EXERCISES

DYNAMICAL MATHEMATICS

St. Kotsios

1. Solve the equations:

$$t\dot{x} = x(1-t) \quad , \quad (t_0, x_0) = (1, 1/e)$$

$$x\dot{x} = t \quad , \quad (t_0, x_0) = (\sqrt{2}, 1)$$

2. Solve the differential equations:

$$(a) \quad (x^2 + x)y' = 2y + 1 \qquad (b) \quad y' = 2\sqrt{y}\ln x \quad , \quad y(e) = 1$$

$$(c) \quad y' - \frac{3y}{x} = x \qquad (e) \quad \frac{ds}{dt} = \frac{s}{t} - \frac{t}{s}$$

$$(c) \quad x^2 y' = y^2 + xy$$

3. Solve the differential equations:

$$(a) \quad y' x^3 = 2y \qquad (b) \quad y' = 2\sqrt{y}\ln x \quad , \quad y(e) = 1$$

$$(c) \quad (1 + x^2)y' = 1 + y^2 \qquad (e) \quad t^2 \frac{ds}{dt} = 2ts - 3, \quad s(-1) = 1$$

$$(c) \quad x^2 y' = y^2 + xy$$

4. Solve the differential equations:

$$(a)\quad y' + y = x - e^x$$

$$(b)\quad y' + \frac{1}{x}y = x^2$$

5. Solve the differential equations:

$$(a)\quad 2xy' + y^2 - 1 = 0$$

$$(b)\quad y' + y^2 e^x = 0 \quad , \quad y(0) = 1$$

6. Solve the differential equations:

$$(a)\quad xy' - y + x^3 y^2 = 0$$

$$(b)\quad y' - 2ye^x = 2\sqrt{ye^x}$$

7. Solve the differential equations:

$$(a)\quad y^{(4)}(t) + y''(t) = 4$$

$$(b)\quad y''(t) + y'(t) + 3y(t) = 2t\sin(3t) \quad , \quad y(1) = 0 \quad , \quad y'(1) = 1$$

8. Solve the differential equations:

$$(a)\quad y'' + y' - 2y = 6x^2 \quad , \quad (b)\quad y'' + 2y' + y = (x-1)e^x$$

$$(c)\quad y''' - y' = 1 \quad , \quad (d)\quad \frac{d^2 s}{dt^2} + 2\frac{ds}{dt} + 2s = 2t^3 - 2$$

9. Solve the differential equations:

$$(a)\quad y'' + y' - 2y = e^x \quad , \quad (b)\quad y'' + 2y' + y = x$$

$$(c)\quad y''' - y' = 5 \quad , \quad (d)\quad \frac{d^2 x}{dt^2} + k^2 x = 2k\sin kt$$

10. Solve the differential equations:

$$(a)\quad y''=y'+(y')^2$$

$$(b)\quad xy'+y\ln\left(\frac{y}{x}\right)=0$$

$$(c)\quad yy''-2(y')^2=0$$

11. Let $X(t)$ denotes the national product, $K(t)$ the capital stock, $L(t)$ the labor. Suppose that for any positive time instant:

$$X=AK^{1-a}L^{a}\quad,\quad \dot{K}=sX\quad,\quad L=L_0e^{\lambda t}\quad,\quad K(0)=K_0\quad.$$

Find an expression for K(t).

12. Solve the differential equations:

$$(a)\quad y''=y'+(y')^2$$

$$(b)\quad xy'+y\ln\left(\frac{y}{x}\right)=0$$

13. Solve the equations: $t\dot{y}+(1-t)y=e^{2t}$, $\dot{x}=4x+2e^{t}\sqrt{x}\quad,\quad x>0$

14. In a macroeconomic model we have:

$$Y(t)=C(t)+I(t)\quad,\quad I(t)=k\dot{C}(t)\quad,\quad C(t)=aY(t)+b$$

Compute the limit : $\lim_{t\to\infty}\frac{Y(t)}{I(t)}$

15. Solve completely the equation: $x''+4x=4t+1\quad,\quad x(\pi/2)=0, x'(\pi/2)=0$.

16. An economic model due to T. Haavelmo leads to the differential equation: $p''(t)=\gamma(a-\alpha)p(t)+k$. Solve the equation. Is it possible to choose the constants so that the equation is stable?

