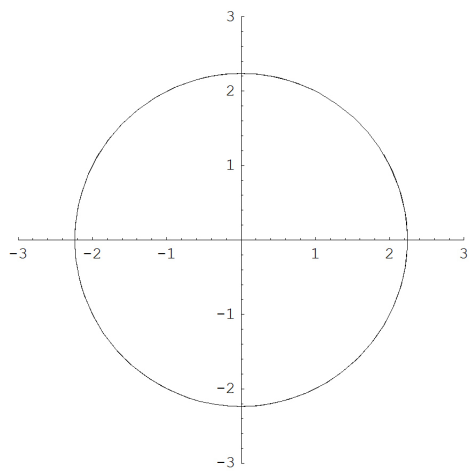


Figure 4.2: A trajectory of the phase space

We construct the system:

$$\begin{pmatrix} z' \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{b}{a}y' - \frac{c}{a}y \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{b}{a}z - \frac{c}{a}y \\ z \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} z' \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{b}{a} & -\frac{c}{a} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z \\ y \end{pmatrix}$$

To solve the system we have to find the eigenvalues of A.
The eigenvalues are the roots of the equation:

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\frac{b}{a} - \lambda & -\frac{c}{a} \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \frac{b\lambda}{a} + \lambda^2 + \frac{c}{a} = 0 \Rightarrow$$

$\Rightarrow a\lambda^2 + b\lambda + c = 0$

This is the **characteristic** equation, appeared in the solution of the linear differential equations.

4.3 Exercise: *Solve, the system:*

$$3x''(t) = 2y'(t) - y(t)$$

$$y''(t) = 2x'(t) + x(t)$$

Solution: This is not in the typical form.
To bring it in a typical form we use the relations $\boxed{w = x'}$, $\boxed{s = y'}$.

The original system becomes: $\begin{aligned} w' &= \frac{2}{3}s - \frac{1}{3}y \\ s' &= 2w + x \end{aligned}$

By introducing the new state vectot $(w, s, x, y)^T$ we get:

$$\begin{pmatrix} w' \\ s' \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ s \\ x \\ y \end{pmatrix}$$

The eigenvalues of this matrix are:

$$-1 \quad , \quad 1 \quad , \quad -\frac{\sqrt{3}}{3} \quad , \quad \frac{\sqrt{3}}{3}$$

The corresponding eigenvectors are:

$$\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1/3 \\ 1 \\ 1/3 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{2+\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ -\frac{3+2\sqrt{3}}{3} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{2-\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{3-2\sqrt{3}}{3} \\ 1 \end{pmatrix}$$

Thus, the final solution is:

$$\begin{pmatrix} w \\ s \\ x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1/3 \\ 1 \\ 1/3 \\ 1 \end{pmatrix} + c_3 e^{-\frac{\sqrt{3}}{3}} \begin{pmatrix} \frac{2+\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ -\frac{3+2\sqrt{3}}{3} \\ 1 \end{pmatrix} + c_4 e^{-\frac{\sqrt{3}}{3}} \begin{pmatrix} \frac{2-\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{3-2\sqrt{3}}{3} \\ 1 \end{pmatrix}$$

Chapter 5

General Theory of Differential Equations

Any differential equation can be written as:

$$\boxed{\textbf{x'=F(t,x)}}$$

$$\mathbf{F} : U \to \mathbf{R}^n, \qquad U \subseteq \mathbf{R} \times \mathbf{R}^n \quad \textit{open}$$

• $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ is the state vector.

• $\mathbf{R}$ is called the **time-space**.

• $\mathbf{R}^n$ is called the **phase-space**.

• For $n = 1$ we have a single or an ordinary differential equation.

• For $n \neq 1$ we have a system of differential equations.

• If $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ then we have an **autonomous** differential equation.

• **Solution** of a differential equation is a function $\mathbf{\Phi} : \mathbf{R} \to \mathbf{R}^n$, $\mathbf{\Phi}(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ such that $\mathbf{\Phi}'(t) = \mathbf{F}(t, \mathbf{\Phi}(t))$

• A point $(t_0, \mathbf{x}_0)$ is called an **initial condition** for the solution $\mathbf{\Phi}$ if $\mathbf{\Phi}(t_0) = \mathbf{x}_0$.

QUESTIONS	1)When do we have a solution? 2)How many solutions do we have? 3)How do we find the solutions?

EXISTENCE THEOREM

PEANO'S THEOREM. Let $\mathbf{x}' = \mathbf{F}(t, \mathbf{x})$ be a differential equation and $(t_0, \mathbf{x}_0)$ an initial condition. If $\mathbf{F} : U \to \mathbf{R}^n$ is continuous, then there is AT LEAST ONE solution $\mathbf{\Phi}(t)$ such that $\mathbf{\Phi}(t_0) = \mathbf{x}_0$.

UNIQUENESS THEOREM

Let $\mathbf{x}' = \mathbf{F}(t, \mathbf{x})$ be a differential equation and $(t_0, \mathbf{x}_0)$ an initial condition. If $\left\| \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \right\| \leq L < 1$, then there is ONE AND ONLY ONE solution $\mathbf{\Phi}(t)$, such that $\mathbf{\Phi}(t_0) = \mathbf{x}_0$.

5.1 Exercise: *Solve, the equation:* $x'(t) = \sqrt[3]{x(t)}$ *at* $(0, 0)$.

Solution: Successively we have:

$$x'(t) = \sqrt[3]{x(t)} \Rightarrow x'(t) = x^{1/3}(t) \Rightarrow \frac{dx}{dt} = x^{\frac{1}{3}} \Rightarrow$$

$$\frac{dx}{x^{\frac{1}{3}}} = dt \quad \text{with} \quad x \neq 0$$

$$\Rightarrow \int \frac{dx}{x^{\frac{1}{3}}} = \int dt \Rightarrow x(t) = \frac{2}{3}\sqrt{\frac{2}{3}}\sqrt[3]{(t+c)^2}$$

To calculate c we must use the initial condition $(0, 0)$. But $x \neq 0$, so, this condition cannot be used and c cannot be determined.

Hence, we have an infinite number of solutions passing through the point $(0, 0)$. IT IS A POINT WITHOUT UNIQUENESS.

Chapter 6

One Dimension Phase Portraits

Example: Let us have the equation: $y'(t) = y(t) - 2 \Rightarrow y(t) = ce^t + 2$
But , $y(0) = y_0$, and thus

$$\boxed{y(t) = (y_0 - 2)e^t + 2}$$

We are endowed with two kinds of information: QUANTITATIVE and QUALITATIVE

QUANTITATIVE INFORMATION

I can calculate $y(t)$ at any time instant for any initial condition y_0.

$$y(5) = (y_0 - 2)e^5 + 2$$

QUALITATIVE INFORMATION

I can calculate either constant solutions, or where the solution is increasing or decreasing.

CONSTANT SOLUTIONS

$$y(t) = k \Rightarrow (y_0 - 2)e^t + 2 = k \Rightarrow y_0 - 2 = 0$$

$$y_0 = 2 \Rightarrow y(t) = 2$$

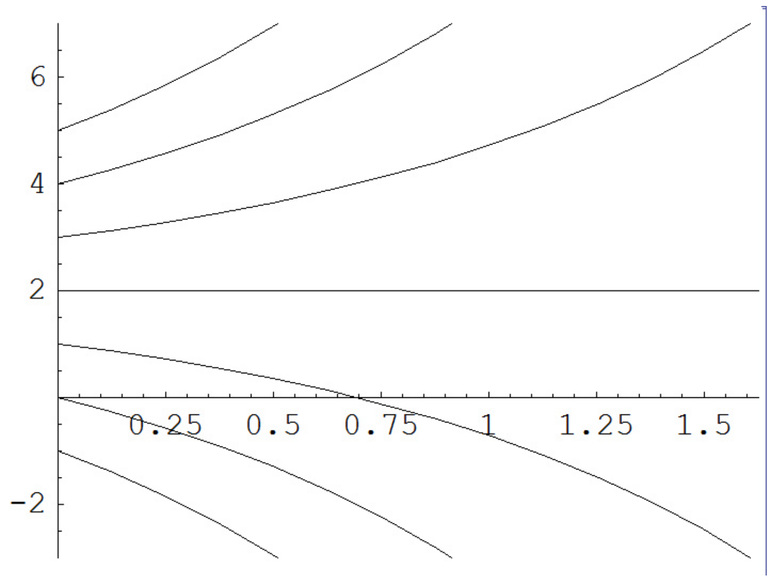


Figure 6.1: Trajectories

INCREASING OR DECREASING SOLUTIONS

Let $t_1 > t_2$. If $y_0 - 2 > 0 \Leftrightarrow y_0 > 2$ then:

$$(y_0 - 2)e^{t_1} + 2 > (y_0 - 2)e^{t_2} + 2 \Rightarrow y(t_1) > y(t_2)$$

INCREASING

If $y_0 - 2 < 0 \Leftrightarrow y_0 < 2$ then:

$$(y_0 - 2)e^{t_1} + 2 < (y_0 - 2)e^{t_2} + 2 \Rightarrow y(t_1) < y(t_2)$$

DECREASING

$y_0 = 2$	CONSTANT SOLUTION, $y(t) = 2$
$y_0 > 2$	INCREASING SOLUTION
$y_0 < 2$	DECREASING SOLUTION

CLAIM: I can get the qualitative information **WITHOUT SOLVING THE EQUATION**

CONSTANT SOLUTION: $y'(t) = 0 \Rightarrow y(t) - 2 = 0 \Rightarrow y_0 = 2$

INCREASING SOLUTION: $y'(t) > 0 \Rightarrow y(t) - 2 > 0 \Rightarrow y_0 > 2$

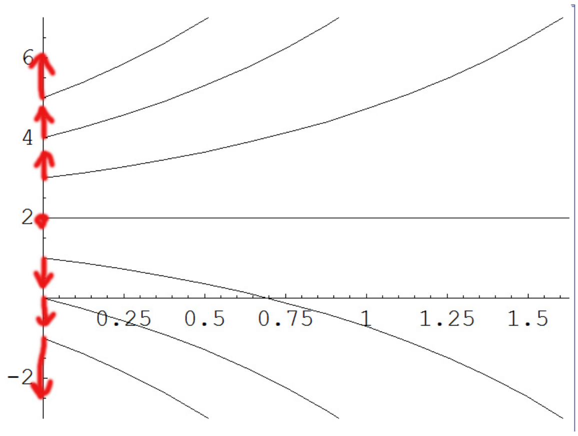


Figure 6.2: Phase Space

Figure 6.3: Classical Phase Space

DECREASING SOLUTION: $y'(t) < 0 \Rightarrow y(t) - 2 < 0 \Rightarrow y_0 < 2$

6.1 One Dimension Phase Portrait Terminology

$$x'(t) = f(t, x(t)) \quad , \quad f : I \times \Phi \subseteq \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

$$\boxed{\Phi \text{ The Phase Space}}$$

Phase Space = The projection of the trajectories onto the y-axis

BASIC DEFINITIONS

The projections of the trajectories are called **orbits**. We denote them by $\varphi(t)$.

Let us have the differential equation $x' = f(x)$. A point of the phase space, $x_0 \in \Phi$ is called **equilibrium point** if and only if $f(x_0) = 0$

An equilibrium point x_0 is called **stable**, if for each neighbourhood U of x_0, we can find a neighbourhood $V \subset U$, such that any orbit starting into V remains into U.

An equilibrium point x_0 is called **unstable** if it is not stable.

An equilibrium point x_0 is called **asymptotical stable** if it is stable and $\lim_{t\to+\infty} x(t) = x_0$.

We call **basin of attraction** or **region of attraction** of an asymptotical stable equilibrium point x_0, all the points which are attracted by x_0.

An unstable equilibrium point x_0 is called **unstable repeller** if it is asymptotical stable for the equation $x' = -f(x)$.

An unstable equilibrium point x_0 is called **shunt** if any orbit at his lefts, approahes it and any orbit, at his rights, goes away from it and vice versa.

An orbit $\varphi(t)$ is called **periodic** if we can find a time instant T such that: $\varphi(t + T) = \varphi(t)$.

6.2 Theorems

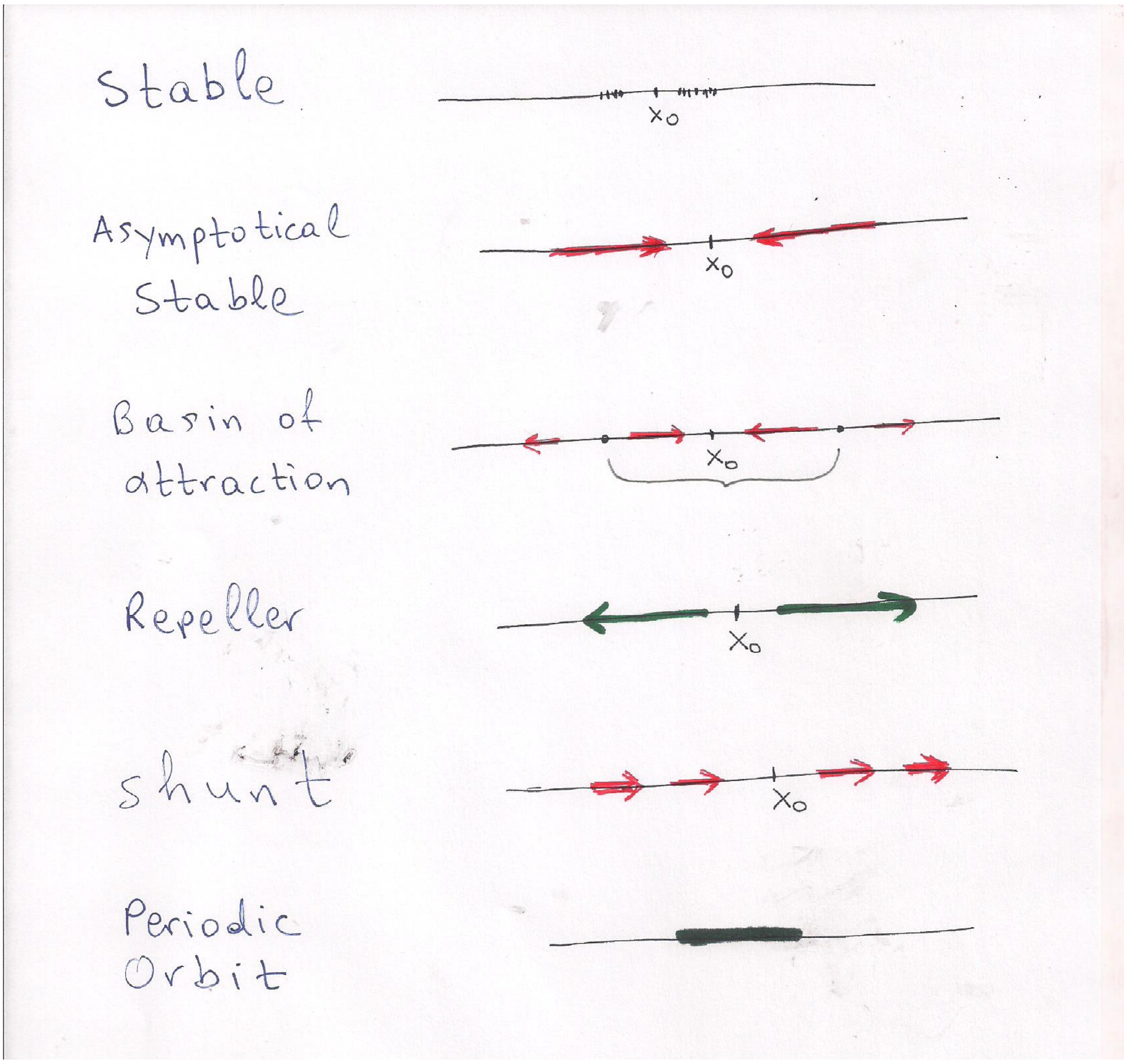


Figure 6.4: Different Equilibrium Points

Theorem 6.1 The Sign Criterion. *Let x_0 be an equilibrium point of $x' = f(x)$. Let (a, x_0) and (x_0, b) a "left" and a "right" interval. The next table is valid:*

The sign of f at (a, x_0)	*The sign of f at (x_0, b)*	*RESULT*
$+$	$-$	*x_0 is as. stable*
$-$	$+$	*x_0 is repeller*
$-$	$-$	*x_0 is shunt*
$+$	$+$	*x_0 is shunt*

Theorem 6.2 The Derivative Criterion. *Let x_0 be an equilibrium point of $x' = f(x)$.*

- *If $f'(x_0) < 0$ then x_0 is asymptotical stable.*
- *If $f'(x_0) > 0$ then x_0 is unstable.*

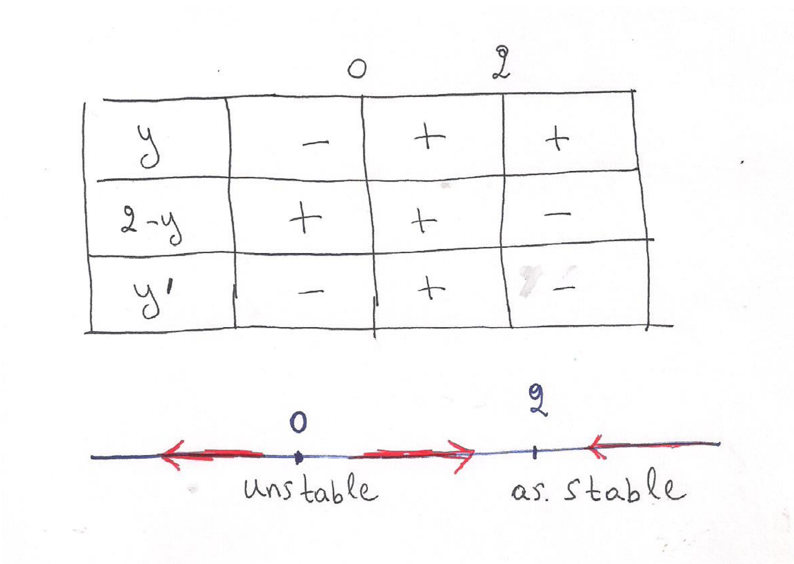
How do we draw an one dimensional Phase Portrait?

We have the differential equation $y' = f(y)$

1. We find the equilibrium points, by solving the equation $f(y) = 0$.
2. If we have an analytic expression for f, we determine the sign of f.
3. If we have a "theoretical" expression for f we use the "First Derivative Criterion"
4. We draw the phase portrait.

6.1 Exercise: *Draw the phase portrait of the equation: $y' = y(2 - y)$.*

Solution: Equilibrium Points: $y(2 - y) = 0 \Rightarrow y = 0, y = 2$. For their stability analysis see the figure below.



Basin of Attraction: $(0, +\infty)$

6.2 Exercise: *Draw the phase portrait of the equation:* $y' = e^y(y - y^3)$.

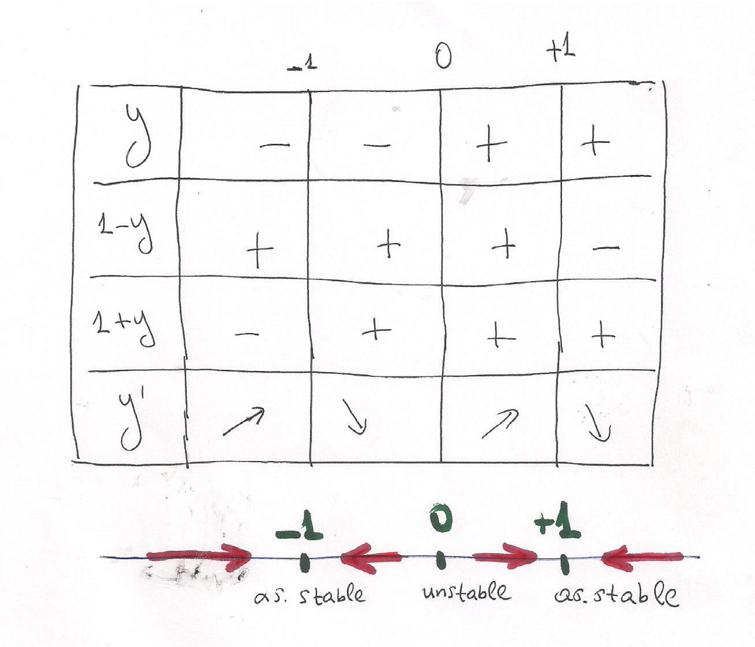
(This ODE is unsolvable)

Solution: Equilibrium Points:

$$e^y(y - y^3) = 0 \Rightarrow y - y^3 = 0 \Rightarrow y(y-1)(y+1) = 0$$

$$\Rightarrow y = 0 \quad , \quad y = 1 \quad , \quad y = -1$$

Table of signs, as in the figure below.