17. Solve the system:

$$\begin{aligned} x'&=-2x+5y \\ y'&=-2x+4y \end{aligned}$$

18. Solve the system:

$$\begin{aligned} &x'+3x+y=0 \\ &y'-x+y=0 \\ &x(0)=1, y(0)=1 \end{aligned}\quad.$$

19. Solve the system: $\begin{array}{l} \dfrac{dx}{dt}+y=0 \\ \dfrac{dx}{dt}-\dfrac{dy}{dt}=3x+y \end{array}$.

20. Transform the equation $y''-y'-2y=0$ to a system of differential equations and solve it.

21. Transform the equation $y''-5y'+6y=0$ to a system of differential equations and solve it.

22. Transform the equation $y''-2y'+3y=0$ to a system of differential equations and solve it.

23. We suppose that the price of a good is a function over time: $p(t)$. We also have the next demand and supply functions:

$$D(t)=\alpha-\beta p(t)+mp'(t)+np''(t) \qquad (\alpha,\beta>0)$$
$$S(t)=-\gamma+\delta p(t) \qquad (\gamma,\delta>0)$$

If the market follows the rule: $\dfrac{dp}{dt}=j(D-S)$, $(j>0)$. Find the price path and its equilibrium point, if any. Find the condition which ensures an oscillating price path. Would we had an oscillated price path if $n>0$?

24. Find the extrema of the integrals:

(a) $\int_0^2 [2ye^t+y^2+(y')^2]dt$ subject to $y(0)=2$ and $y(2)=2e^2+e^{-2}$

(b) $\int_1^2 [x+tx'-(x')^2]dt$ subject to $x(1)=3$ and $x(2)=4$

25. Find the extremals, if any, of the integrals:

$$(a)\quad \int_0^1 [ty + 2(y')^2]dt, \quad y(0)=1, \quad y(1)=2$$

$$(b)\quad \int_0^1 tyy'dt, \quad y(0)=0, \quad y(1)=1$$

$$(c)\quad \int_0^2 (y^2 + t^2 y')dt, \quad y(0)=0, \quad y(2)=2$$

26. Find the extremals, if any, of the integrals:

$$(a)\quad \int_0^2 [ty + (y')^2]dt, \quad y(0)=1, \quad y(2)=0$$

$$(b)\quad \int_0^a (y'^2 + 2yy' - 16y^2)dt, \quad y(0)=0, \quad y(a)=0$$

$$(c)\quad \int_1^2 \frac{x^3}{y'^2}dx, \quad y(1)=1, \quad y(2)=4$$

27. By transforming it to a linear system in standard form, solve the system:

$$x'' + y' = 2$$
$$x' + y' = 1$$

28. A monopolist believes that the number of units $x(t)$ he can sell depends on the price $p(t)$ with respect to the relation: $x(t) = a_0 p(t) + b_0 + c_0 p'(t)$. His cost of producing at rate x is: $C(x) = b_1 x + c_1$. Given the initial price $p(0) = p_0$ and required final price $p(T) = p_1$ find the price policy to maximize profits over the time path $0 \le t \le T$.

29. A monopolist believes that the number of units $x(t)$ he can sell depends on the price $p(t)$ with respect to the relation: $x(t) = 2p(t) + 3p'(t)$. His cost of producing at rate x is: $C(x) = (1/4)x^2 + (1/2)x + c_1$. Given the initial price $p(0) = 10$ and required final price $p(2) = 20$ find the price policy to maximize profits over the time path $0 \le t \le 2$.

30. Solve the system: $\begin{matrix} x' = y + x \\ y' = x - y \end{matrix}$.

31. Solve the system: $\begin{matrix} x'' = 2x' + 5y \\ y' = -x' - 2y \end{matrix}$ by bringing it to a canonical form.

32. Consider a country, which we shall call Home, which is open to a global free market in bonds. We refer to the rest of the world as Foreign and treat it as having a common currency. Let r be the interest rate in Home and r^* the interest rate in the Foreign. Let the price levels at Home and in Foreign be P and P^* let their natural logarithms be p and p^*. Let M be the stock of money in Home. The next relations hold:

$$\frac{dq}{dt} = r - r^* \quad , \quad \frac{dp}{dt} = \sigma(q + p^* - p) \quad , \quad \frac{M}{P} = ae^{-\beta r}$$

Form a system of differential equations with unknowns the quantities $p(t), q(t)$. Solve the system and find conditions which guarantee that the system will be stable. Explain "economically" your results.

33. A principal P is compounded continuously with interest rate r.
 (i) What is the rate of change of P?
 (ii) Solve for P at time t, i.e., $P(t)$, given $P(0) = P_0$.
 (iii) If $P_0 = £2{,}000$ and $r = 7.5\%$ annually, what is P after 5 years?

a. (Hint:

We know that an initial deposit, P_0, compounded continuously at a rate of r per cent per period will grow to

$$P(t) = P_0 e^{rt}$$

Now assume that in addition to the interest received, rP, there is a constant rate of deposit, d. Thus

$$\frac{dP}{dt} = rP + d$$

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34.

A simple model for a national economy is given by

$$\begin{aligned} I' &= I - \alpha C \\ C' &= \beta(I - C - G), \end{aligned}$$

where

I	denotes the national income,
C	denotes the rate of consumer spending, and
G	denotes the rate of government expenditure.

The model is restricted to $I, C, G \geq 0$, and the constants α, β satisfy $\alpha > 1$, $\beta \geq 1$.

Suppose that the government expenditure is related to the national income according to $G = G_0 + kI$, where G_0 and k are positive constants.

Let $k = 0$, and let (I_0, C_0) denote the equilibrium point. Introduce the new variables $I_1 = I - I_0$ and $C_1 = C - C_0$.

Find analytic expressions for I_1, C_1.

35. Draw the phase portrait of the differential equations:

$$(i) \quad y' = e^{3-y} - 1$$
$$(ii) \quad y' = (3+2y)(y-2)(1-3y)$$

36. Draw the phase portrait of the differential equations:

$$(i) \quad y' = \ln(y^2 - 1)$$
$$(ii) \quad y' = \frac{(2-y)(3y+1)}{(y+2)}$$

37. Draw the phase portrait of the differential equations:

$$(i) \quad y' = (y-1)\sin y \quad , \quad y \in [-2\pi, 2\pi]$$
$$(ii) \quad y' = y^3 + y^2 - y - 1$$

38. We have the differential equation: $y' = f(ay^2 + by)$, where f is a differentiable function with $f(0) = 0$. Find the conditions which guarantee that the set $(0, +\infty)$ is a region of attraction for an appropriate equilibrium point.

39. Let $p(t)$ be a price flow. We consider that $p'(t)$ is a function of the excess demand $D(p)-S(p)$, D(p) the demand function and S(p) the supply one. In other words, $p'(t)=H[D(p(t))-S(p(t))]$. We assume that H is strictly increasing and $H(0)=0$. Let p^e be the equilibrium point, show that it is asymptotically stable.

40. We have the differential equation: $y'=\alpha y^2+\beta y+\gamma$. Find conditions, if any, which guarantee the asymptotic stability of the equilibrium points.

41. By transforming the next equations to a system of differential equations show that:
(i) $x''+ax'+bx=0$ is stable if and only if $a>0$ and $b>0$.
(ii) the equilibrium point of the equation $x''+\kappa x=\lambda,\quad \kappa\neq 0$ is always unstable.