Basins of Attraction: $(-\infty, 0)$ to -1 and $(0, +\infty)$ to 1.

6.3 Exercise: *Find the equilibrium points of the equation* $y' = \dfrac{y^2-1}{y^2+1}$ *and characterize them.*

Solution: Equilibrium Points:

$$\frac{y^2-1}{y^2+1} = 0 \Rightarrow y^2 - 1 = 0 \Rightarrow y_1 = 1, \quad y_2 = -1$$

$$f'(y) = \left(\frac{y^2-1}{y^2+1}\right)' = \frac{4y}{(y^2+1)^2} \Rightarrow f'(y_1) = 1 > 0, f'(y_2) = -1 < 0$$

$\Rightarrow$, $y_1 = 1$, unstable, $y_2 = -1$, asymptotical stable.

6.4 Exercise: *Find the equilibrium points of the equation* $ay' + by = c$ *and characterize them.*

Solution: The equation is written: $y' = -(b/a)y + c/a$. Equilibrium point:

$$-\frac{b}{a}y + \frac{c}{a} = 0 \Rightarrow y^* = \frac{c}{b}$$

REMARK The particular solution is $y_p = A \Rightarrow a(A)' + bA = c \Rightarrow y_p = c/b = y^*$ (equal to the eq. point)

Stability

$$\left(-\frac{b}{a}y + \frac{c}{a}\right)' = -\frac{b}{a} \Rightarrow \begin{cases} b/a > 0 & \text{as. stability} \\ b/a < 0 & \text{instability} \end{cases}$$

6.5 Exercise: *The price of a good depends from the time, $p(t)$. The demand function is $D(t) = a + bp(t), b < 0$. The supply function is $S(t) = \gamma + \delta p(t)$, $\delta > 0$. Considering that the rate of change of the price is proportional of the current available quantity of the good, find the equilibrium point and study its stability.*

Solution: The rate of change of the price is $\dfrac{dp(t)}{dt} = \lambda(D - S)$, $\lambda > 0$.

By substituting we get the differential equation: $\boxed{\dfrac{dp(t)}{dt} - \lambda(b - \delta)p(t) = \lambda(a - \gamma)}$

$\Leftrightarrow \boxed{p' = \lambda(b - \delta)p + \lambda(a - \gamma)}$

Equilibrium point: $\lambda(b - \delta)p + \lambda(a - \gamma) = 0 \Rightarrow$

$$\boxed{p^* = \frac{\gamma - a}{b - \delta}}$$

Stability of the equilibrium point

$$(\lambda(b - \delta)p + \lambda(a - \gamma))' = \lambda(b - \delta) < 0 \quad \Rightarrow$$

$\Rightarrow p^*$ asymptotical stable

Remark: By solving the equation we get the information of how much exactly the price is, at any time instant. By using phase space analysis we do not know the value of the price but its limit behaviour. This is achieved WITHOUT solving the equation.

6.6 Exercise: *Find the equilibrium points of the equation $y' + \lambda y = sy^a$ and characterize them.*

Solution: We rewrite the equation as follows: $y' = sy^a - \lambda y$.

Equilibrium Points.

$sy^a - \lambda y = 0 \Rightarrow$

$\boxed{y^* = 0}$, $\boxed{y^* = \sqrt[a-1]{\frac{\eta}{s}}}$

Stability We shall use the Derivative Criterion $f' = say^{a-1} - \eta$

$$f'(0) = -\eta \Rightarrow \begin{cases} \eta > 0 & \text{as. stability} \\ \eta < 0 & \text{instability} \end{cases}$$

We work now with the other equilibrium point:

$$f'(y^*) = sa\left(\sqrt[a-1]{\frac{\eta}{s}}\right)^{a-1} - \eta = a\eta - \eta \Rightarrow$$

$$\Rightarrow \begin{cases} \eta < 0, a < 1 & \text{instability} \\ \eta > 0, a < 1 & \text{as. stability} \\ \eta > 0, a > 1 & \text{instability} \end{cases}$$

6.7 The Solow Model *We consider the production function* $Q = K^a L^b$, $a + b = 1$ *and the rates of change:* $\frac{dK}{dt} = \dot{K} = sQ$, $\frac{dL}{dt} = \dot{L} = \eta L$, $1 > s, \eta > 0$. *Characterize the equilibrium point.*

Solution: We form, as in previous exercise, the equation: $\dot{k} + \eta k = sk^a$.

Equilibrium Point: $sk^a - \eta k = 0 \Rightarrow k^* = \sqrt[a-1]{\frac{\eta}{s}}$

Stability: Since $\eta > 0$ and $0 < a < 1$, we have asymptotic stability (last exercise).

Chapter 7

Two Dimensions Phase Spaces

7.1 Definitions

Definition. Let us have the differential equation: $\vec{x}' = F(t, \vec{x})$ with $F : I \times \Phi \subset \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$. The set Φ is called the Phase Space.

Equivalently, We have the system of ODE:

$$x'(t) = f(x(t), y(t))$$

$$y'(t) = g(x(t), y(t))$$

Phase Space= THE PROJECTION of the trajectories (solutions) $x(t)$, $y(t)$ to the plane xy. $\Leftrightarrow$ We eliminate the t-parameter, the time. The projections are called ORBITS.

EXAMPLE: We have the system: $x' = 0.02x - y$, $y' = 4x - 0.03y$. The trajectories-solutions space is:

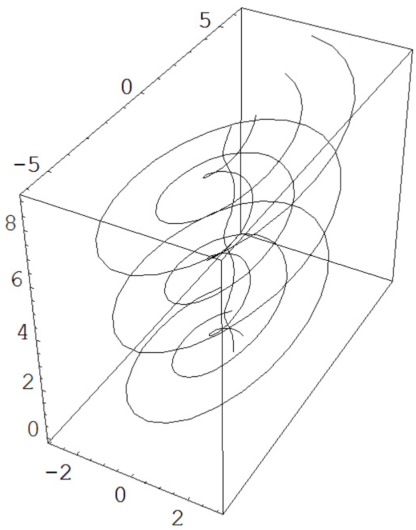


Figure 7.1: The Trajectories

The Phase space (the projection of the trajectories) is:

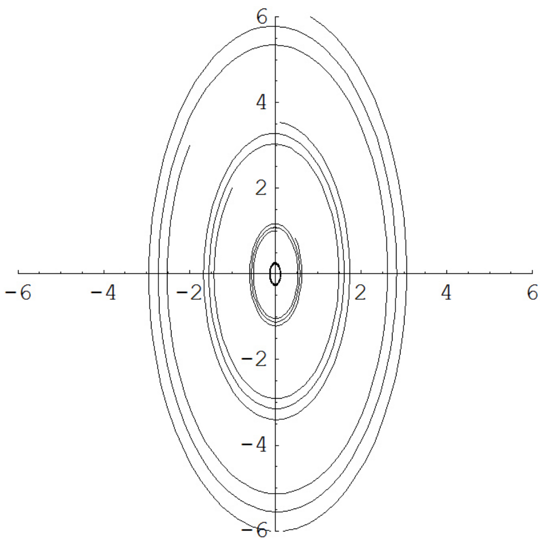


Figure 7.2: The Phase Space

Both of them together:

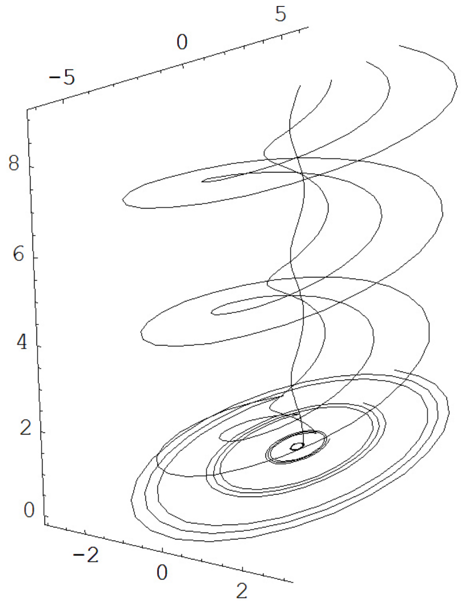


Figure 7.3: Trajectories-Orbits

Remarks:

1. Trajectories of different kind may correspond to the same orbit.
2. We can draw adequate phase spaces only for two dimensions.
3. Phase Spaces can be drawn WITHOUT SOLVING THE EQUATION
4. The phase space provides only qualitative properties, NEVER quantitative.

7.2 Phase Space Entities

1. The Orbits
2. The Circles or Closed Orbits
3. The Invariant Sets
4. The Equilibrium Points

The Orbits

Definition 7.1 *Orbits are the projections of the trajectories onto the xy-plane.*

We denote them by $C : x = x(t), y = y(t)$ or $\varphi(t) = (x(t), y(t))$

To indicate that an orbit passes through the point (x_0, y_0) at the time instant t_0, we write:

$C : x_0 = x(t_0), y_0 = y(t_0)$ or $\varphi(t_0) = (x_0, y_0)$

Theorem 7.1 *There is only ONE orbit passing from a given point.*

Theorem 7.2 *Two orbits cannot be intersected.*

A Circle or a Closed Orbit

Definition 7.2 *A circle is an orbit which meets itself. 0r in other words, $\varphi(t + T) = \varphi(t)$, for some T*

Invariant Set

Definition 7.3 *A subset M of the phase space, is called* **invariant** *if every orbit which starts inside M remains into M.*

Equilibrium Points

Definition 7.4 *A point (x_0, y_0) is called* **equilibrium point** *if and only if the set $\{(x_0, y_0)\}$ is invariant.*

MAIN THEOREM

Theorem 7.3 *Let us have the system:*

$$x' = f(x, y) \quad , \quad y' = g(x, y)$$

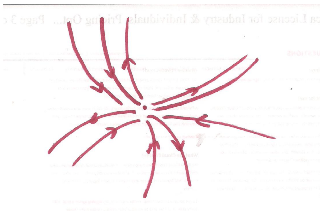
A point (x_0, y_0) is equilibrium point if and only if

$$f(x_0, y_0) = 0 \quad , \quad g(x_0, y_0) = 0$$

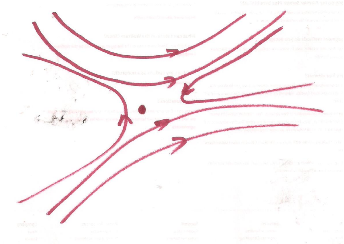
Definition 7.5 *An equilibrium point is called* **isolated** *if we can find a neighbourhood of it which contains no other equilibrium points.*

7.3 Different Kinds of Equilibrium Points

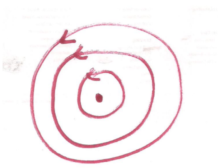
A Node. The orbits go "inside" the eq. point



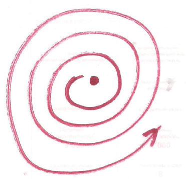
A Saddle. The orbits "approach" the eq. point.



A Center. Closed orbits surround the eq. point.



A Focus. The orbits approach the eq.point in a spiral way.



7.4 Stability

An equilibrium point (x_0, y_0) is called **stable**, if for each neighborhood U of (x_0, y_0), we can find a neighborhood $V \subset U$, such that any orbit starting into V remains into U.

An equilibrium point (x_0, y_0) is called **unstable** if it is not stable.

An equilibrium point is called **locally asymptotical stable** if it is stable and there is a neighborhood of it V, with the property: ”any orbit starting into V, approaches the eq.point.”

We call **basin of attraction** or **region of attraction** of an asymptotical stable equilibrium point, the set of all the points which are attracted by it.

An equilibrium point is called **globally asymptotical stable** if it is asymptotical stable and the region of attraction is the whole phase space.

Chapter 8

Basic tools for Studying a Phase Space

8.1 Summary

1. The Tangent Element - Vector Fields
2. The Isoclines and the Sign (The most popular)
3. The ”x and y” Differential Equations
4. The First Integrals
5. The Polar Coordinates
6. The Nabla Technique for Inavariant Sets
7. Poincare-Bendixson Theorem
8. Linear Systems (Complete Results)
9. Linearization of Nonlinear Systems
10. Lyapunov Functions (Advance)

8.2 The Tangent Element - The Vector Fields

Basic Remark: We have the system:

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \Rightarrow \boxed{\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}}$$

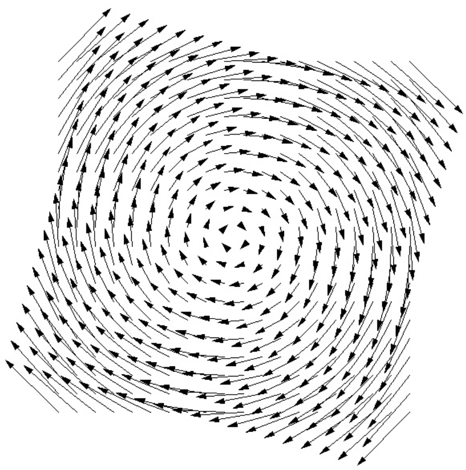
At any point (x_0, y_0) we can calculate the slope of the tangent of the orbit, WITHOUT SOLVING THE SYSTEM.

TANGENT ELEMENT - TANGENT VECTOR

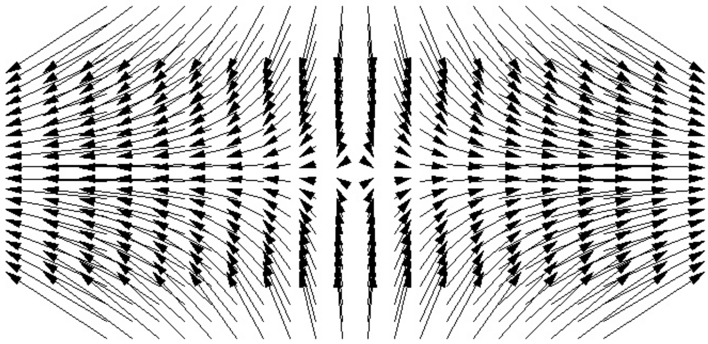
At any point we draw either a VECTOR or a line segment, with slope equal to $g(x_0, y_0)/f(x_0, y_0)$ and length arbitrary small. These vectors will give us a FIRST IMPRESSION of the phase portrait.

It is an elementary, initial method relied on graphs.

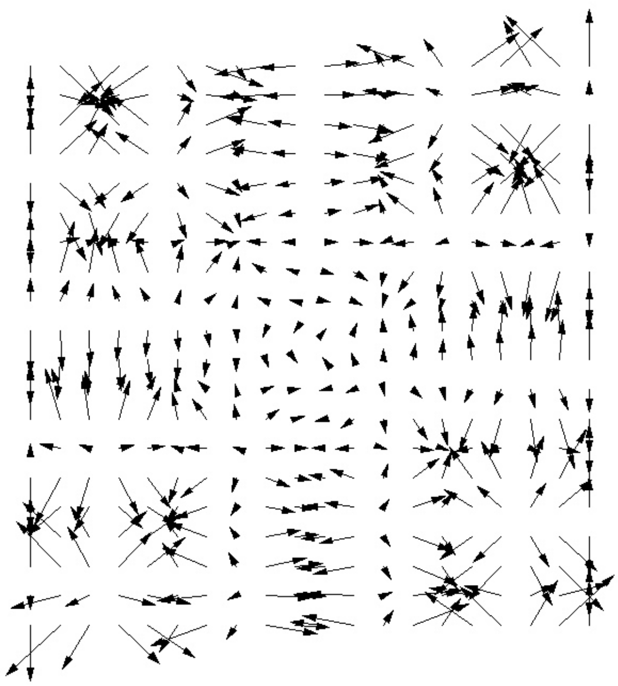
EXAMPLE: $x' = y$, $y' = -x$



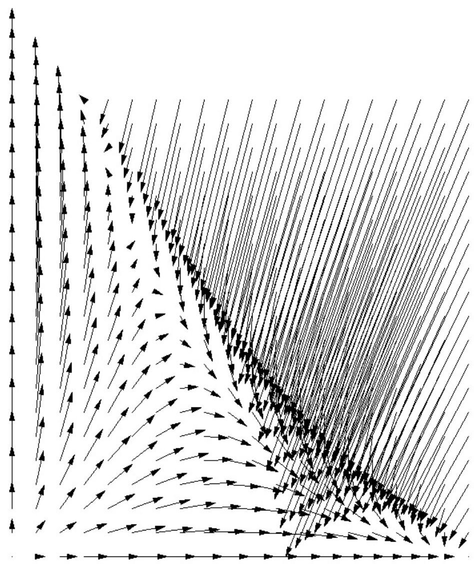
EXAMPLE: $x' = 2x$, $y' = -2y$



EXAMPLE: $x' = y\sin x$, $y' = x\cos y$



EXAMPLE: $x' = x(4 - x - y)$, $y' = y(6 - y - 3x)$



8.3 Isoclines

Definition 8.1 *A* **λ-isocline** *is a line of the phase space, at each point of which, the passing orbit has slope equal to λ.*

$$\lambda\text{-isocline} \Leftrightarrow \frac{g(x,y)}{f(x,y)} = \lambda$$

Definition 8.2 *A* **vertical isocline** *is an isocline with slope equal to $+\infty$. The orbits cross it vertically.*

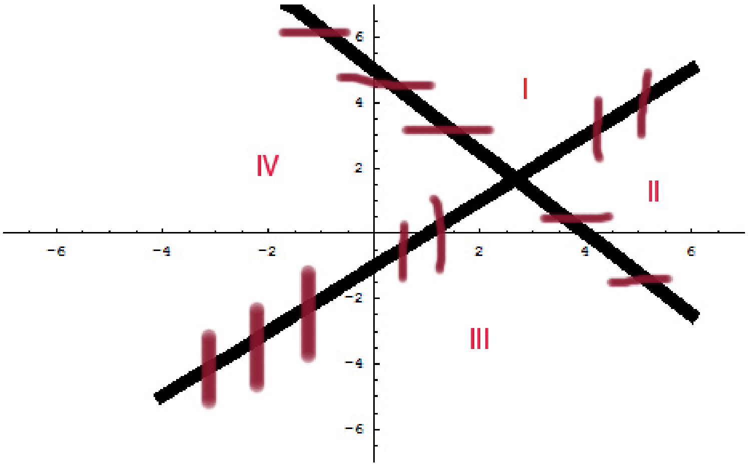
Definition 8.3 *A* **horizontal isocline** *is an isocline with slope equal to* 0. *The orbits cross it horizontally.*

$$\text{Vertical isocline} \Leftrightarrow f(x,y) = 0.$$
$$\text{Horizontal isocline} \Leftrightarrow g(x,y) = 0.$$

ATTENTION: The isoclines are NOT orbits. THEY DO NOT BELONG TO THE PHASE SPACE. They help us to draw the phase space.

EXAMPLE: $$\begin{aligned} x' &= x + y - 1 \\ y' &= 5x + 4y - 20 \end{aligned}$$

The equilibrium point is $\left(\frac{8}{3}, \frac{5}{3}\right)$. Horizontal isocline: $5x + 4y - 20 = 0$. Vertical isocline: $x + y - 1 = 0$



They separate the phase portrait in four regions.

8.4 The Sign

At each region we calculate the sign of the functions $f(x, y)$, $g(x, y)$. This is achieved by substituting arbitrary points. The sign of the above functions determine the sign of the derivatives x', y' and thus the monotony of the solutions $x(t), y(t)$.