42. Draw the phase portrait of the following system: $\begin{aligned} x'&=y(1-x)\\ y'&=-x(1-y)\end{aligned}$

43. Draw the phase portrait of the following system: $\begin{aligned} x'&=-x+y^2\\ y'&=-y(x+1)\end{aligned}$

44. Draw the phase portrait of the following system: $\begin{aligned} x'&=-2y^3\\ y'&=x\end{aligned}$

45. Draw the phase portrait of the following system: $\begin{aligned} x'&=y\\ y'&=-x+5y\end{aligned}$

46. Draw the phase portrait of the following system: $\begin{aligned} x'&=x(6-2y)\\ y'&=y(2x-4)\end{aligned}$

47. Draw the phase portrait of the following systems:

$$x'=-2x-5y,\quad y'=2x+2y$$

$$x'=x+3y,\quad y'=-6x-5y$$

48. Draw the phase portrait of the following system: $\begin{aligned} x'&=2xy-2y^2\\ y'&=x-y^2+2\end{aligned}$

49. A prey-predatory system may be modelled by:

$$\dot{x}_1 = x_1(1 - x_1 - ax_2) \quad , \quad \dot{x}_2 = bx_2(x_1 - x_2)$$

Where the variables x_1 and x_2 denote the prey and predator populations respectively , a and b are positive constants. Find all the equilibrium points and determine their type. Construct the phase portrait in the first quadrant when a=1, b=0.5 and discuss the qualitative behavior of the system.

50. A prey-predatory system may be modelled by:

$$x' = (a - by)x \quad , \quad y' = (cx - d)y - h \quad , a, b > 0$$

Where, x represents the number of hares and y the amount of foxes into an isolated forest. Construct the phase portrait in the first quadrant when a=0.4, b=0.01, c=0.003, d=0.3 , h=10. What do you think it says about the forest?

51. Draw the phase portrait of the following systems:

a. $$x' = -x + 2x^3 + y, \quad y' = -x - y$$

b. $$x' = 2x - xy, \quad y' = 2x^2 - y$$

c. $$x' = y, \quad y' = -x + \frac{1}{16}x^5 - y$$

52. Examine, for the various values of μ, the phase portrait of the system:

$$x' = \mu x + y$$
$$y' = 2x + (\mu - 1)y$$

53. We have the next IS-LM model

$$e = a + c(1-t)y - hr + jy$$
$$m^d = ky - ur$$
$$\dot{y} = A(e - y)$$
$$\dot{r} = \beta(m^d - m_0)$$
$$a > 0, 0 < c < 1, 0 < t < 1, h > 0, j > 0$$
$$k > 0, u > 0, A > 0, \beta > 0$$

Where the quantities are as in the lesson. Explain what j is. Study the phase portrait and deduce, through it, some conclusions.

54. Draw the phase portrait of the following differential equations:

$$y' = (y-1)(y+2)(y-3)e^{y}$$
$$y' = e^{y}\cos y$$
$$y' = \frac{y(y-1)}{y^2+1}$$

55. Study the dynamics of the following growth population differential equation for the various values of the parameters,

$$y' = ry^{a}(1-\frac{y}{K}), \quad a \geq 1$$

56. We assume that the production Y depends on the capital K and the labor L. That is, Y=f (K, L), where f is a given continuous and first degree homogeneous function. We also assume that the capital accumulation follows the rule: $\dot{K} = \theta Y$, θ>0 and the labor grows at rate ρ: $\dot{L} = \rho L$. Form a differential equation indicating the evolution of the quantity: $\kappa = \frac{K}{L}$. Find conditions that guarantee that the equilibrium point of the above equation is asymptotical stable.

57. Draw the phase portrait of the following linear system: $\begin{aligned} x' &= y \\ y' &= -x+5y \end{aligned}$

58. Draw the phase portrait of the following nonlinear system: $\begin{aligned} x' &= x(6-2y) \\ y' &= y(2x-4) \end{aligned}$

59. Draw the phase portrait of the following nonlinear system: $\begin{aligned} x' &= 2xy-2y^2 \\ y' &= x-y^2+2 \end{aligned}$

60. Draw the phase portrait of the following system:

$$x_1' = 5x_1 - x_1^2 - x_1x_2$$

$$x_2' = -2x_2 + x_1x_2.$$

61. For each of the following systems, show that the system has no limit cycles:
(i) $\dot{x}_1 = 1 - x_1 x_2{}^2 \quad , \quad \dot{x}_2 = x_1$

(ii) $\dot{x}_1 = x_1 x_2 \quad , \quad \dot{x}_2 = x_2$

62. The relations of an open access fishery model are:

Fishery production function	(1) $H = eES$
Net (of harvest) growth of fish stock	(2) $\frac{dS}{dt} = G(S) = g\left(1 - \frac{S}{S_{MAX}}\right)S - H$
Fishery profit	(3) $NB = B - C = PeES - wE$
Open access entry rule	(4) $\frac{dE}{dt} = \delta \cdot (NB)$

Where H defines the fishery harvest, S the stock, E the effort, NB the profit and e,P,g,δ,w parameters. Using the above equations form a system of two differential equations with unknowns the quantities S and E. Substitute to the above system the values:

$$g = 0.15$$

$$S_{MAX} = 1$$

$$e = 0.015$$

$$P = 200$$

$$\delta = 0.4$$

$$w = 0.6$$

63.