We have the next cases:

• If, somewhere, $f > 0 \Rightarrow x' > 0 \Rightarrow x(t)$ increasing.
We denote it by $\longrightarrow$.

• If, somewhere, $f < 0 \Rightarrow x' < 0 \Rightarrow x(t)$ decreasing.
We denote it by $\longleftarrow$.

• If, somewhere, $g > 0 \Rightarrow y' > 0 \Rightarrow y(t)$ increasing.
We denote it by $\uparrow$.

• If, somewhere, $g < 0 \Rightarrow y' < 0 \Rightarrow y(t)$ decreasing.
We denote it by $\downarrow$.

EXAMPLE: $$\begin{aligned} x' &= -x + y + 1 \\ y' &= 5x + 4y - 20 \end{aligned}$$

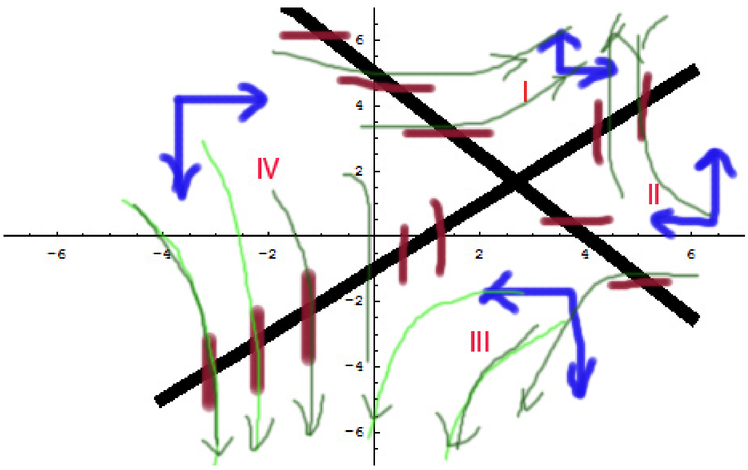
Here, $f(x, y) = -x + y + 1$, $g(x, y) = 5x + 4y - 20$

Region I: Let us take the point $(3, 4)$. We see, $f(3, 4) = 2 > 0 \Rightarrow \longrightarrow$ and $g(3, 4) = 11 > 0 \Rightarrow \uparrow$.

Region II: Let us take the point $(6, 2)$. We see, $f(6, 2) = -3 < 0 \Rightarrow \longleftarrow$ and $g(6, 2) = 18 > 0 \Rightarrow \uparrow$.

Region III: Let us take the point $(3, -4)$. We see, $f(3, -4) = -6 < 0 \Rightarrow \longleftarrow$ and $g(3, -4) = -21 < 0 \Rightarrow \downarrow$.

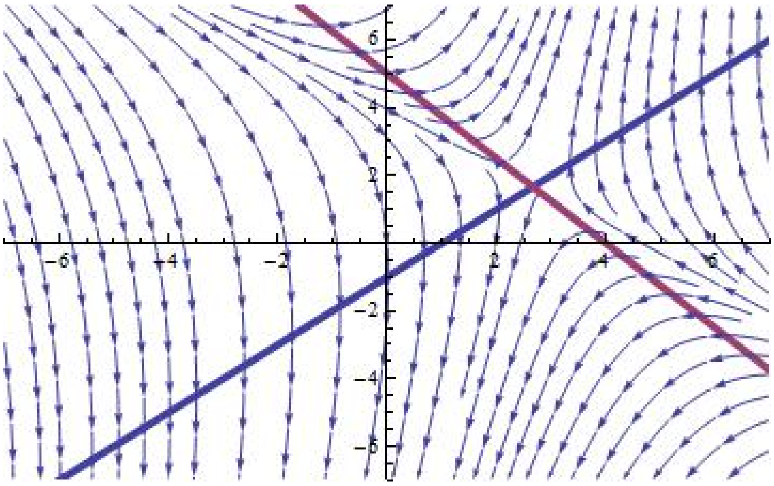
Region IV: Let us take the point $(-2, 2)$. We see, $f(-2, 2) = 5 > 0 \Rightarrow \longrightarrow$ and $g(-2, 2) = -22 < 0 \Rightarrow \downarrow$.



The equil. point is unstable

ATTENTION: The green lines ARE NOT THE ORBITS. They are just an indication of how the real orbits will behave.

Here we have the REAL phase space, designed by a proper software



8.1 Exercise: *Draw the phase portrait of the system:*

$$x' = -2x - y + 9$$
$$y' = x - y + 3$$

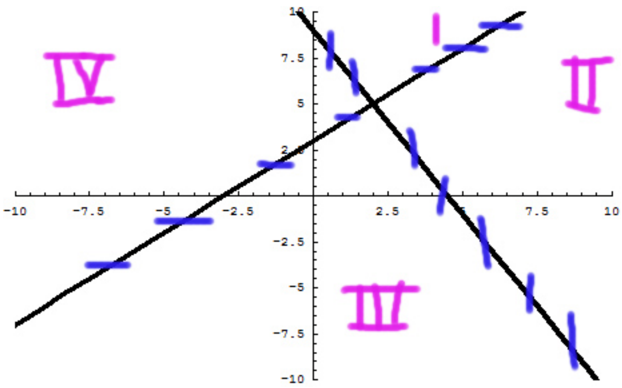
.

Solution:

Step 0: We find the eq.point. $-2x - y + 9 = 0, x - y + 3 = 0 \Rightarrow$ $\boxed{x_0 = 2, y_0 = 5}$.

Step 1: The vertical isocline: $-2x - y + 9 = 0$. The horizontal isocline $x - y + 3 = 0$.

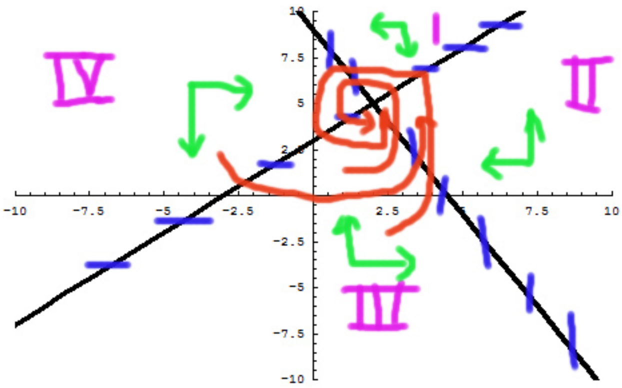
Step 2: We sketch the isoclines:



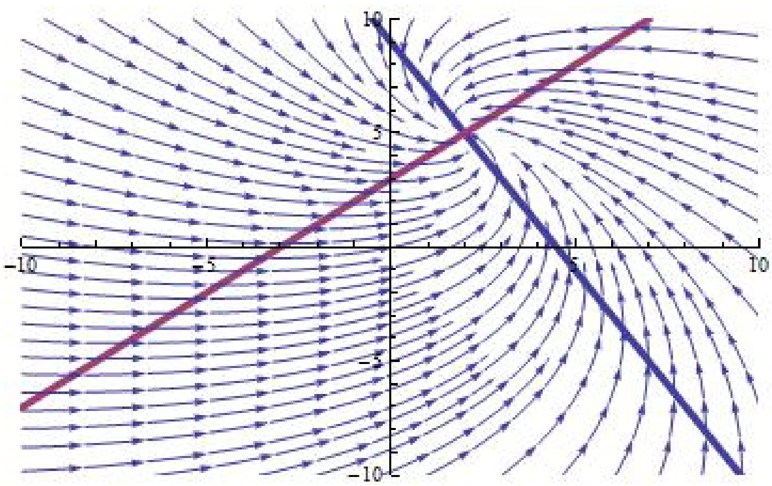
Step 3: We examine the sign at the several regions:

Region	**Point**	$-2x-y+9$	$x-y+3$	x'	y'	x	y
I	$(2,8)$	-3	-3	$-$	$-$	$\leftarrow$	$\downarrow$
II	$(5,4)$	-5	4	$-$	$+$	$\leftarrow$	$\uparrow$
III	$(0,0)$	$+9$	$+3$	$+$	$+$	$\rightarrow$	$\uparrow$
IV	$(0,5)$	$+4$	-2	$+$	$-$	$\rightarrow$	$\downarrow$

Step 4: We put all the above information onto the phase portrait.



The eq. point is asymptotical stable (exponentially).



8.2 Exercise: *Draw the phase portrait of the system:*

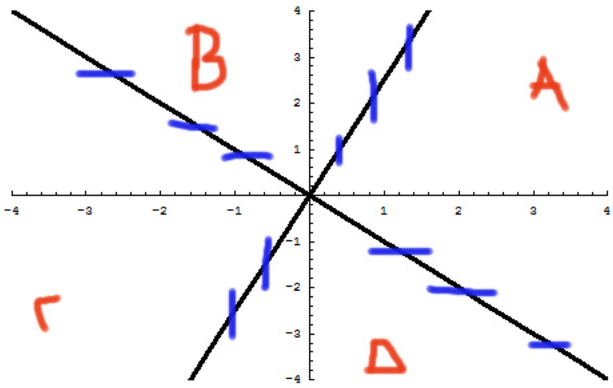
$$x' = 5x - 2y$$
$$y' = x + y$$

Solution:

Step 0: We find the eq.point. $5x - 2y = 0, x + y = 0 \Rightarrow \boxed{x_0 = 0, y_0 = 0}$.

Step 1: The vertical isocline: $5x - 2y = 0$. The horizontal isocline $x + y = 0$.

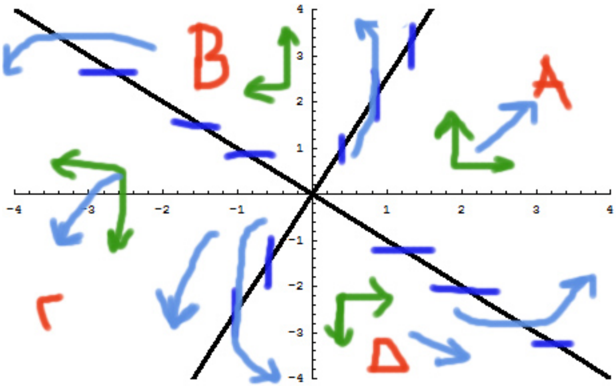
Step 2: We sketch the isoclines:



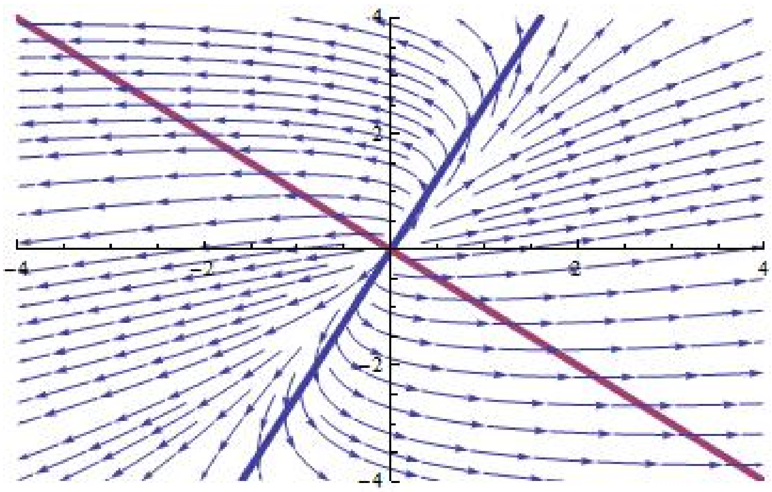
Step 3: We examine the sign at the several regions:

Region	**Point**	$5x - 2y$	$x + y$	x'	y'	x	y
A	$(1, 0)$	5	1	+	+	$\rightarrow$	$\uparrow$
B	$(0, 1)$	-2	1	$-$	+	$\leftarrow$	$\uparrow$
Γ	$(-1, 0)$	-5	-1	$-$	$-$	$\leftarrow$	$\downarrow$
Δ	$(0, -1)$	$+2$	-1	+	$-$	$\rightarrow$	$\downarrow$

Step 4: We put all the above information onto the phase portrait.



The eq. point is unstable.



8.5 The ”x-y” Differential Equation Method

If we are able to solve the differential equation: $\dfrac{dy}{dx} = \dfrac{g(x,y)}{f(x,y)}$ we get a relation $\boxed{\Phi(x,y) = 0}$ between x and y which permits us to draw the phase space.

8.3 Exercise: *Draw the phase portrait of the system:*

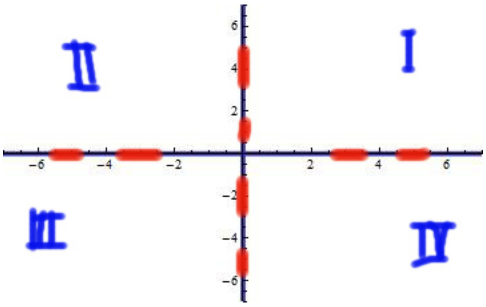
$$x' = 2x$$
$$y' = y$$

Solution:

Step 0: We find the eq.point. $2x = 0, y = 0 \Rightarrow \boxed{x_0 = 0, y_0 = 0}$.

Step 1: The vertical isocline: $2x = 0$. The horizontal isocline $y = 0$.

Step 2: We sketch the isoclines:

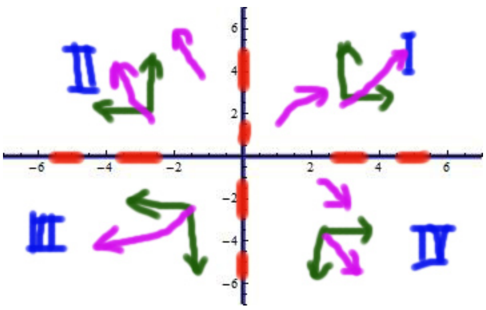


Remark: This is a special case, where the orbits cannot cross the isoclines.

Step 3: We examine the sign at the several regions:

Region	Point	$2x$	y	x'	y'	x	y
I	$(1, 1)$	2	1	+	+	$\rightarrow$	$\uparrow$
II	$(-1, 1)$	-2	1	$-$	+	$\leftarrow$	$\uparrow$
III	$(-1, -1)$	-2	-1	$-$	$-$	$\leftarrow$	$\downarrow$
IV	$(1, -1)$	$+2$	-1	+	$-$	$\rightarrow$	$\downarrow$

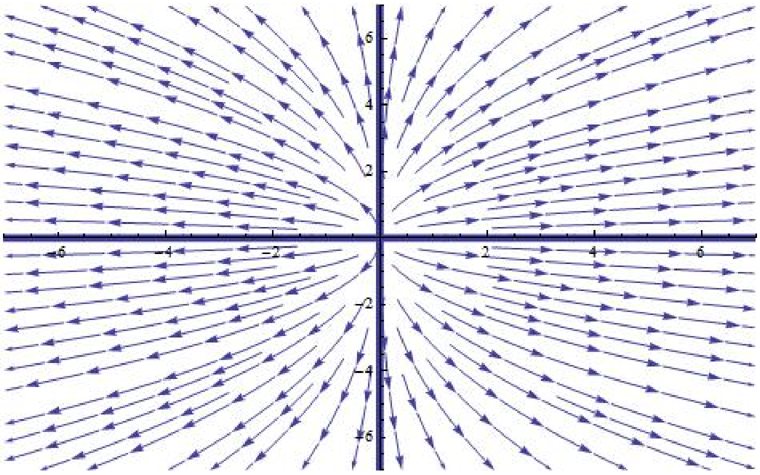
Step 4: We put all the above information onto the phase portrait.



Step 5: To find the "shape" of the orbits we solve the dif. eq. :

$$\frac{dy}{dx} = \frac{y}{2x} \Rightarrow \frac{dy}{y} = \frac{dx}{x} \Rightarrow \ln|y| = \frac{1}{2}\ln|x| + C$$

, , , $\Rightarrow \boxed{y^2 = cx}$, parabola



8.6 The First Integral Technique

We seek for a relation between x and y: $\Phi(x, y) = 0$.

Successively we have:

$$\Phi(x, y) = 0 \Rightarrow \frac{d}{dt}\Phi(x, y) = 0 \Rightarrow \frac{\partial \Phi}{\partial x}\frac{dx}{dt} + \frac{\partial \Phi}{\partial y}\frac{dy}{dt} = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial \Phi}{\partial x}x' + \frac{\partial \Phi}{\partial y}y' = 0 \Rightarrow \frac{\partial \Phi}{\partial x}f + \frac{\partial \Phi}{\partial y}g = 0$$

Φ is called FIRST INTEGRAL. To find Φ usually we take:

$$\frac{\partial \Phi}{\partial x} = -g \quad , \quad \frac{\partial \Phi}{\partial y} = f$$

8.4 Exercise: *Draw the phase portrait of the system:*

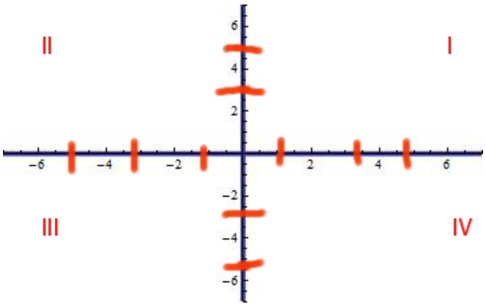
$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$$

Solution:

Step 0: We find the eq.point. $2x = 0, y = 0 \Rightarrow \boxed{x_0 = 0, y_0 = 0}$.

Step 1: The vertical isocline: $y = 0$. The horizontal isocline $-x = 0$.

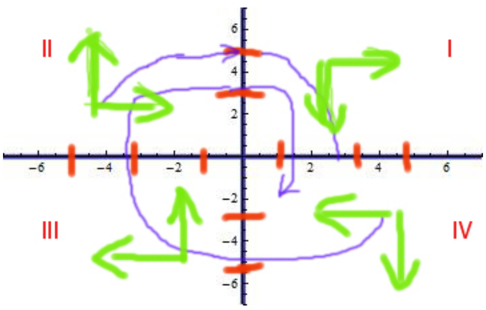
Step 2: We sketch the isoclines:



Step 3: We examine the sign at the several regions:

Region	**Point**	y	$-x$	x'	y'	x	y
I	$(1, 1)$	1	-1	$+$	$-$	$\rightarrow$	$\downarrow$
II	$(-1, 1)$	1	1	$+$	$+$	$\rightarrow$	$\uparrow$
III	$(-1, -1)$	-1	1	$-$	$+$	$\leftarrow$	$\uparrow$
IV	$(1, -1)$	-1	-1	$-$	$-$	$\leftarrow$	$\downarrow$

Step 4: We put all the above information onto the phase portrait.



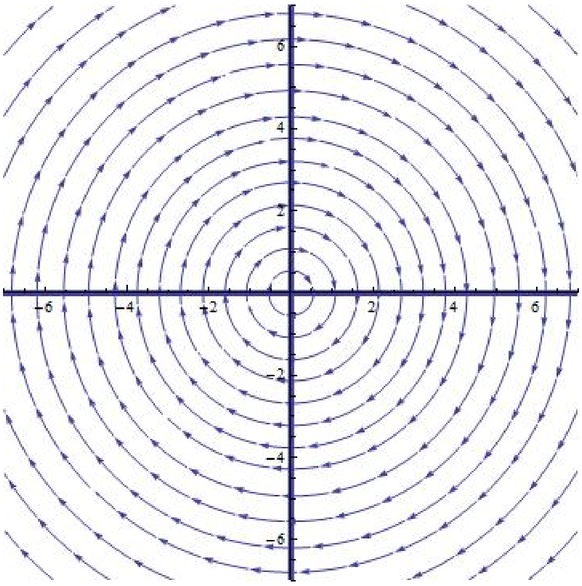
Step 5: To find the "shape" of the orbits we use the girst integral technique:

$$\frac{\partial \Phi}{\partial x} = -g \Rightarrow \frac{\partial \Phi}{\partial x} = x \Rightarrow \Phi = \int x dx = \frac{x^2}{2} + F(y) \Rightarrow$$

$$\frac{\partial \Phi}{\partial y} = f \Rightarrow \frac{\partial \Phi}{\partial y} = y \Rightarrow 0 + F'(y) = y \Rightarrow F(y) = \frac{y^2}{2} + C$$

$$\Phi(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + C$$

The orbits are CIRCLES



Chapter 9

Linear Systems

9.1 The General Super Theorem

Theorem 9.1 *Let us have the Linear system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where A is an $n \times n$ matrix, then:*

1. The only equilibrium point is the $\vec{0}$.

2. If all the eigenvalues of A have negative real part, then $\vec{0}$ is **globally asymptotical stable**.

3. If there is at least one eigenvalue of A with positive real part, then $\vec{0}$ is **unstable**.

4. If there are some eigenvalues of A with zero real parts and linear independent basic eigenvectors and the rest eigenvalues have negative real part, then $\vec{0}$ is **stable**.