And draw the phase portrait. Give economic explanation.

64. Consider a competitive market composed of two commodities with prices p_1 and p_2 respectively. If D_i and S_i, $i = 1,2$, denote the demand and supply function for each commodity, then define $E_i = D_i - S_i$ as the excess demand functions. The dynamics of the system are described as prices changing in proportion to their excess demand, formally: $\frac{dp_i}{dt} = k_i E_i$. If $E_1 = 3 - 6p_1 + 3p_2$, $E_2 = 16 + 4p_1 - 8p_2$, $k_1 = 2, k_2 = 3$ draw the phase portrait of the system. Give an economical interpretation of your mathematical results.

65. The growth rate of any economical function $y(t)$, is defined as follows: $r_y = \frac{\frac{dy}{dt}}{y}$.

We define the next functions of time (flows):

Q: The output level. M_d : the money demand

p: The price level. M_s : the money supply

Y: The national product.

ρ: the rate of inflation, i.e. $\rho = r_p$

μ: the money demand-supply ratio, i.e. $\mu = \frac{M_d}{M_s}$.

If $Y = pQ$, $M_d = aY, \quad a > 0$, and $\frac{d\rho}{dt} = h\left(\frac{M_S - M_d}{M_d}\right), \quad h > 0$, then:

Show that:
$$\frac{d\rho}{dt} = h(1-\mu)$$
$$\frac{d\mu}{dt} = (\rho + r_Q - r_{M_s})\mu$$

Draw the phase portrait of the above system. Give an economical explanation of your results.

66. In a duopoly model we have:

$$p(t) = 20 - 5Q(t)$$
$$Q(t) = q_1(t) + q_2(t)$$
$$TC_1(t) = 4q_1^2(t)$$
$$TC_2(t) = 4q_2^2(t)$$

We assume that the rates of change of the outputs of the
Firms, are proportional to the differences between their desired levels and the actual levels. That is:

Firm 1 $\quad \frac{dq_1(t)}{dt} = k_1(x_1(t) - q_1(t)) \quad k_1 > 0$

Firm 2 $\quad \frac{dq_2(t)}{dt} = k_2(x_2(t) - q_2(t)) \quad k_2 > 0$

The desired level of output (x_1(t),x_2(t)) for each firm, is the output level that maximises profits under the assumption that the other firm does not alter its output level. If k_1=k_2=0.2, contruct the phase diagram of the system. Give economical explanation.

67. Solve by optimal control the problem:

$$\max \int_1^5 (ux - u^2 - x^2)dt$$

68. $s.t. \quad x' = x + u, \quad x(1) = 2, \quad x(5) = free$

69. . Solve by optimal control:

$$\max \int_0^2 (2x - u^2 - 3u)dt$$

$$s.t. \quad x' = x + u$$

$$x(0) = 5$$

and the control constraint $u(t) \in [0,2]$

70. By using the Hamilton-Jacobi-Bellman equation solve the problem:

$$\mathrm{m}\iota\nu \quad \int_0^T (c_1 u^2 + c_2 x)dt$$

$$s.t. \quad x' = u, \quad x(0) = 0, \quad x(T) = B$$

(Hint: try a solution of the form: $J(t,x) = a + bxt + \frac{hx^2}{t} + kt^3$)

71. We get the model:

$$\max \int_0^T U(C(t))e^{-\rho t}dt$$

$$s.t. \quad W'(t) = rW(t) - C(t), \quad W(0) = W_0$$

72. where W(t)=Wealth, C(t)=Consumption rate, U(C)=Utility of Consumption, T= Terminal Time, W_0 =Initial Wealth, ρ = Discount Rate, r=Interest Rate. By using optimal control write down the necessary conditions for solving the problem.