9.2 2×2 Linear Systems

Theorem 9.2 *Let us have the Linear system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where A is a 2×2 matrix, then the origin is global asymptotical stable if and only if $det(A) > 0$ and $tr(A) < 0$.*

Remark: By using the above theorem we can check the stability of a 2×2 linear system, without computing the eigenvalues.

Theorem 9.3 *Let us have the Linear system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, where A is a 2×2 matrix. Let λ_1, λ_2 be the eigenvalues of A (which are considered to be real* *numbers) and (a_1, b_1), (a_2, b_2) the corresponding eigenvectors, then the orbits of the phase space are "given" by the next relation:*

$$(-1)^{\lambda_2-\lambda_1} \begin{vmatrix} x & a_2 \\ y & b_2 \end{vmatrix}^{\lambda_2} \begin{vmatrix} a_1 & x_0 \\ b_1 & y_0 \end{vmatrix}^{\lambda_1} = \begin{vmatrix} a_1 & x \\ b_1 & y \end{vmatrix}^{\lambda_1} \begin{vmatrix} x_0 & a_2 \\ y_0 & b_2 \end{vmatrix}^{\lambda_2}$$

where, (x_0, y_0) is the initial point.

Theorem 9.4 *Let us have the Linear system $\vec{x}' = A\vec{x}$, where A is a 2×2 matrix. Let λ_1, λ_2 be the eigenvalues of A (which are considered to be real numbers) and $\vec{u}_1 = (a_1, b_1)$, $\vec{u}_2 = (a_2, b_2)$ the corresponding eigenvectors. Then, the straight lines $\vec{OM} = r\vec{u}_1$, $\vec{ON} = s\vec{u}_1$, $r, s \in \mathbf{R}$, are invariant sets. Furthermore, depending from the sign of λ_i, their orbits approach or go away from the origin.*

Definition 9.1 *If the orbits of some of the said straight lines approach the origin, it is called the* **stable** *line, otherwise the* **unstable** *one.*

Sketch of the proof: We suppose that $(x_0, y_0) \in \vec{OM}$. This means that $x_0 = ra_1$ and $y_0 = rb_1$ for some $r \in \mathbf{R}$.

Using the relation of the theorem we get:

$$0 = \begin{vmatrix} a_1 & x \\ b_1 & y \end{vmatrix}^{\lambda_1} \begin{vmatrix} ra_1 & a_2 \\ rb_1 & b_2 \end{vmatrix}^{\lambda_2}$$

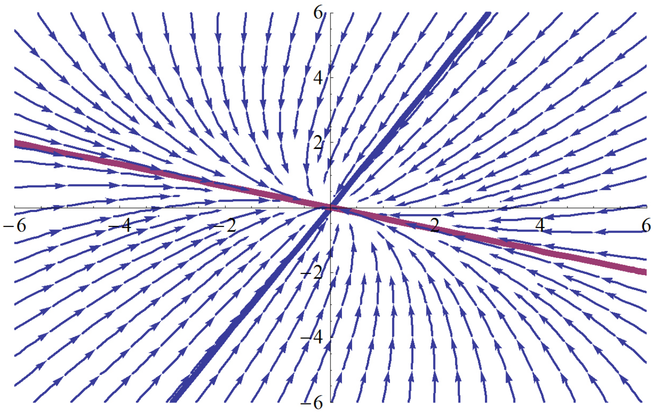
Which gives: $\begin{vmatrix} a_1 & x \\ b_1 & y \end{vmatrix} = 0$. This is achieved by setting $x = wa_1$, $y = wb_1$, for some $w \in \mathbf{R}$.
The latter implies that $(x, y) \in \vec{OM}$ and thus $\vec{OM}$ is invariant. We also know that $x(t) = re^{\lambda_1 t}a_1$ and $x(t) = re^{\lambda_1 t}b_1$. Hence if $\lambda_1 < 0$, $x(t), y(t) \to 0$, otherwise the orbits go away.

9.3 The Phase Space of the 2×2 Linear Systems

CASE 1. $\lambda_1 < \lambda_2 < 0 \Rightarrow$ GLOBAL ASYMPTOTICAL STABILITY OF THE ORIGIN, A NODE. The line u_1 is called the **fast** one. The line u_2 is called **slow**.

Example: $A = \begin{pmatrix} -8/7 & -3/7 \\ -2/7 & -13/7 \end{pmatrix}$, $\lambda_1 = -2$, $\lambda_2 = -1$,

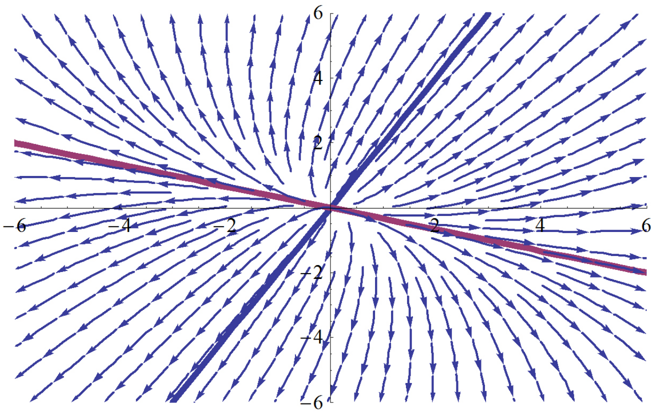
$u_1 = (1, 2)$, $u_2 = (1, -1/3)$.



> **CASE 2.** $\lambda_1 > \lambda_2 > 0 \Rightarrow$ INSTABILITY, A NODE. The line u_1 is called the **fast** one. The line u_2 is called **slow**.

Example: $A = \begin{pmatrix} 8/7 & 3/7 \\ 2/7 & 13/7 \end{pmatrix}$, $\lambda_1 = 2$, $\lambda_2 = 1$,

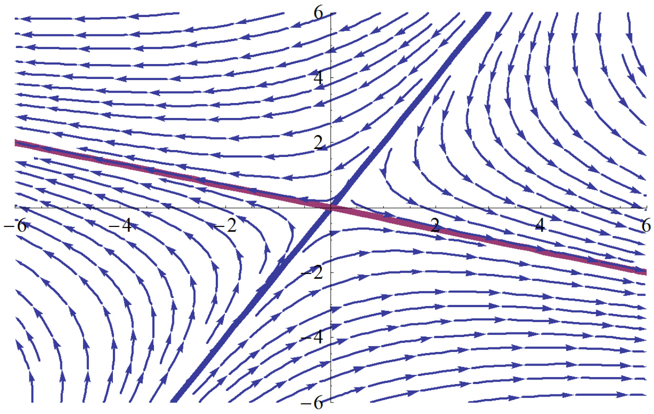
$u_1 = (1, 2)$, $u_2 = (1, -1/3)$.



> **CASE 3.** $\lambda_2 < 0 < \lambda_1 \Rightarrow$ INSTABILITY, **A SADDLE**. The line u_1 is called the **unstable** one. The line u_2 is called **stable**.

Example: $A = \begin{pmatrix} 16/7 & -15/7 \\ -10/7 & -9/7 \end{pmatrix}$, $\lambda_1 = -2$, $\lambda_2 = 3$,

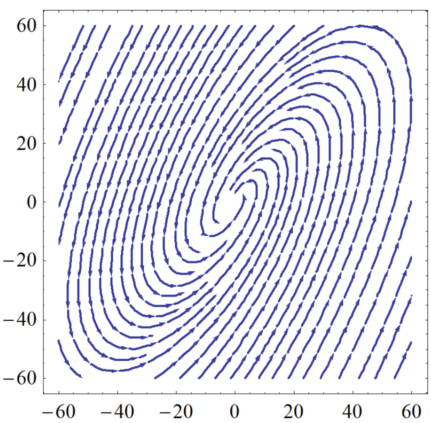
$u_1 = (1, 2)$, $u_2 = (1, -1/3)$.



CASE 4. $\lambda_1 = a + bi, \lambda_2 = a - bi$, $a < 0 \Rightarrow$ GLOBAL ASYMPTOTICAL STABLE, **A SINK**.

Example: $A = \begin{pmatrix} 4 & -7 \\ 14 & -10 \end{pmatrix}$, $\lambda_1 = -3 + 7i$, $\lambda_2 = -3 - 7i$,

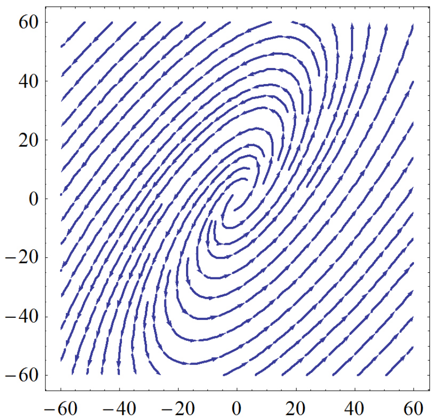
$u_1 = (1 + i, 2)$, $u_2 = (1 - i, 2)$.



CASE 5. $\lambda_1 = a + bi, \lambda_2 = a - bi$, $a > 0 \Rightarrow$ INSTABILITY, **A SOURCE**.

Example: $A = \begin{pmatrix} 10 & -7 \\ 14 & -4 \end{pmatrix}$, $\lambda_1 = 3 + 7i$, $\lambda_2 = 3 - 7i$,

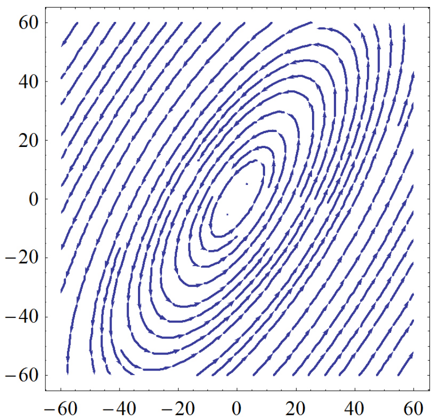
$u_1 = (1 + i, 2)$, $u_2 = (1 - i, 2)$.



CASE 6. $\lambda_1 = bi$, $\lambda_2 = -bi$, $\Rightarrow$ STABILITY, **A CENTER**.

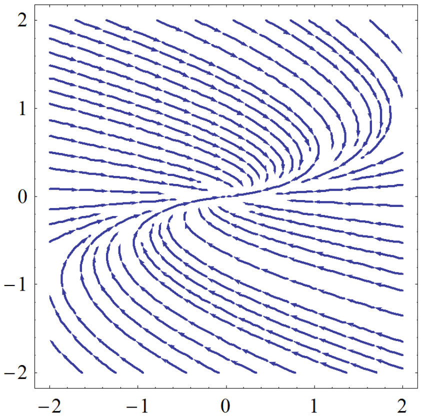
Example: $A = \begin{pmatrix} 7 & -7 \\ 14 & -7 \end{pmatrix}$, $\lambda_1 = +7i$, $\lambda_2 = -7i$,

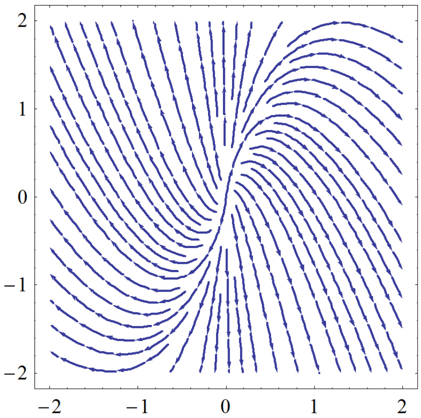
$u_1 = (1 + i, 2)$, $u_2 = (1 - i, 2)$.



CASE 7. $\lambda_1 = \lambda_2 = \lambda \neq 0 \in \mathbf{R}$, $\Rightarrow$ GLOBAL ASYMPTOTIC STABILITY IF $\lambda < 0$, INSTABILITY IF $\lambda > 0$.

Example: $A = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$, $\lambda_1 = -1$, $\lambda_2 = -1$.

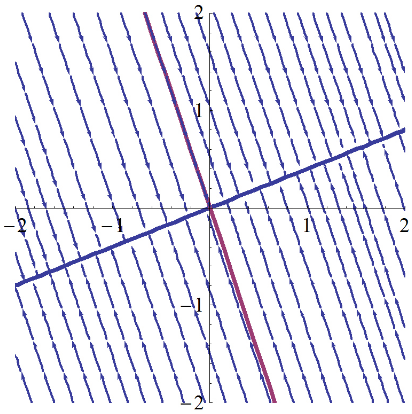


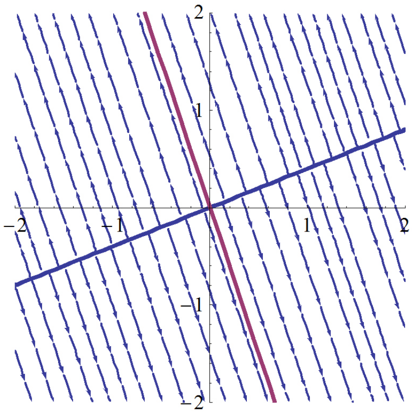


CASE 8. $\lambda_1 = 0, \lambda_2 \neq 0 \in \mathbf{R}$, $\Rightarrow$ GLOBAL ASYMPTOTIC STABILITY <u>TO THE LINE</u> $\vec{u}_1$ IF $\lambda_2 < 0$, INSTABILITY IF $\lambda_2 > 0$.

Example: $A = \begin{pmatrix} -6/17 & 15/17 \\ 18/17 & -45/17 \end{pmatrix}$, $\lambda_1 = 0$, $\lambda_2 = -3$,

$u_1 = (5, 2)$, $u_2 = (-1, 3)$.

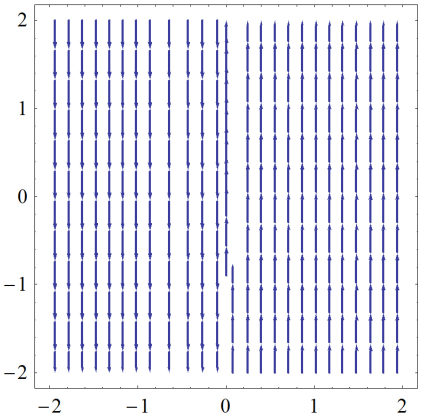




CASE 8. $\lambda_1 = 0, \lambda_2 = 0, \Rightarrow$ PARALLEL SYSTEM.

Example: $A = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$, $\lambda_1 = 0$, $\lambda_2 = 0$,

$u_1 = (0, 1)$, $u_2 = (0, 1)$.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Chapter 10

Nonlinear Systems

10.1 The Main Theorem

Theorem 10.1 *Let us have the Nonlinear system* $\vec{\mathbf{x}}' = \mathbf{F}(\vec{\mathbf{x}})$, *where F is a function* $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$. *Let* $\vec{\mathbf{x}}_0$ *be an equilibrium point,* $\mathbf{J}(\vec{\mathbf{x}}_0)$ *the Jacobian, evaluated at this equilibrium point and* λ_i *its eigenvalues. Then:*

1. If all the eigenvalues of the Jacobian have negative *real part, then* $\vec{\mathbf{x}}_0$ *is* **locally asymptotical stable**.

2. If there is at least one eigenvalue of the Jacobian with positive *real part, then* $\vec{\mathbf{x}}_0$ *is* **locally unstable**.

3. If there are some eigenvalues with zero *real parts and linear independent basic eigenvectors and the rest eigenvalues have* negative *real parts, then* $\vec{\mathbf{x}}_0$ *is* **locally stable**.

10.2 The Phase Space of a 2×2 Nonlinear System

Theorem 10.2 *Let us have the* 2×2 *Nonlinear system* $\vec{\mathbf{x}}' = \mathbf{F}(\vec{\mathbf{x}})$, *where F is a function* $\mathbf{F} : \mathbf{R}^2 \to \mathbf{R}^2$. *and* $\vec{\mathbf{x}}_0$ *be an equilibrium point. Around of this equilibrium point (locally), the phase space is the same with the phase space of the linear system* $\vec{\mathbf{x}}' = \mathbf{J}(\vec{\mathbf{x}}_0)\vec{\mathbf{x}}$, *where* $\mathbf{J}(\vec{\mathbf{x}}_0)$ *is the Jacobian, evaluated at this equilibrium point.*

10.3 The Methodology

1. We find the equilibrium points.
2. Using the theorem we characterize them.
3. We find the isoclines and the SIGN at the different regions.
4. We use the differential equation or the first integral method, if possible.
5. We draw the phase space

10.1 Exercise: *Draw the phase portrait of the system:*

$$x' = -y + yx$$
$$y' = -x$$

Solution:

Step 0: We find the eq.point. $-y + yx = 0, -x = 0 \Rightarrow \boxed{x_0 = 0, y_0 = 0}$.

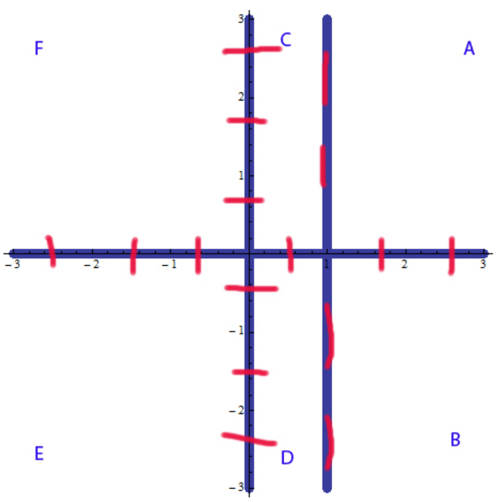
Step 1. The Jacobian:

$$J(0,0) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(0,0)} = \begin{pmatrix} y & -1+x \\ -1 & 0 \end{pmatrix}_{(0,0)} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

The eigenvalues are $1, -1$ so, we have instability and a saddle.

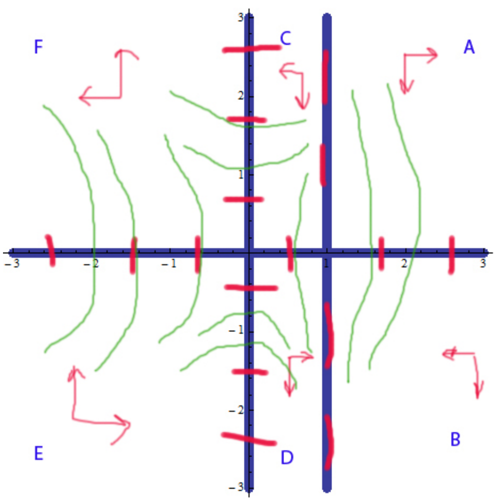
Step 2. Horizontal Isoclines: $-x = 0 \Rightarrow x = 0$.

Vertical Isoclines, $-y + yx = 0 \Rightarrow y = 0$, $x = 1$.



Step 3. The sign,

Region	The Point	$-y + yx$	$-x$	x'	y'	x	y
A	$(2, 2)$	2	-2	$+$	$-$	$\rightarrow$	$\downarrow$
B	$(2, -2)$	-2	-2	$-$	$-$	$\leftarrow$	$\downarrow$
C	$(1/2, 1)$	-1/2	-1/2	$-$	$-$	$\leftarrow$	$\downarrow$
D	$(1/2, -1)$	1/2	-1/2	$+$	$-$	$\rightarrow$	$\downarrow$
E	$(-1, -1)$	2	1	$+$	$+$	$\rightarrow$	$\uparrow$
F	$(-1, 1)$	-2	1	$-$	$+$	$\leftarrow$	$\uparrow$

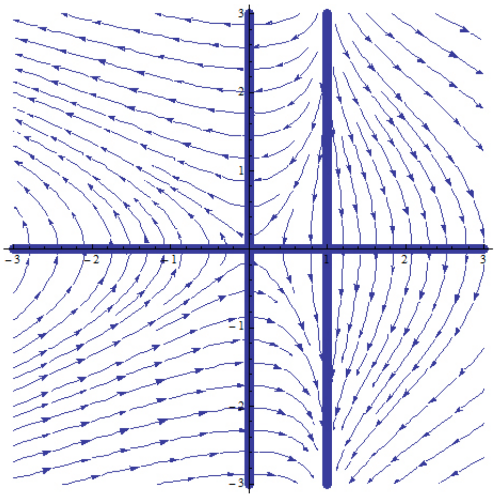


Step 4. We solve the differential equation:

$$\frac{dy}{dx} = \frac{-x}{yx - y} \Rightarrow y^2 = 2x_0 + y_0^2 - 2x + 2\ln(x_0 - 1) - 2\ln(x - 1)$$

where x_0, y_0 the initial conditions.

So, finally, the phase portrait is:



10.2 Exercise: *Draw the phase portrait of the system:*

$$\begin{aligned} x' &= x(4 - x - y) \\ y' &= y(6 - y - 3x) \end{aligned}$$

Solution:

Step 0: We find the equilibrium points:

$$\left.\begin{aligned} x(4 - x - y) = 0 \\ y(6 - y - 3x) = 0 \end{aligned}\right\} \Rightarrow (0,6), \quad (1,3), \quad (4,0), \quad ,(0,0)$$

Step 1. We characterize them. To do so we calculate the Jacobian:

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 4 - 2x - y & -x \\ -3y & 6 - 3x - 2y \end{pmatrix} \Rightarrow$$

$J(0,6) = \begin{pmatrix} -2 & 0 \\ -18 & -6 \end{pmatrix}$ with eigenvalues $-2, -6 \Rightarrow$ local asymptotical stable.

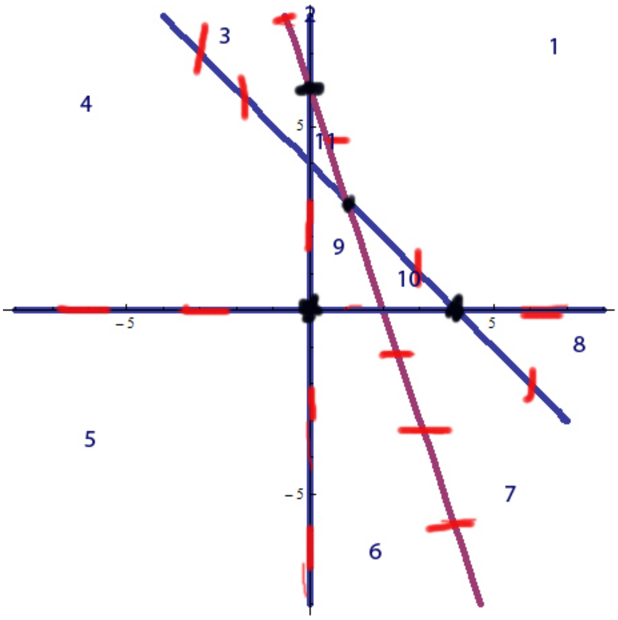
$J(1,3) = \begin{pmatrix} -1 & -1 \\ -9 & -3 \end{pmatrix}$ with eigenvalues $-2 - \sqrt{10}, -2 + \sqrt{10} \Rightarrow$ unstable

$J(4,0) = \begin{pmatrix} -4 & -4 \\ 0 & -6 \end{pmatrix}$ with eigenvalues $-4, -6 \Rightarrow$ local asymptotical stable.

$J(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$ with eigenvalues $4, 6 \Rightarrow$ unstable

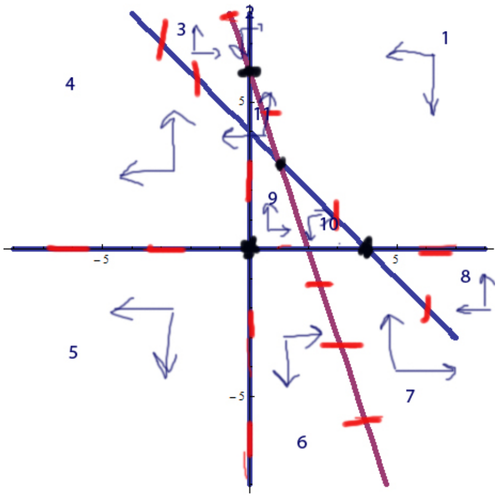
Step 2. Horizontal Isoclines: $y(6 - y - 3x) = 0 \Rightarrow$ either $y = 0$ or $6 - y - 3x = 0$.

Vertical Isoclines, $x(4 - x - y) = 0 \Rightarrow$ either $x = 0$ or $4 - x - y = 0$.



Step 3. The sign,

R	P	$x(4-x-y)$	$y(6-y-3x)$	x'	y'	x	y
1	$(5,5)$	-30	-70	$-$	$-$	$\leftarrow$	$\downarrow$
2	$(-1/4,8)$	$15/16$	-10	$+$	$-$	$\rightarrow$	$\downarrow$
3	$(-1,6)$	1	18	$+$	$+$	$\rightarrow$	$\uparrow$
4	$(-5,1)$	-40	20	$-$	$+$	$\leftarrow$	$\uparrow$
5	$(-5,-1)$	-50	-22	$-$	$-$	$\leftarrow$	$\downarrow$
6	$(1,-5)$	8	-40	$+$	$-$	$\rightarrow$	$\downarrow$
7	$(4,-1)$	4	5	$+$	$+$	$\rightarrow$	$\uparrow$
8	$(6,-1/2)$	-9	$23/4$	$-$	$+$	$\leftarrow$	$\uparrow$
9	$(1/2,1)$	$5/4$	$7/2$	$+$	$+$	$\rightarrow$	$\uparrow$
10	$(3,1/2)$	$3/2$	$-7/4$	$+$	$-$	$\rightarrow$	$\downarrow$
11	$(1/2,4)$	$-1/4$	2	$-$	$+$	$\leftarrow$	$\uparrow$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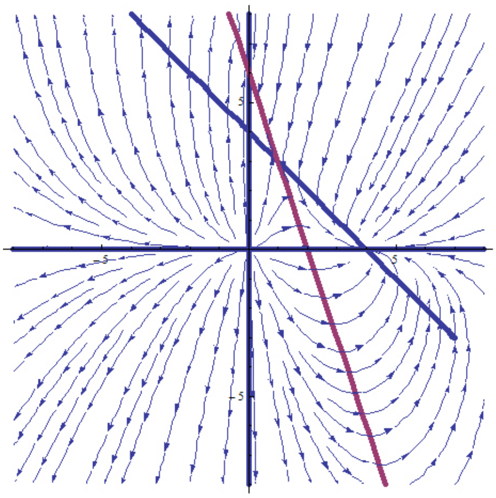
−5
5
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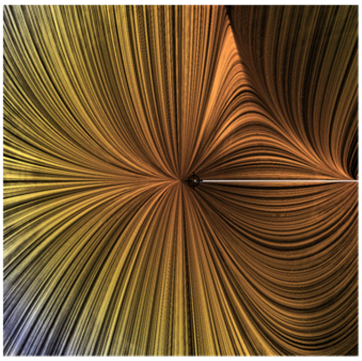
5



Chapter 11

Invariant Sets

11.1 The Basic Theorems

Theorem 11.1 *Let us have the system:* $x' = f(x,y)$, $y' = g(x,y)$ *and* $M = \{(x,y) : \Phi(x,y) \leq c\}$ *a set. This set is invariant for the system if and only if*

$$f\frac{\partial \Phi}{\partial x} + g\frac{\partial \Phi}{\partial y} \leq 0$$

on the boundary $\Phi(x,y) = c$.

Theorem 11.2 *Let us have the system:* $x' = f(x,y)$, $y' = g(x,y)$ *and the region* $M = \{(x,y) : W(x,y) \geq c_1, \Phi(x,y) \leq c_2\}$. *This set is invariant for the system if and only if*

$$f\frac{\partial \Phi}{\partial x} + g\frac{\partial \Phi}{\partial y} \leq 0 \quad , \quad f\frac{\partial W}{\partial x} + g\frac{\partial W}{\partial y} \geq 0$$

on the boundaries $\Phi(x,y) = c_2$ *and* $W(x,y) = c_1$

EXAMPLE: We have the system $x' = y, y' = -x$. Show that the set $M = \{(x,y) : a \leq x^2 + y^2 \leq b\}$, $b > a > 0$ is invariant.

Proof: We calculate the quantity:

$$f\frac{\partial (x^2+y^2)}{\partial x} + g\frac{\partial (x^2+y^2)}{\partial y} = 2xy - 2xy = 0$$

So, by the previous theorem the set is invariant.

11.2 Poincare-Bendixson Theorem

Theorem 11.3 *Let us have the system:* $x' = f(x, y)$, $y' = g(x, y)$*and let M be a closed subset of the plane such that:*

- *M contains no equilibrium points or it contains ONE equilibrium point which is unstable focus or unstable node.*
- *M is invariant.*

Then, M contains a periodic orbit.

Theorem 11.4 *(***Bendixson Criterion.***) If, on a simply connected region D of the plane, the expression* $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ *is not identically zero and does not change sign, then the system has no periodic orbits lying entirely in D.*

EXAMPLE: We have the system $x' = y, y' = -x$. Show that it has an infinite number of periodic orbits.

Proof: We consider the set $M = \{(x, y) : a \leq x^2 + y^2 \leq b\}$, $b > a > 0$. This set is invariant (previous example).

The set M is closed, bounded and free of equilibrium points, since the only equilibrium point $(0, 0)$ does not belong to M.

Hence, by the Poincare-Bendixson Theorem we conclude that there is a periodic orbit in M

Since we can take any $a, b > 0$ to define M , we have an infinite number of closed orbits.

11.1 Exercise: *Draw the phase portrait of the system:*

$$x' = x + y - x^3 - xy^2$$
$$y' = -2x + y - yx^2 - y^3$$

Solution:

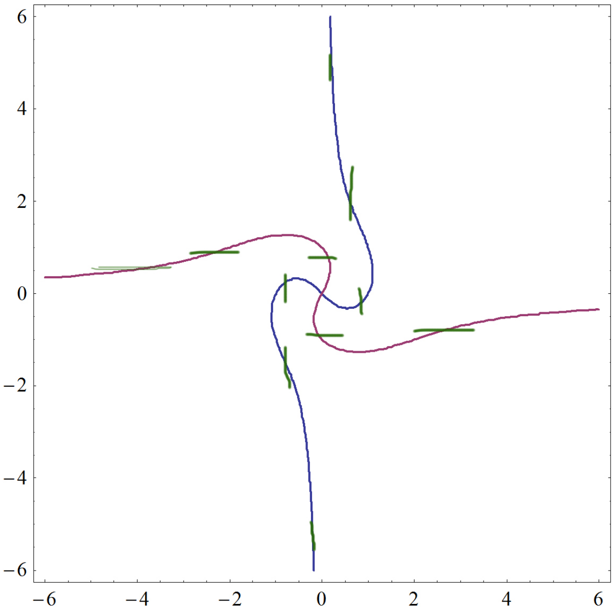
Step 0: We find the equilibrium points: $(0, 0)$

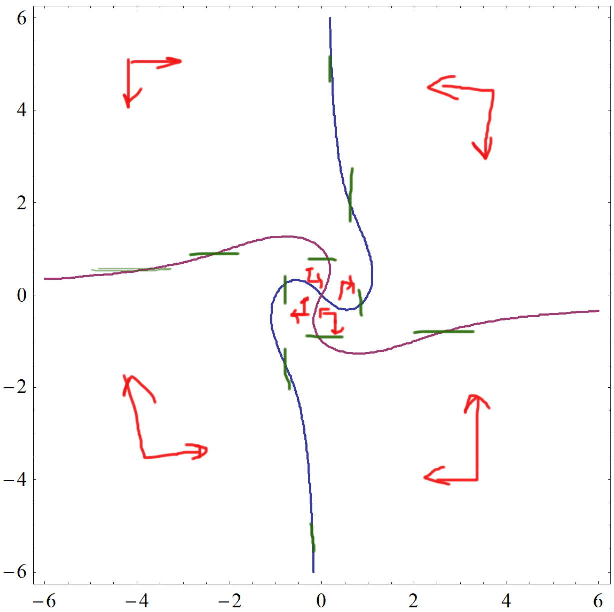
Step 1. The Jacobian is:

$$J(x, y) = \begin{pmatrix} 1 - 3x^2 - y^2 & 1 - 2xy \\ -2 - 2xy & 1 - x62 - 3y^2 \end{pmatrix} \Rightarrow J(0, 0) = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

The eigenvalues are $1 + i\sqrt{2}$, $1 - i\sqrt{2}$. It is unstable.

Isoclines-Sign-Phase Portrait





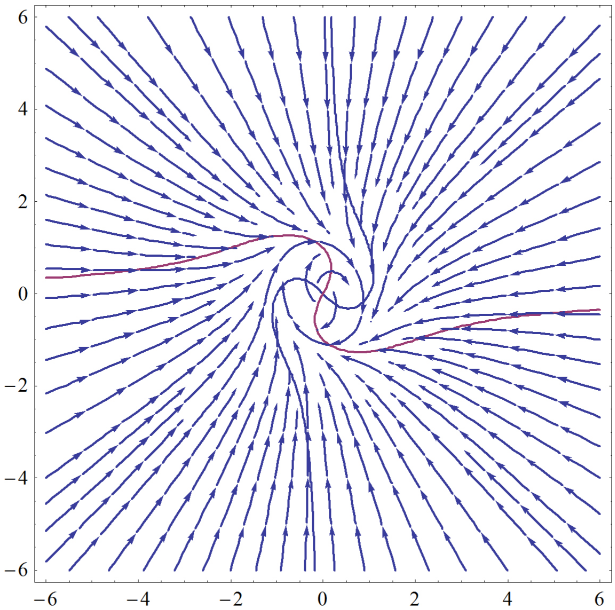
Periodic Orbits

We consider the set $M = \{x^2+y^2 \leq c\}$. It is closed and bounded. It contains the only equilibrium point $(0,0)$ which is an unstable focus. Successively we have:

$$\frac{\partial(x^2+y^2)}{\partial x}f + \frac{\partial(x^2+y^2)}{\partial y}g = 2(x^2+y^2) - 2(x^2+y^2)^2 - 2xy =$$

$$\leq 2(x^2+y^2) - 2(x^2+y^2)^2 + (x^2+y^2) = 3c - 2c^2$$

By choosing $c = 3/2$ the above expression becomes negative and thus M is invariant. Hence, by the Poincare-Bendixson theorem, we conclude that there is a periodic orbit in M.



Chapter 12

IS-LM Models

12.1 A Simple Model

12.1 Exercise: *We consider the next economic quantities:*

$e(t)$, *real expediture*	c, *marginal rate of consuming*
$Y(t)$, *real income*	t_1, *marginal tax rate*
$r(t)$, *interest rate*	h, *marginal rate of investment*
$m^d(t)$, *money demand*	M_0, *nominal Money supply*
A, β, k, u, *constants*	P, *constant price level*

and the next relations:

$$e(t) = a + c(1 - t_1)Y(t) - hr(t) \quad , \quad \dot{Y}(t) = A[e(t) - Y(t)]$$

$$\dot{r}(t) = \beta[m^d(t) - m_0] \quad , \quad m^d(t) = kY(t) - ur(t), \quad m_0 = \frac{M_0}{P}$$

with $a > 0$, $0 < c < 1$, $0 < t_1 < 1$, $h > 0$, $k, u > 0$, $A, \beta > 0$.

Examine the stability of the model.

Solution: Step 0: After a simple substitution we get the system:

$$Y'(t) = A[c(1 - t_1) - 1]Y(t) - Ahr(t) + aA$$

$$r'(t) = \beta kY(t) - \beta ur(t) - \beta m_0$$

We consider that the phase-space is $Y - r$.

Step 1: Equilibrium Point:

$$Y^* = \frac{au + hm_0}{hk + u - cu + cut_1} \quad , \quad r^* = \frac{ak - m_0 + cm_0 - cm_0 t_1}{hk + u - cu + cut_1}$$

Step 2: The Jacobian is:

$$J(Y^*, r^*) = \begin{pmatrix} A[c(1-t_1)-1] & -Ah \\ \beta k & -\beta u \end{pmatrix}$$

$$tr(J) = A[c(1-t_1)-1] + (-\beta u), \; det(J) = -\beta u A[c(1-t_1)-1] + Ah\beta k$$

Successively now, we have:

$$\left.\begin{array}{c} 0 < t_1 < 1 \\ 0 < c < 1 \end{array}\right\} \Rightarrow \left.\begin{array}{c} 0 > -t_1 > -1 \\ 0 < c < 1 \end{array}\right\} \Rightarrow \left.\begin{array}{c} 1 > 1 - t_1 > 0 \\ 1 > c > 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow c(1-t_1) < 1 \Rightarrow \left.\begin{array}{c} c(1-t_1) - 1 < 0 \\ A > 0 \end{array}\right\} \Rightarrow \left.\begin{array}{c} A[c(1-t_1) - 1] < 0 \\ \beta u > 0 \end{array}\right\}$$

$\Rightarrow tr(J) < 0.$

Similarly, we get $det(J) > 0 \Rightarrow (Y^*, r^*)$ local asymptotical stable

Step 3: Vertical Isocline (IS).

$$A[c(1-t_1)-1]Y - Ahr + aA = 0$$

The slope is $\dfrac{dr}{dY} = -\dfrac{[1 - c(1-t_1)]}{h} < 0$

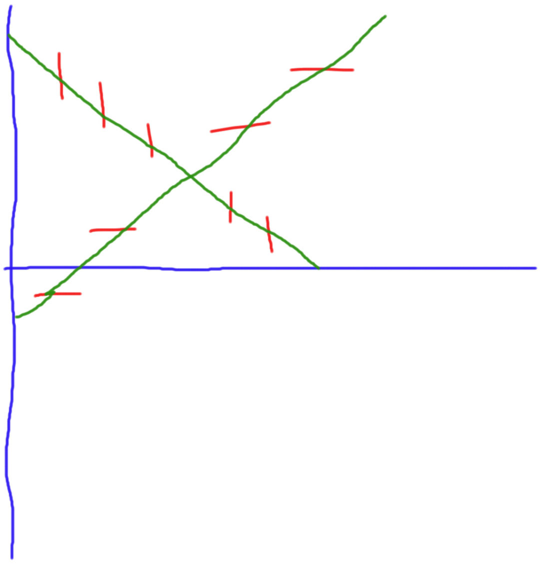
Intersection Points: $\left(0, \dfrac{a}{h}\right), \left(\dfrac{-a}{[c(1-t_1)-1]}, 0\right)$.

with $\dfrac{a}{h} > 0$ and $\dfrac{-a}{[c(1-t_1)-1]} > 0$

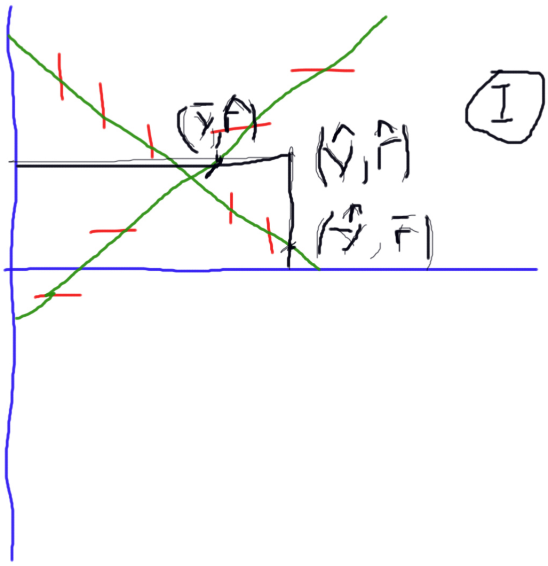
Horizontal Isocline (LM). $\beta kY - \beta ur - \beta m_0 = 0$

Slope: $\dfrac{dr}{dY} = \dfrac{k}{u} > 0.$

Intersection Points: $\left(0, -\dfrac{m_0}{u}\right)$ with $-\dfrac{m_0}{u} < 0$ and $\left(\dfrac{m_0}{k}, 0\right)$ with $\dfrac{m_0}{k} > 0$



Step 4. The Sign. We consider a point in region I, with coordinates $(\hat{Y}, \hat{r})$. Drawing vertical lines from this point, they met the vertical isocline (IS) at the point $(\hat{Y}, \bar{r})$ and the horizontal isocline (LM) at the point $(\bar{Y}, \hat{r})$.