73. Solve by optimal control:

$$\max_u \int_0^1 5xdx$$

$$s.t. \quad x' = x + u$$

$$x(0) = 2, \quad x(1) \quad free$$

$$u(t) \in [0,3]$$

74. Solve by optimal control:

$$\max_u \int_0^1 5xdx$$

$$s.t. \quad x' = 2x - u$$

$$x(0) = 2, \quad x(1) \quad free$$

$$u(t) \in [0,3]$$

75. (a) By applying the maximum principle, find $c(t)$ to maximize:

$$V = \int_0^T \ln(c(t))dt$$

$$s.t. \quad s'(t) = -c(t)$$

$$s(0) = s_O, \quad s(T) = s_T \quad with \quad s_O > s_T$$

(b) By applying the maximum principle, find $c(t)$ to maximize:

$$V = \int_0^T e^{-t} \ln(c(t))dt$$

76. $s.t. \quad s'(t) = -c(t)$

$$s(0) = s_O, \quad s(T) = s_T \quad with \quad s_O > s_T$$

Compare the two results.

77. Solve the problem:

$$\max_u \int_0^1 u^2 dt$$

$$s.t. \quad x' = -u$$

$$x(0) = 1, \quad x(1) \geq 1$$

78. Solve the optimal control problem:

$$\min_u \int_0^1 dt$$

$$s.t. \quad x_1' = 2x_2$$

$$x_2' = x_1 - u$$

$$x_1(0) = 0, \quad x_2(0) = 6$$

$$x_1(1) = 1, \quad x_2(1) = 2$$

$$|u(t)| \leq 1$$

79. Solve by optimal control:

$$\max \int_0^2 (2x - u^2 - 3u)dt$$

$$s.t. \quad x' = x + u$$

$$x(0) = 5$$

80. A water reservoir is leaking and its water height $x(t)$ is governed by $x'(t) = -0.1x(t) + u(t)$ with $x(0) = 10$, where the control function $u(t)$ denotes the net inflow at time t and $0 \le u(t) \le 3$. Find the optimal control which maximizes the quantity

$$\int_0^{100} (x - 5u)dt .$$

81. Use optimal control to find the shortest distance between the point A and the point B.

82. Solve the problem:

$$\begin{aligned} &\max_u \int_0^1 u^2 dt \\ &s.t. \quad x' = -u \\ &\qquad x(0) = 1, \quad x(1) = 0 \end{aligned}$$

83. Solve the optimal control problem:

$$\begin{aligned} &\min_u \int_0^{t_1} dt \\ &s.t. \quad x_1' = x_2 \\ &\qquad x_2' = x_1 + u \\ &\qquad x_1(0) = 0, \quad x_2(0) = 6 \\ &\qquad |u(t)| \le 1 \end{aligned}$$

84. Consider the optimal growth problem:

$$\begin{aligned} &\max_C \int_0^{+\infty} 2e^{-0.02t}\sqrt{C}dt \\ &s.t. \quad K' = K^{1/4} - 0.06K - C \\ &\qquad K(0) = 2 \end{aligned}$$

Study it by using phase-space analysis.

85. Determine the control $u(t)$ which minimizes $\int_{t_o}^{t_1} x^2 dx$, with $x' = -ax + bu$, a and b positive constants, x_0 given, and $|u(t)| \le 1$, by employing (a) the maximum principle (b) dynamic programming.

86. Let $I=\int_0^1 (3x^2+u^2)dt$ be a cost function over the period $[0,1]$ to be minimized. We suppose that $x'=x+u$, $x(0)=1$. Write Bellman's equation associated with the above functional. Assuming that the solution to Bellman's equation in part is quadratic in x: $J(x,t)=x^2J(1,t)$, find a solution to Bellman's equation.

87. Let $I=\int_0^1 (3x^2+u^2)dt$ be a cost function over the period $[0,1]$ to be minimized. We suppose that $x'=x+u$, $x(0)=1$. Write Bellman's equation associated with the above functional. Assuming that the solution to Bellman's equation in part is quadratic in x: $J(x,t)=x^2J(1,t)$, find a solution to Bellman's equation.