Obviously, $\hat{r} > \bar{r} \Rightarrow -Ah\hat{r} < -Ah\bar{r} \Rightarrow$

$$A[c(1-t_1)-1]\hat{Y} - Ah\hat{r} + aA < A[c(1-t_1)-1]\hat{Y} - Ah\bar{r} + aA$$

Since, $(\hat{Y}, \bar{r})$ belongs to the vertical isocline we conclude that $A[c(1-t_1)-1]\hat{Y} - Ah\bar{r} + aA = 0$ and finally,

$$\boxed{Y'(\hat{Y}, \hat{r}) < 0}$$

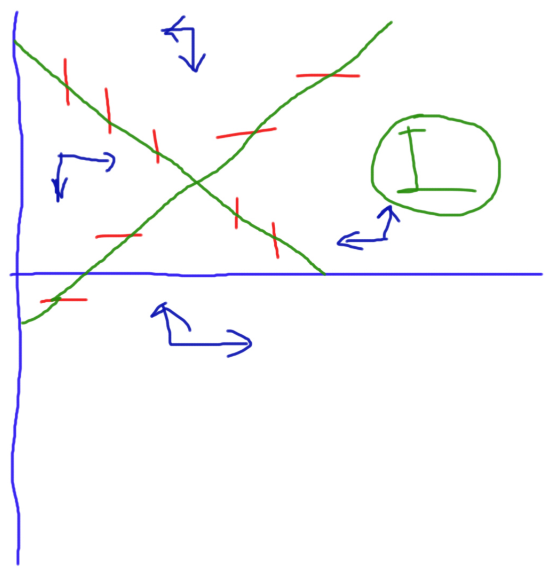
Obviously, $\hat{Y} > \bar{Y} \Rightarrow \beta k\hat{Y} > \beta k\bar{Y} \Rightarrow$

$$\beta k\hat{Y} - \beta u\hat{r} - \beta m_0 > \beta k\bar{Y} - \beta u\hat{r} - \beta m_0$$

Since, $(\bar{Y}, \hat{r})$ belongs to the horizontal isocline we conclude that $\beta k\bar{Y} - \beta u\hat{r} - \beta m_0 = 0$ and finally,

$$\boxed{r'(\hat{Y}, \hat{r}) > 0}$$

Working, similarly we get the next phase portrait, which implies asymptotic stability of the eq. point.



12.2 A General IS-LM model

12.2 Exercise: *By* $c(y)$ *we denote the consumption. It depends only from* y*. By* $i(r, y)$ *we denote the investments, depending from the income* y *and the rate* r*. By* g *we denote the goverment expenditure.* m^d *is the money* *demand function and* m_0 *the supply of real money balance. We suppose that the next relations hold:*

$$e = c(y) + i(r, y) + g \qquad m^d = l(y, r)$$
$$y' = A(e - y) \qquad r' = \beta(m^d - m^0)$$

with $0 < c_y < 1$, $i_r < 0$, $i_y > 0$, $l_y > 0$, $l_r < 0$, $A, \beta > 0$. *Prove that: if the slope of the IS curve is negative and less steep than LM, then the equilibrium point* (y^*, r^*) *is locally asymtpotical stable.*

Solution:

Step 0: The equilibrium point is (y^*, r^*) which is undetermined.

Step 1: The Jacobian is:

$$J = \begin{pmatrix} A(c_y + i_y - 1) & Ai_r \\ \beta l_y & \beta l_r \end{pmatrix}_{(y^*, r^*)}$$

$\boxed{tr(A) = A(c_y + i_y - 1) + \beta l_r}$, $\boxed{det(A) = -A\beta[l_r(1 - c_y - i_y) + i_r l_y]}$

Step 2: The vertical isocline (IS).

$$f(y, r) = 0 \Rightarrow e = y \Rightarrow c(y) + i(y, r) + g - y = 0$$

The slope is:

$$\frac{dr}{dy} = -\frac{F_y}{F_r} = \frac{1 - c_y - i_y}{i_r}$$

The horizontal isocline (LM).

$$g(y, r) = 0 \Rightarrow l = m_0 \Rightarrow l - m_0 = 0$$

The slope is:

$$\frac{dr}{dy} = -\frac{G_y}{G_r} = -\frac{l_y}{l_r}$$

Step 3: (The Proof.)

slope of IS $< 0 \Rightarrow \dfrac{1 - c_y - i_y}{i_r} < 0$ but $i_r < 0 \Rightarrow 1 - c_y - i_y > 0 \Rightarrow$ $c_y + i_y - 1 < 0 \Rightarrow \boxed{tr(J) < 0}$.

IS less steep than LM $\Rightarrow$

$$\frac{1 - c_y - i_y}{i_r} < -\frac{l_y}{l_r} \Rightarrow l_r(1 - c_y - i_y) + l_y i_r < 0$$

$\Leftrightarrow \boxed{det(J) > 0}$.

The above two relations imply the local asymptotical stability of the eq.point.

Chapter 13

Optimal Control

13.1 Basic Notions

- Optimal Control treats with the problem of maximizing or minimizing integrals under the presence of constraints.
- There two ways of approaching: The Maximum Principle analysis and The Dynamic Programming analysis.
- Differential equations are used to continuous-time analysis, difference equations are used to discrete-time analysis.

Example

$$\max_u \int_0^5 (y + 5y^2 + y' - 3u^2)dt$$

$$s.t. \quad y' = 5y + 6u - 1$$

$$y(0) = 3, \quad y(5) \quad \texttt{free}$$

13.2 Syllabus

1. Basic Terminology
2. One State - One Control-Maximum Principle

3. Explanation
4. Fixed End Points - Transversality conditions
5. Many States - Many Controls-Maximum Principle
6. Infinite Horizon- Phase Space Analysis
7. Dynamic Programming
8. Differential Games

13.3 The Simple Optimal Control Problem

Find a function $u(t)$ such that

$$\max_u \quad V = \int_{t_0}^{T} F(t, x(t), u(t))dt$$

$$\text{subject to} \quad x'(t) = f(t, x(t), u(t))$$

$$x(t_0) = x_0 \quad , \quad x(T) \quad \text{free}$$

$$u(t) \in \mathcal{U}, \quad \text{for all} \quad t \in [t_0, T]$$

$x(t)$, the state function, $x(t) : [t_0, T] \to \mathbf{R}$

$u(t)$, the control function, $u(t) : [t_0, T] \to \mathbf{R}$

$\mathcal{U}$, the control set, (compact and convex), usually $\mathcal{U} = [0, 1]$

$x(T)$ free, Terminal State Conditions.

13.4 The Alternative Expression

Find a function $u(t)$ such that

$$\max_u \quad V = \int_{t_0}^{T} F(t, x(t), u(t))dt + \phi(x(T), T)$$

$$\text{subject to} \quad x'(t) = f(t, x(t), u(t))$$

$$x(t_0) = x_0 \quad , \quad \Psi(x(T), T) = 0 \quad \text{free}$$

$$u(t) \in \mathcal{U}, \quad \text{for all} \quad t \in [t_0, T]$$

$\varphi(x(T), T)$ is a penalty function representing the cost associated with the value of the state variables at the terminal time T.

φ is a scalar function, $\Psi(x(T), T)$ is a p-dimensional vector of functions describing the terminal conditions.

13.5 Calculus of Variation is transformed to Optimal Control

$$\max \int_{t_0}^{t_1} f(t, x(t), x'(t))dt \qquad \Longleftrightarrow \qquad \max \int_{t_0}^{t_1} f(t, x(t), u(t))dt$$

$$s.t. \quad x(t_0) = x_0 \qquad\qquad s.t. \quad x'(t) = u(t)$$

$$x(t_0) = x_0$$

Chapter 14

The Maximum Principle

We define the Hamiltonian:

$$\mathcal{H}(t, x, u, \lambda) = F(t, x, u) + \lambda(t) f(t, x, u)$$

$\lambda(t)$ a function,

1st order conditions:

$$\boxed{\begin{array}{c} \max_u \mathcal{H}(t, x, u, \lambda) \quad , \quad t \in [t_0, T] \\ \\ x'(t) = \dfrac{\partial \mathcal{H}}{\partial \lambda} \quad , \quad \lambda'(t) = -\dfrac{\partial \mathcal{H}}{\partial x} \quad , \quad \lambda(T) = 0 \end{array}}$$

2nd order conditions: $F(t, x, u)$ to be concave

14.1 Exercise: *Solve the problem:*

$$\max \int_0^1 (x + u) dt$$

$$s.t. \quad x' = 1 - u^2$$

$$x(0) = 1$$

Solution:

$$\mathcal{H} = (x + u) + \lambda(t)(1 - u^2)$$

Maximization of $\mathcal{H}$

$$\left.\begin{array}{l} \frac{\partial \mathcal{H}}{\partial u} = 1 - 2\lambda u = 0 \\ \frac{\partial^2 \mathcal{H}}{\partial u^2} = -2\lambda < 0 \end{array}\right\} \Rightarrow \boxed{u^*(t) = \frac{1}{\lambda(t)}}$$

Calculation of $\lambda(t)$

$$\lambda'(t) = -\frac{\partial \mathcal{H}}{\partial x} \Rightarrow \lambda'(t) = -1 \Rightarrow \lambda(t) = -t + c_1$$

$$\lambda(1) = 0 \Rightarrow c_1 = 1 \Rightarrow \boxed{\lambda(t) = 1 - t} \Rightarrow \boxed{u^*(t) = \frac{1}{2(1-t)}}$$

Calculation of $x^*(t)$

$$x'(t) = 1 - u^2 \Rightarrow x'(t) = 1 - \left(\frac{1}{2(1-t)}\right)^2 \Rightarrow$$

$$x^*(t) = \int \left[1 - \left(\frac{1}{2(1-t)}\right)^2\right] dt = t - \frac{1}{4(1-t)} + C$$

$$x(0) = 1 \Rightarrow C = 5/4 \Rightarrow \boxed{x^*(t) = t - \frac{1}{4(1-t)} + \frac{5}{4}}$$

14.2 Exercise: *Find the curve with the shortest distance from a given point $(0, A)$ to a given vertical straight line.*

Solution: Let $y(t)$ be an arbitrary curve such that $y(0) = A$ and $y(T)$ free. The distance is given by the quantity:

$$d = \int_0^T \sqrt{1 + \left(\frac{dy}{dt}\right)^2} dt$$

so our problem becomes:

$$\boxed{\begin{array}{c} \min \int_0^T \sqrt{1 + (y')^2} dt \\ \text{s.t.} \quad y(0) = A, \quad y(T) \text{free} \end{array}}$$

By setting $y' = u$, the problem is transformed to the next optimal control problem:

$$\boxed{\begin{array}{c} \max_u - \int_0^T \sqrt{1+u^2}dt \\ \\ \text{s.t.} \quad y'(x) = u \\ y(0) = A, \quad y(T) \quad \text{free} \end{array}}$$

The Hamiltonian is: $\mathcal{H} = -\sqrt{1+u^2} + \lambda(t)u$ We have to maximize $\mathcal{H}$:

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \Rightarrow -\frac{1}{2}(1+u^2)^{-1/2}2u + \lambda = 0$$

$$\frac{\partial^2 \mathcal{H}}{\partial u^2} = -(1+u^2)^{-3/2}$$

$\Rightarrow u^*(t) = \lambda(1-\lambda^2)^{-1/2}$

But $\dot{\lambda} = -\dfrac{\partial \mathcal{H}}{\partial y} = 0 \Rightarrow \dot{\lambda} = 0 \Rightarrow \lambda(t) = C$

But $\lambda(T) = 0 \Rightarrow \lambda(t) = 0 \Rightarrow \boxed{u^*(t) = 0}$

$\Rightarrow y'(t) = 0 \Rightarrow y^*(t) = K$

Since $y(0) = A \Rightarrow \boxed{y^*(t) = A}$

So, the requested curve is the perpendicular straight line from the point to the line.

14.3 Exercise: *Solve the problem:*

$$\max_u \int_0^5 (ux - u^2 - x^2)dt$$

$$s.t. \quad x' = x + u$$

$$x(1) = 2, \quad x(5) = 0$$

Solution: The Hamiltonian is:

$$\mathcal{H} = (ux - u^2 - x^2) + \lambda(x + u)$$

To maximize $\mathcal{H}$ we have successively:

$$\left.\begin{array}{l} \frac{\partial \mathcal{H}}{\partial u} = x - 2u + \lambda = 0 \\ \\ \frac{\partial^2 \mathcal{H}}{\partial u^2} = -2 < 0 \end{array}\right\} \Rightarrow \boxed{u^*(t) = \frac{x(t)+\lambda(t)}{2}}$$

Calculation of $\lambda(t)$

$$\lambda'(t) = -\frac{\partial \mathcal{H}}{\partial x} \Rightarrow \lambda'(t) = -u^* + 2x - \lambda \Rightarrow \boxed{2\lambda'(t) = x - 2\lambda}$$

But, $x'(t) = x + u^* = x + \frac{x+\lambda}{2} \Rightarrow \boxed{2x'(t) = 3x + \lambda}$

By solving this system we get:

$$\boxed{\lambda^*(t) = \frac{e^{\frac{1}{4}\left(t-\sqrt{29}t\right)}\left(2c_1\left(e^{\frac{\sqrt{29}t}{2}}-1\right)+c_2\left(\left(\sqrt{29}-5\right)e^{\frac{\sqrt{29}t}{2}}+\sqrt{29}+5\right)\right)}{2\sqrt{29}}}$$

$$\boxed{x^*(t) = \frac{e^{\frac{1}{4}\left(t-\sqrt{29}t\right)}\left(c_1\left(\left(5+\sqrt{29}\right)e^{\frac{\sqrt{29}t}{2}}+\sqrt{29}-5\right)+2c_2\left(e^{\frac{\sqrt{29}t}{2}}-1\right)\right)}{2\sqrt{29}}}$$

By using the initial conditions $x(1) = 2$, $x(5) = 0$ we calculate the constants. Finally:

$$\boxed{u^*(t) = \frac{x^*(t)+\lambda^*(t)}{2}}$$

14.4 Exercise: *Solve the problem:*

$$\max_u \int_0^2 (2y - 3u)dt$$

$$s.t. \quad y' = y + u$$

$$y(0) = 4, \quad y(2) \quad free$$

$$and \quad u(t) \in \mathcal{U} = [0, 2]$$

Solution: The Hamiltonian is:

$$\mathcal{H} = (2y(t) - 3u(t)) + \lambda(t)(y(t) + u(t))$$

Step 1: Calculation of u^*

$$\frac{\partial \mathcal{H}}{\partial u} = \lambda - 3 = \begin{cases} > 0 & \lambda > 3 \quad so \quad \mathcal{H} \uparrow \quad \Rightarrow \quad u^* = 2 \\ < 0 & \lambda < 3 \quad so \quad \mathcal{H} \downarrow \quad \Rightarrow \quad u^* = 0 \end{cases}$$

$$\Rightarrow \boxed{u^*(t) = \begin{cases} 2, & \lambda > 3 \\ 0, & \lambda < 3 \end{cases}}$$

Step 2: Calculation of λ

$$\lambda' = -\frac{\partial \mathcal{H}}{\partial y} \Rightarrow \lambda' = -\lambda - 2 \Rightarrow \lambda' + \lambda = -2 \Rightarrow \lambda(t) = ke^{-t-2}$$

Using the condition $\lambda(2) = 0$ we get: $\boxed{\lambda^*(t) = 2e^{2-t} - 2}$

Step 3: Further Calculation of u^*

$\lambda^*(t)$ is a decreasing function $\Rightarrow \lambda^*(2) \leq \lambda(t) \leq \lambda^*(0) \Leftrightarrow 0 \leq \lambda(t) \leq 12.778$ there is a number $\tau : \lambda^*(\tau) = 3 \Rightarrow \tau \approx 1.084 \Rightarrow$

$$u^*(t) = \begin{cases} 2, & 0 \leq t \leq \tau \\ 0, & \tau \leq t \leq 2 \end{cases}$$

Step 4: Calculation of $y(t)$

$$y' = y + u \Rightarrow \begin{cases} y' = y + 2, & y(0) = 4, & 0 \leq t \leq \tau \\ y' = y, & y(0) = 4,, & \tau \leq t \leq 2 \end{cases}$$

$\Rightarrow$

$$\boxed{y^*(t) = \begin{cases} 2(3e^{t} - 1), & 0 \leq t \leq \tau \\ 5.324e^{t}, & \tau \leq t \leq 2 \end{cases}}$$

Chapter 15

Maximum Principle Explanation

We know that $\dot{x} = f(x, u, t) \Rightarrow \dot{x} - f(x, u, t) = 0 \Rightarrow \lambda(t)[\dot{x} - f(x, u, t)] = 0$

For the objective function we have:

$$\int_0^T I(x, u, t)dt = \int_0^T \{I(x, u, t) + \lambda(t)[f(x, u, t) - \dot{x}]\}\, dt =$$

$$= \int_0^T I(x, u, t)dt + \int_0^T \lambda(t)f(x, u, t)dt - \int_0^T \lambda(t)\dot{x}dt$$

Using integration by parts we get:

$$\int_0^T \lambda(t)\dot{x}dt = \lambda(t)x(t)]_0^T - \int_0^T \dot{\lambda}(t)xdt =$$

$$= \lambda(T)x(T) - \lambda(0)x(0) - \int_0^T \dot{\lambda}(t)xdt$$

So, the objective function becomes:

$$\int_0^T [I(x, u, t) + \lambda f(x, u, t) + \dot{\lambda}x]dt - \lambda(T)x(T) + \lambda(0)x(0)$$

Pertubations

Let $u(t) = u^*(t) + \epsilon h(t)$, where $u^*(t)$ is the requested optimal control, $h(t)$ any function and ϵ a small number.

Let $x(t, \epsilon)$ denote the state variable, generated by the state equation with control: $u(t) = u^*(t) + \epsilon h(t)$, $0 \leq t \leq T$. Hence,

$$x(t, 0) = x^*(t), \quad x(T, \epsilon) = x(T), \quad x(0, \epsilon) = x_0$$

Thus

$$J(\epsilon) = \int_0^T \underbrace{[I(x(t,\epsilon), u^* + \epsilon h, t) + \lambda f((x(t,\epsilon), u^* + \epsilon h, t) + \dot{\lambda} x(t,\epsilon)]}_{\Omega} dt -$$

$$-\lambda(T)x(T,\epsilon) + \lambda(0)x(0,\epsilon) \Rightarrow$$

$$\Rightarrow J(\epsilon) = \int_0^T \Omega dt - \lambda(T)x(T,\epsilon) + \lambda(0)x(0,\epsilon)$$

Since u^* is a maximizing control, the function $J(\epsilon)$ assumes its maximum at $\epsilon = 0 \Rightarrow \left(\frac{dJ}{d\epsilon}\right)(0) = 0$

$$\frac{dJ}{d\epsilon} = \int_0^T \left[\frac{\partial \Omega}{\partial t} \cdot \frac{\partial t}{\partial \epsilon} + \frac{\partial \Omega}{\partial x} \cdot \frac{\partial x}{\partial \epsilon} + \frac{\partial \Omega}{\partial u} \cdot \frac{\partial u}{\partial \epsilon}\right] dt -$$

$$-\lambda(T)\frac{\partial x(T,\epsilon)}{\partial \epsilon} + \lambda(0)\frac{\partial x(0,\epsilon)}{\partial \epsilon}$$

∂But

$$\frac{\partial t}{\partial \epsilon} = 0 \quad , \quad \frac{\partial x(0,\epsilon)}{\partial \epsilon} = \frac{\partial x_0}{\partial \epsilon} = 0$$

Hence,

$$\frac{dJ(0)}{d\epsilon} = \int_0^T \left[[I_x + \lambda_x f + \lambda f_x + \dot{\lambda}] \left(\frac{\partial x}{\partial \epsilon}\right)_{\epsilon=0} + \right.$$

$$\left. + [I_u + \lambda_u f + \lambda f_u + 0] \left(\frac{\partial u}{\partial \epsilon}\right)_{\epsilon=0} \right] dt - \lambda(T)\frac{\partial x(T,0)}{\partial \epsilon} = 0$$

But, $\lambda_x = 0$, $\lambda_u = 0$, $\frac{\partial u}{\partial \epsilon} = h$ and so:

$$\int_0^T \left[[I_x + \lambda f_x + \dot{\lambda}] \left(\frac{\partial x}{\partial \epsilon}\right)_{\epsilon=0} + [I_u + \lambda f_u] h \right] dt - \lambda(T)\frac{\partial x(T,0)}{\partial \epsilon} = 0$$

The latter becomes equal to zero iff

$$\left.\begin{array}{c} I_x + \lambda f_x + \dot{\lambda} = 0 \\ I_u + \lambda f_u = 0 \\ \lambda(T) = 0 \\ \mathcal{H} = I + \lambda f \end{array}\right\} \Rightarrow \begin{array}{l} \dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial x} \\ \frac{\partial \mathcal{H}}{\partial u} = 0 \quad \Leftrightarrow \mathcal{H} \quad maximum \\ \lambda(T) = 0 \end{array}$$

Interpretation of $\lambda(t)$

Let x^*, u^* be the state and the control functions providing the maximum and let $\lambda^*(t)$ be the corresponding multiplier.

We define: $V(x_0, t_0) = \int_{t_0}^{T} F(t, x^*, u^*)dt$, then:

$$\boxed{\lambda^*(t_0) = \frac{\partial V}{\partial x^*}(x_0, t_0)}$$

Chapter 16

Transversality Conditions

Boundary Condition	Transversality Condition
$x(T)$ free	$\lambda(T) = 0$
$x(T) = k$	$\lambda(T) = b$, b to be determined
$x(T) \geq k$	$\lambda(T)[k - x^*(T)] = 0$, $\lambda(T) \geq 0$
$x(T) \leq k$	$\lambda(T)[k - x^*(T)] = 0$, $\lambda(T) \leq 0$

16.1 Exercise: *Solve the problem:*

$$\max_u \int_0^2 (-x)dt$$

$$s.t. \quad x' = u$$

$$x(0) = 1, \quad x(2) \geq 0$$

$$and \quad u(t) \in \mathcal{U} = [-1, 1]$$

Solution: The Hamiltonian is: $\mathcal{H} = -x + \lambda u$

Step 1: Calculation of u^*

$$\frac{\partial \mathcal{H}}{\partial u} = \lambda \Rightarrow u = \begin{cases} -1, & \lambda < 0 \\ ?, & \lambda = 0 \\ 1 & \lambda > 0 \end{cases}$$

Step 2: Calculation of λ

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial x} \Rightarrow \dot{\lambda} = 1 \Rightarrow \boxed{\lambda(t) = t + c_1}$$

hence,

$$u = \begin{cases} -1, & 0 \leq t \leq t^* \\ 1, & t^* < t \leq 2 \end{cases}$$

t^* to be determined. Obviously:

$$\lambda(t^*) = 0 \Rightarrow 0 = t^* + c_1 \Rightarrow c_1 = -t^* \Rightarrow$$

$$\Rightarrow \boxed{\lambda(t) = t - t^*}$$

Step 3: Calculation of $x(t)$

If $0 \leq t \leq t^*$ then $u(t) = -1$ and $x'(t) = -1 \Rightarrow x(t) = -t + c_2$. But $x(0) = 1 \Rightarrow c_2 = 1$ and finally:

$$\boxed{x(t) = 1 - t\,, \quad \text{for } 0 \leq t \leq t^*}$$

If $t^* \leq t \leq 2$ then $u(t) = 1$ and $x'(t) = 1 \Rightarrow x(t) = t + c_3$. But $x(t^*) = x^*$ $\Rightarrow c_3 = x^* - t^*$ and finally:

$$\boxed{x(t) = t - t^* + x^*\,, \quad \text{for } t^* \leq t \leq 2}$$

But $x(t) = 1 - t \Rightarrow x^* = x(t^*) = 1 - t^* \Rightarrow$

$$\boxed{x(t) = t - t^* + x^* = t + 1 - 2t^*, \quad \text{for } t^* < t \leq 2}$$

Step 4:Transversality Condition

$\lambda(2)[0 - x^*(2)] = 0$. But $\lambda(2) = 2 - t^* \neq 0$ which means $\boxed{x^*(2) = 0}$.

$$x^*(2) = 0 \Rightarrow 0 = 3 - 2t^* \Rightarrow t^* = 3/2$$

$$\Rightarrow u(t) = \begin{cases} -1, & 0 \leq t \leq 3/2 \\ 1, & 3/2 < t \leq 2 \end{cases}$$

Chapter 17

The General Optimal Control Problem

17.1 The Description

Find the functions $u_1(t), u_2(t), \ldots, u_m(t)$ such that

$$\max_{u_1, u_2, \ldots, u_m} \quad V = \int_{t_0}^{T} F(t, x_1(t), x_2(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t))dt$$

subject to

$$x_1'(t) = f_1(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t))$$

$$x_2'(t) = f_2(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t))$$

$$\vdots$$

$$x_n'(t) = f_n(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t))$$

$$x_1(t_0) = x_{01}, x_2(t_0) = x_{02}, \ldots, x_n(t_0) = x_{0n}$$

$$x_1(T), x_2(T), \ldots, x_n(T) \qquad \text{free}$$

$$u_1(t), u_2(t), \ldots, u_m(t) \in \mathcal{U}, \quad \text{for all} \quad t \in [t_0, T]$$

Or, using vectors notation

Find the vector function $\vec{u}(t)$ such that

$$\max_{\vec{u}} \quad V = \int_{t_0}^{T} F(t, \vec{x}(t), \vec{u}(t))dt$$

subject to

$$\vec{x}'(t) = \vec{f}(t, \vec{x}(t), \vec{u}(t))$$

$$\vec{x}(0) = \vec{x_0}, \qquad \vec{x}(T) \qquad \text{free}$$

$$\vec{u}(t) \in \mathcal{U} \subset \mathbf{R}^m, \quad \text{for all} \quad t \in [t_0, T]. \quad \text{(compact and convex)}$$

17.2 The General Maximum Principle

We define the Hamiltonian:

$$\mathcal{H}(t, \vec{x}, \vec{u}, \vec{\lambda}) = F(t, \vec{x}, \vec{u}) + \vec{\lambda}(t) \cdot \vec{f}(t, \vec{x}, \vec{u})$$

$\vec{\lambda}(t)$ a vector function, and $\cdot$ the inner-product.

1st order conditions:

$$\max_{\vec{u}} \mathcal{H}(t, \vec{x}, \vec{u}, \vec{\lambda}) \quad , \quad t \in [t_0, T]$$

$$\frac{dx_i}{dt} = \frac{\partial \mathcal{H}}{\partial \lambda_i}, \quad \frac{d\lambda_i}{dt} = -\frac{\partial \mathcal{H}}{\partial x_i}, \quad \vec{\lambda}(T) = 0, \quad i = 1, \dots, n$$

2nd order conditions: $F(t, \vec{x}, \vec{u})$ to be concave

ATTENTION

<u>Remark:</u> The above maximization is a multivariable one

17.1 Exercise: *Solve the problem:*

$$\min_{u} \int_{t_0}^{t_1} dt$$

$$s.t. \quad x_1' = x_2$$

$$x_2' = u$$

$$x(t_0) = x_0, \quad x(t_1) \quad \textit{free}$$

$$and \quad u(t) \in \mathcal{U} = [-1, 1]$$

Solution: First, we change the objective function:

$$\max_u \int_{t_0}^{t_1} (-1)dt$$

The Hamiltonian is:

$$\mathcal{H} = -1 + \lambda_1(t)x_2 + \lambda_2(t)u$$

Step 1: Calculation of u^*

$$\frac{\partial \mathcal{H}}{\partial u} = \lambda_2 \Rightarrow u^* = \begin{cases} -1, & \lambda_2 < 0 \\ 1 & \lambda_2 \geq 0 \end{cases}$$

Step 2: Calculation of λ's

$$\frac{d\lambda_1}{dt} = -\frac{\partial \mathcal{H}}{\partial x_1} = 0 \quad \Rightarrow \quad \lambda_1 = c_1$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial \mathcal{H}}{\partial x_2} = -\lambda_1 \quad \Rightarrow \quad \lambda_2 = c_2 - c_1 t$$

thus

$$u^* = \begin{cases} -1, & \lambda_2 < 0 \\ 1 & \lambda_2 \geq 0 \end{cases} \quad \Rightarrow \quad u^* = \begin{cases} -1, & t > c_2/c_1 \\ 1 & t \leq c_2/c_1 \end{cases}$$

Step 3: Calculation of x's

Let $u^* = 1$, $\Rightarrow$

$$x_2' = 1 \quad \Rightarrow \quad x_2 = t + k_2$$

$$x_1' = x_2 \quad \Rightarrow \quad x_1 = \frac{t^2}{2} + k_2 t + k_1$$

Let $u^* = -1$, $\Rightarrow$

$$x_2' = -1 \quad \Rightarrow \quad x_2 = -t + k_2$$

$$x_1' = x_2 \quad \Rightarrow \quad x_1 = -\frac{t^2}{2} - k_2 t + k_1$$

Step 4: The Phase Space

The $u = 1$ orbits $\Rightarrow \frac{dx_2}{dx_1} = \frac{1}{x_2} \Rightarrow \boxed{x_1 = \frac{1}{2}x_2^2 + C}$

The $u = -1$ orbits $\Rightarrow \frac{dx_2}{dx_1} = -\frac{1}{x_2} \Rightarrow \boxed{x_1 = -\frac{1}{2}x_2^2 + \hat{C}}$

17.2 Exercise: *We have the production function* $Q = AK^{1-a}R^a$, $0 < a < 1$, *A a constant.*

$K(t)$: *is the capital function.*

$R(t)$: *an extractive resource.*

The product may be consumed or invested.

$C(t)$: *the consumption function.*

$X(t)$: *the remaining stock of the extractive resource.*

$U(C) = \ln C$ *the utility function.*

Initial conditions: $K(0) = K_0$, $K(T) = 0$, $X(0) = X_0$, $X(T) = 0$.

We wish to find a policy which maximizes the total utility over a period $[0, T]$

Solution: We have the problem:

$$\max_{C,R} \int_0^T \ln C \quad dt$$

$$s.t. \quad X' = -R$$

$$K' = AK^{1-a}R^a - C$$

$$X(0) = X_0, \quad X(T) = 0, \quad K(0) = K_0, \quad K(T) = 0$$

$$C \geq 0, \quad R(t) \geq 0$$

state variables: K, X, control variables: C, R

Transformation of the Problem. It is convenient to define: $y(t) = R/K$, the ratio of resource to capital. The problem becomes:

$$\max_{C,y} \int_0^T \ln C \quad dt$$

$$s.t. \quad X' = -Ky$$

$$K' = AKy^a - C$$

$$X(0) = X_0, \quad X(T) = 0, \quad K(0) = K_0, \quad K(T) = 0$$

$$C \geq 0, \quad y(t) \geq 0$$

state variables: K, X, control variables: C, y

Step 1: The Hamiltonian

$$\mathcal{H} = \ln C - \lambda_1 Ky + \lambda_2(AKy^a - C)$$

Step 2: Maximization of the Hamiltonian

$$\frac{\partial \mathcal{H}}{\partial C} = 0 \quad , \quad \frac{\partial \mathcal{H}}{\partial y} = 0 \quad \Rightarrow\Rightarrow$$

$$\frac{1}{C} - \lambda_2 = 0 \tag{17.1}$$

$$-\lambda_1 K + \lambda_2 AKay^{a-1} = 0 \tag{17.2}$$

Step 4: The λ-equations

$$\lambda_1' = -\frac{\partial \mathcal{H}}{\partial X} \Rightarrow \boxed{\lambda_1' = 0} \tag{17.3}$$

$$\lambda_2' = -\frac{\partial \mathcal{H}}{\partial K} \Rightarrow \boxed{\lambda_2' = \lambda_1 y - \lambda_2 Ay^a} \tag{17.4}$$

Step 5: The Manipulations

A From (17.3) we have:

$$\lambda_1' = 0 \Rightarrow \lambda_1(t) = c \underbrace{\Longrightarrow}_{\lambda_1(T)=0} \boxed{\lambda_1(t) = 0} \tag{17.5}$$

B From (17.2) and (17.5) we have:

$$\lambda_2 y^{a-1} = 0 \Rightarrow [\lambda_2 y^{a-1}]' = 0 \Rightarrow$$

$$\lambda_2' y^{a-1} + \lambda_2(a-1)y^{a-2}y' = 0 \Rightarrow \boxed{\frac{\lambda_2'}{\lambda_2} = (1-a)\frac{y'}{y}} \tag{17.6}$$

C From (17.2) we have: $\lambda_1 = \lambda_2 Aay^{a-1}$. Substituting into (17.4) we get:

$$\lambda_2' = \lambda_2 Aay^{a-1}y - \lambda_2 Ay^a \Rightarrow \boxed{\frac{\lambda_2'}{\lambda_2} = -(1-a)Ay^a} \tag{17.7}$$

D From (17.6) and (17.7) we have:

$$y' = -Ay^{a+1} \Rightarrow y^a = \frac{1}{A(c_1 + at)} \tag{17.8}$$

E From (17.7) and (17.8) we have:

$$\boxed{\hat{\lambda}_2(t) = \frac{(c_1 + \alpha t)^{\frac{\alpha-1}{\alpha}}}{c_2}} \tag{17.9}$$

Z Finally, from (17.1) and (17.9) we get:

$$C(t) = c_2(c_1 + \alpha t)^{\frac{\alpha-1}{\alpha}} \tag{17.10}$$

With further calculations we can find $X(t)$ and $K(t)$.

Chapter 18

Current Value Hamiltonian

The Problem:

$$\max_u \int_0^T \varphi(x,t,u)e^{-\rho t}dt$$

$$\text{s.t. } x' = g(t,x,u), \quad x(0) = x_0$$

$\rho \geq 0$ be a constant continuous discount rate.

We define the Hamiltonian: $H = \varphi(x,t,u) + \lambda(t)g(t,x,u)$, $\lambda(t)$ a function,

1st order conditions:

$$\boxed{\begin{array}{c} \max_u H(t,x,u,\lambda) \quad , \quad t \in [0,T] \\ x'(t) = \dfrac{\partial \mathcal{H}}{\partial \lambda} \quad , \quad \lambda'(t) = \rho\lambda - \dfrac{\partial \mathcal{H}}{\partial x} \quad , \quad \lambda(T) = 0 \end{array}}$$

Explanation: $H^s = \varphi(x,t,u)e^{-\rho t} + \lambda^s g(t,x,u)$,

maximize H^s, $\dot{\lambda}^s = -\partial H^s/\partial x$, $\lambda^s(T) = 0$

Let $\lambda = e^{\rho t}\lambda^s \Rightarrow \lambda^s = e^{-\rho t}\lambda$

$H^s = e^{-\rho t}(\varphi(x,t,u) + \lambda g(x,u,t)) = e^{-\rho t}H \Rightarrow \max H^s \iff \max H$

$\lambda = e^{\rho t}\lambda^s \Rightarrow \dot{\lambda} = \rho e^{\rho t}\lambda^s + e^{\rho t}\dot{\lambda}^s = \rho e^{\rho t}\lambda^s + e^{\rho t}\left(-e^{-\rho t}\dfrac{\partial H}{\partial x}\right) = \rho\lambda - \dfrac{\partial H}{\partial x}$

$\lambda^s(T) = 0 \Rightarrow \lambda(T) = 0$

18.1 Exercise: *Solve the following consumption problem:*

$$\max_{C(t)} \int_0^T e^{-\rho t} \ln C(t) dt$$

$$s.t. \quad \dot{W}(t) = \rho W(t) - C(t)$$

$$W(0) = W_0 > 0 \quad , \quad W(T) = 0$$

Solution: The current-value Hamiltonean is:

$$H = \ln C(t) + \lambda(t)(\rho W(t) - C(t))$$

The adjoint equation is:

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial W} = \rho\lambda - \rho\lambda = 0$$

Transversality condition: $\lambda(T) = \beta$, β to be determined.

$$\dot{\lambda} = 0 \Rightarrow \lambda(t) = constant \Rightarrow \lambda(t) = \beta, \quad t \in [0, T]$$

Maximization of H:

$$\frac{\partial H}{\partial C} = 0 \Rightarrow \frac{1}{C} - \lambda = 0 \Rightarrow C = \frac{1}{\lambda} = \frac{1}{\beta}$$

$$\dot{W} = \rho W - \frac{1}{\beta} \Rightarrow W(t) = W_0 e^{\rho t} - \frac{1}{\beta\rho}(e^{\rho t} - 1)$$

$$W(T) = 0 \Rightarrow \beta = \frac{1 - e^{-\rho T}}{\rho W_0} \Rightarrow \boxed{C^*(t) = \frac{\rho W_0}{1 - e^{-\rho T}}}$$

18.2 Exercise: *Solve the following consumption problem:*

$$\max_{C(t)} \int_0^T e^{-0.05t} \ln C(t) dt$$

$$s.t. \quad \dot{s}(t) = q(t) - C(t)$$

$$s(0) = s_0 > 0 \quad , \quad s(T) = s_T$$

where, $c(t)$ *the consumption,* $s(t)$ *the stock of capital and* $q(t) = 2s^{1/2}$*, the output.*

Solution: The current-value Hamiltonian is:

$$H = \ln C(t) + \lambda(t)(2s^{1/2} - C(t))$$

The adjoint equations are:

$$\dot{\lambda} = 0.05\lambda - \frac{\partial H}{\partial s}, \quad s' = 2s^{1/2} - C(t), \quad \lambda(T) = 0$$

Step 1: $\max_{C(t)} H \Rightarrow \frac{\partial H}{\partial C} = \frac{1}{C} - \lambda = 0 \Rightarrow \boxed{C = \frac{1}{\lambda}}$

Step 2: $\dot{\lambda} = 0.05\lambda - \frac{\partial H}{\partial s} = 0.05\lambda - \lambda \cdot 2 \cdot \frac{1}{2} \cdot s^{-1/2} = \lambda(0.05 - s^{-1/2})$

$\lambda = \frac{1}{C} \Rightarrow \lambda' = -\frac{C'}{C^2} \Rightarrow$

$$-\frac{C'}{C^2} = \frac{1}{C}(0.05 - s^{-1/2}) \Rightarrow C' = C(s^{-1/2} - 0.05)$$

Step 3: The system:

$$s' = 2s^{1/2} - C$$

$$C' = C(s^{-1/2} - 0.05)$$

We use phase-space analysis

Step 4: The equilibrium point: $(s, c) = (400, 40)$

Step 5: The stability of the eq. point.

$$J(s, C) = \begin{pmatrix} s^{-1/2} & -1 \\ C(-1/2)s^{-3/2} & s^{-1/2} - 0.05 \end{pmatrix} \Rightarrow$$

$$J(400, 40) = \begin{pmatrix} \frac{1}{20} & -1 \\ -\frac{1}{400} & 0 \end{pmatrix} \Rightarrow eigen = \frac{1}{40}\left(1 + \sqrt{5}\right), \frac{1}{40}\left(1 - \sqrt{5}\right)$$

so, we have instability.

Step 6: Isoclines

Horizontal: $C(s^{-1/2} - 0.05) = 0 \Rightarrow C = 0, s = 400$

Vertical : $2s^{1/2} - C = 0 \Rightarrow C = 2s^{1/2}$

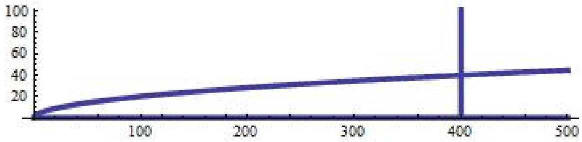


Figure 18.1:

Step 7: We examine the sign at the several regions:

Region	**Point**	$2s^{1/2} - C$	$C(s^{-1/2} - 0.05)$	s'	C'	s	C
I	$(300, 60)$	$-25, \ldots$	0.46	$-$	$+$	$\leftarrow$	$\uparrow$
II	$(450, 80)$	$-37.5..$	-0.2287	$-$	$-$	$\leftarrow$	$\downarrow$
III	$(450, 20)$	$22\ldots$	-0.06	$+$	$-$	$\rightarrow$	$\downarrow$
IV	$(300, 10)$	$+24.641$	0.077	$+$	$+$	$\rightarrow$	$\uparrow$

Step 8: We sketch the phase portrait.....

Step 9 Conclusion: Everything depends from the initial condition s_0, if, for instance $s_0 > 400$ and $C < 40$, then we must decrease C to have an optimum path, as s increases.

Chapter 19

The Infinite Horizon Case

Since the time is infinite we can discover steady (or equilibrium) states.

Each problem must be faced independently.

-1-

Exercise: $x(t)$: a productive capital

$P(x)$: The profit rate

$u(t)$: The investment rate

$C(u)$: The investment cost

Assumptions: $P'(0)>0$, $P''(x)<0$

$C(u)$ convex , $C'(u)>0$, $C''(u)>0$, $C'(0)=0$

Find the investment rate $u(t)$ that maximizes the present value of the profit stream over the infinite period.

Solution. The problem is:

$$\max \int_0^{+\infty} e^{-rt}\left[P(x)-C(u)\right]dt$$

$$\text{s.t.} \quad x' = u - bx$$

$$x(0) = x_0 > 0$$

$$u(t) \geq 0 \ , \ b, r > 0$$

001.jpg

-2-

Hamiltonean

$$\mathcal{H} = e^{-rt}\left[P(x) - C(u)\right] + \lambda(t)(u - bx)$$

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial x} = -e^{-rt}P'(x) + b\lambda \quad (1)$$

$$\dot{x} = u - bx \quad (2)$$

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \Rightarrow -e^{-rt}C'(u) + \lambda = 0 \quad (3)$$

Manipulations

$$(1) \Rightarrow e^{rt}\dot{\lambda} = -P'(x) + be^{rt}\lambda$$

$$(3) \Rightarrow e^{rt}\lambda = C'(u)$$

$$\Rightarrow e^{rt}\dot{\lambda} = -P'(x) + bC'(u)$$

$$\Rightarrow re^{rt}\lambda + e^{rt}\dot{\lambda} = C''(u) \Rightarrow$$

$$\Rightarrow rC'(u) + e^{rt}\dot{\lambda} = C''(u)\dot{u}$$

-3-

$$\Rightarrow C''(u)\dot{u} = -P'(x) + (b+r)C'(u)$$

$$\Rightarrow \boxed{\begin{aligned} \dot{u} &= \frac{(b+r)C'(u) - P'(x)}{C''(u)} \\ \dot{x} &= u - bx \end{aligned}}$$

Equil. Point Since we have an infinite time period, there is an equilibrium point (at least), (x_s, u_s)

$$\Downarrow$$

$$\frac{(b+r)C'(u_s) - P'(x_s)}{C''(u_s)} = 0$$

$$u_s - bx_s = 0$$

-4-

Vertical isocline

$$x' = 0 \Rightarrow \boxed{u = bx}$$

Horizontal isocline

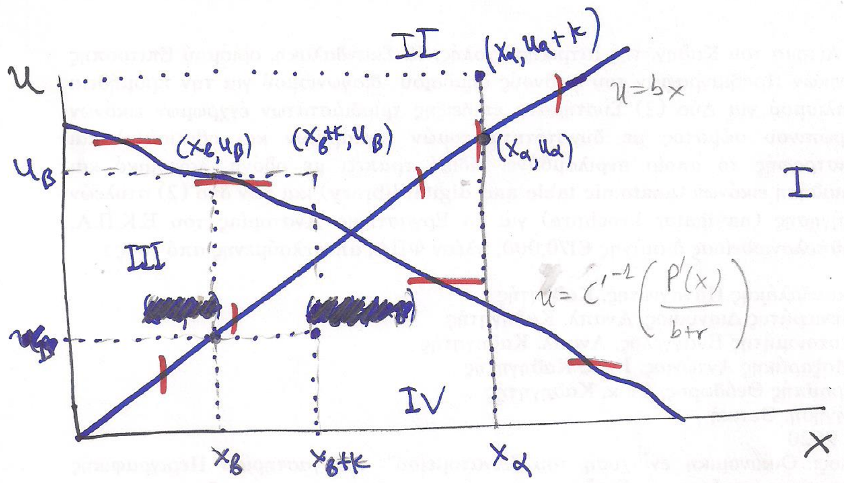
$$(b+r)\,C'(u) - P'(x) = 0 \Rightarrow$$

$$\Rightarrow C'(u) = \frac{P'(x)}{b+r} \xrightarrow{\text{since } C' \text{ is monotone}}$$

$$\Rightarrow \boxed{u = C'^{-1}\left(\frac{P'(x)}{b+r}\right)}$$

$$\frac{du}{dx} = \left[C'^{-1}(\ldots)\right]' \cdot \frac{P''(x)}{b+r} =$$

$$= \frac{1}{C''(\ldots)} \cdot \frac{P''(x)}{b+r} < 0$$ a downward sloping curve



Sign with respect to the vertical locus.

Let $(x_\alpha, u_\alpha+k)$ above the vertical locus

(x_α, u_α) on the vertical locus

$u_\alpha+k > u_\alpha \Rightarrow u_\alpha+k-bx_\alpha > u_\alpha - bx_\alpha \Rightarrow$

$\Rightarrow u_\alpha+k-bx_\alpha > 0 \Rightarrow$ $\boxed{x'(x_\alpha, u_\alpha+k) > 0}$

Similarly $x' < 0$ below the locus

Sign with respect to the horizontal locus

-6-

Let $(x_B + k, u_B)$ at the ~~left~~ right of the horizontal locus.

Let (x_B, u_B) on the horizontal locus

$$x_B + k > x_B \overset{P'\downarrow}{\Longrightarrow} P'(x_B + k) < P'(x_B)$$

$$\Rightarrow \quad -P'(x_B + k) > -P'(x_B) \quad \Rightarrow$$

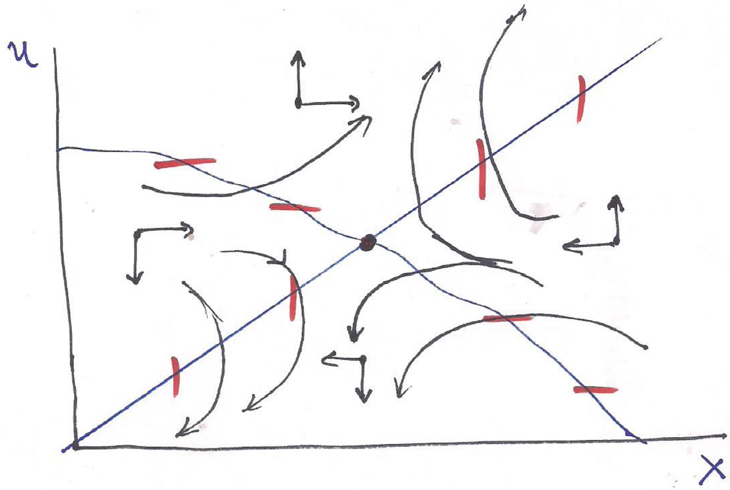
$$\Rightarrow \quad (b+r)C'(u_B) - P'(x_B + k) > (b+r)C'(u_B) - P'(x_B)$$

$$\overset{C''>0}{\Rightarrow} \quad \frac{(b+r)C'(u_B) - P'(x_B + k)}{C''(u_B)} > \frac{(b+r)C'(u_B) - P'(x_B)}{C''(u_B)} = 0$$

$$\Rightarrow \quad \boxed{\dot{u}(x_B + k, u_B) > 0}$$

Similarly $\dot{u} < 0$ at the left of the horizontal locus.

-7-



As x approaches x_s, u (investments) are also decreasing.

steady w.r.t. to r

$$u_s = b x_s \Rightarrow \frac{\partial u_s}{\partial r} = b \frac{\partial x_s}{\partial r}$$

$$(b+r) C'(u_s) - P'(x_s) = 0 \Rightarrow C' + (b+r) C'' \cdot \frac{\partial u_s}{\partial r} - P'' \frac{\partial x_s}{\partial r} = 0$$

$$\Rightarrow \frac{\partial x_s}{\partial r} = \frac{-C''(u_s)}{(b+r)b - P''(x_s)} < 0$$

$$\Rightarrow \frac{\partial u_s}{\partial r} < 0$$

Chapter 20

Dynamic Programming

20.1 The Problem

$$\max_u \int_0^T f(t,x,u)dt$$
$$s.t. \quad x' = g(t,x,u), \quad x(0) = a$$

20.2 The Solution

We define:

$$J(t_0, x_0) = \max_u \int_{t_0}^{T} f(t,x,u)dt$$

$$s.t. \quad x' = g(t,x,u), \quad x(t_0) = x_0$$

Let $\Delta t > 0$ very small, then:

$$J(t_0, x_0) = \max_u \left(\int_{t_0}^{t_0+\Delta t} f(t,x,u)dt + \int_{t_0+\Delta t}^{T} f(t,x,u)dt \right)$$

Due to the principle of optimality, the above integrals are maximized over the same $u(t)$:

$$J(t_0, x_0) = \max_u \left[\int_{t_0}^{t_0+\Delta t} f(t,x,u)dt + max_u \left(\int_{t_0+\Delta t}^{T} f(t,x,u)dt \right) \right]$$

$$s.t. \quad x' = g(t,x,u), \quad x(t_0+\Delta t) = x_0 + \Delta x$$

$$J(t_0, x_0) = max_u \left[\int_{t_0}^{t_0+\Delta t} f(t, x, u)dt + J(t_0 + \Delta t, x_0 + \Delta x) \right] \tag{20.1}$$

Since Δt is very small, u can be considered as constant for the interval $[t_0, t_0 + \Delta t]$, so:

$$\int_{t_0}^{t_0+\Delta t} f(t, x, u)dt = f(t_0, x_0, u)\Delta t \tag{20.2}$$

Using Taylor's theorem we get:

$$J(t_0 + \Delta t, x_0 + \Delta x) \cong J(t_0, x_0) + \frac{\partial J(t_0, x_0)}{\partial t}\Delta t + \frac{\partial J(t_0, x_0)}{\partial x}\Delta x \tag{20.3}$$

From (20.1),(20.2),(20.3) we get:

$$0 = max_u \left[f(t_0, x_0, u) + \frac{\partial J(t_0, x_0)}{\partial t}\Delta t + \frac{\partial J(t_0, x_0)}{\partial x}\Delta x \right]$$

Dividing through by Δt and letting $\Delta t \to 0$, we get:

$$0 = max_u [f(t, x, u) + J_t(t, x) + J_x(t, x)x'] \Rightarrow$$

Hamilton Jacobi Bellman (HJB) equation:

$$\boxed{-\frac{\partial J}{\partial t}(t, x) = max_u \left[f(t, x, u) + \frac{\partial J}{\partial x}(t, x) \cdot g(t, x, u) \right]}$$

20.1 Exercise: *By using the HJB equation, solve the problem:*

$$\min_u \int_0^T e^{-rt}(ax^2 + bu^2)dt$$

$$s.t. \quad x'(t) = u(t) \quad , \quad x(0) = x_0 > 0$$

Solution: We form the HJB - equation:

$$-\frac{\partial J}{\partial t} = \min_u \left[e^{-rt}(ax^2 + bu^2) + \frac{\partial J}{\partial x}u \right] \tag{20.4}$$

To find the minimum value we have:

$$\frac{\partial}{\partial u} \left[e^{-rt}(ax^2 + bu^2) + \frac{\partial J}{\partial x}u \right] = 0 \Rightarrow 2e^{-rt}bu + \frac{\partial J}{\partial x} = 0$$

$$\boxed{\Rightarrow \quad u^* = -\frac{J_x e^{rt}}{2b}}$$

By substituting the latter into (20.4) we get:

$$-J_t = e^{-rt}\left(ax^2 + \frac{J_x^2 e^{2rt}}{4b}\right) - \frac{J_x^2 e^{rt}}{2b}$$

$$\Rightarrow \quad ax^2 - \left(\frac{\partial J}{\partial x}\right)^2 \frac{e^{2rt}}{4b} + e^{rt}\frac{\partial J}{\partial t} = 0 \tag{20.5}$$

This is a partial differential equation. Let us try $J(t, x) = e^{-rt}Ax^2$, where A is a constant to be determined,

$$\frac{\partial J}{\partial x} = 2e^{-rt}Ax \quad , \quad \frac{\partial J}{\partial t} = -re^{-rt}Ax^2$$

By substituting into (20.5), we get:

$$ax^2 - 4e^{-2rt}A^2x^2\frac{e^{2rt}}{4b} + e^{rt}(-re^{-rt}Ax^2) = 0$$

$$\Rightarrow \frac{A^2}{b} + rA - a = 0 \quad \Rightarrow \quad A = \left(-r \pm \sqrt{r^2 + \frac{4a}{b}}\right)\cdot\frac{b}{2}$$

so, $\boxed{\Rightarrow \quad u^* = -\frac{J_x e^{rt}}{2b} = -\frac{2e^{-rt}Axe^{rt}}{2b} = -\frac{Ax}{b}}$

This is a solution in feedback form. Furthermore:

$$x' = u \quad \Rightarrow \quad x' = -\frac{Ax}{b} \quad \Rightarrow \quad x(t) = x_0 e^{-\frac{Ax}{b}}$$

20.2 Exercise: *(The "cake-eating" problem) By using the HJB equation, solve the problem:*

$$\max_C \int_0^T e^{-\rho t} \ln C dt$$

$$s.t. \quad W'(t) = -C(t) \quad , \quad W(0) = W_0 > 0, W(T) = 0$$

Solution: We form the HJB - equation:

$$-\frac{\partial J}{\partial t} = \max_C \left[e^{-\rho t}\ln C - C\frac{\partial J}{\partial W}\right] \tag{20.6}$$

Using derivatives we see that the maximum is achieved at $C^*(t) = e^{-\rho t}/J_W$, therefore (20.6) becomes:

$$-J_t = e^{-\rho t}\ln\left(\frac{e^{-\rho t}}{J_W}\right) - e^{-\rho t}$$

We set: $J(t, W) = e^{-\rho t}(A + B\ln W)$ and HJB becomes:

$$-\rho e^{-\rho t}(A + B\ln W) = e^{-\rho t}\left[\ln\left(\frac{e^{-\rho t}}{e^{-\rho t}\frac{B}{W}}\right) - 1\right] \Rightarrow$$

$$\Rightarrow -\rho(A + B\ln W) = \ln W - \ln B - 1 \Rightarrow B = 1/\rho, A = -(1 - \ln\rho)/\rho$$

$$\Rightarrow J(t, W) = \frac{-1 + \ln\rho + \ln W}{\rho}e^{-\rho t} \Rightarrow \boxed{C^*(t) = \rho W(t)}$$

Chapter 21

HJB and Infinite Horizon

21.1 Exercise: *Solve the problem :*

$$\min_{u(t)} \frac{1}{2}\int_0^{\infty}(x^4+u^2)dt$$

$$s.t. \quad \dot{x} = u$$

Solution: Since the time is infinite, the quantity J does not depend from t and thus, $J_t = 0$. The HJB equation becomes:

$$0 = \min_u \left(\frac{1}{2}x^4 + \frac{1}{2}u^2 + J_x u\right)$$

$$\frac{\partial(....)}{\partial u} = u + J_x = 0 \Rightarrow u^* = -J_x$$

$$\frac{1}{2}x^4 + \frac{1}{2}J_x^2 - J_x^2 = 0 \Rightarrow J_x^2 = x^4$$

$$J_x = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases} \Rightarrow u^* = \begin{cases} -x^2 & x \geq 0 \\ x^2 & x < 0 \end{cases}$$

Chapter 22

HJB, Present Value and Infinite Horizonn

We have the autonomous problem:

$$\max_u \int_0^\infty e^{-rt} f(x,u)dt$$
$$s.t. \quad x'(t) = g(x(t), u(t))$$

The value function $V(x)$ associated to the optimal path verifies the equation:

$$\boxed{rV(x) = \max_u [f(x,u) + V'(x)g(x,u)]}$$

THIS IS AN ODE

Proof: (OPTIONAL)

$$J(t_0, x_0) = \max_u \int_{t_0}^\infty e^{-rt} f(x,u)dt = e^{-rt_0} \max_u \int_{t_0}^\infty e^{-r(t-t_0)} f(x,u)dt$$

The value of the integral on the right depends on the initial state, but is independent of the initial time. We let:

$$V(x_0) = \max_u \int_{t_0}^\infty e^{-r(t-t_0)} f(x,u)dt \Rightarrow J(t,x) = e^{-rt} V(x)$$

$$\Rightarrow \quad J_t = -re^{-rt} V(x) \quad , \quad J_x = e^{-rt} V'(x)$$

$$HJB \Rightarrow \quad -(-re^{-rt} V(x)) = \max_u [e^{-rt} f(x,u) + e^{-rt} V'(x) g(x,u)]$$

$$\Rightarrow \quad rV(x) = \max_u [f(x,u) + V'(x) g(x,u)]$$

22.1 Exercise: *Solve the following problem (Ramsey):*

$$\max_{C(t)} \int_0^{\infty} e^{-\rho t} u(C) dt$$

$$s.t. \quad \dot{K} = f(K) - C$$

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma} \quad , \quad f(K) = AK^{a}$$

Solution: Applying the latter equation, we get:

$$\rho V = \max_{C} \left[\frac{C^{1-\sigma}}{1-\sigma} + V_K'(AK^{a} - C) \right]$$

But, $C^* = (V_K')^{-1/\sigma}$ and thus:

$$\rho V = V_K' \left[\frac{\sigma}{1-\sigma} (V_K')^{-1/\sigma} + AK^{a} \right]$$

For $a = \sigma$ and $V = B_0 + B_1 K^{1-\sigma}$ we get:

$$\rho(B_0 + B_1 K^{1-\sigma}) = (B_1(1-\sigma)K^{-\sigma}) \left[\frac{\sigma}{1-\sigma} (B_1(1-\sigma)K^{-\sigma})^{-1/\sigma} + AK^{\sigma} \right]$$

$$\Rightarrow B_0 = \frac{A(1-\sigma)}{\rho} B_1 \quad , \quad B_1 = \left(\frac{1}{1-\sigma} \right) \left(\frac{\sigma}{\rho} \right)^{\sigma}$$

$$\Rightarrow V = \left(\frac{\sigma}{a} \right)^{\sigma} \left(\frac{A}{\sigma} + \frac{1}{1-\sigma} K^{1-\sigma} \right)$$

$$\Rightarrow C^* = \frac{\rho}{\sigma} K$$

22.2 Exercise: *Solve the following consumption problem:*

$$\max_{C(t)} \int_0^{\infty} e^{-\rho t} \ln C(t) dt$$

$$s.t. \quad \dot{W}(t) = \rho W(t) - C(t)$$

Solution: Applying the latter equation, we get:

$$\rho V(W) = \max_{C} [\ln C + V'(W)(\rho W - C)]$$

But, $C^* = 1/V'$ and thus:

$$\rho V = -\ln V' + V' \rho W - 1$$

which can be solved numerically.