SOLUTION OF THE ADVECTION-DISPERSION EQUATION: CONTINUOUS LOAD OF FINITE DURATION

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ABSTRACT: Field studies of solute fate and transport in streams and rivers often involve an experimental release of solutes at an upstream boundary for a finite period of time. A review of several standard references on surface-water-quality modeling indicates that the analytical solution to the constant-parameter advection-dispersion equation for this type of boundary condition has been generally overlooked. Here an exact analytical solution that considers a continuous load of finite duration is compared to an approximate analytical solution presented elsewhere. Results indicate that the exact analytical solution should be used for verification of numerical solutions and other solute-transport problems wherein a high level of accuracy is required.

INTRODUCTION

Many contemporary water-quality problems involve application of the advection-dispersion equation. Given specific initial and boundary conditions, the advection-dispersion equation describes spatial and temporal variations in solute concentration. A simple form of the governing equation known as the constant-parameter advection-dispersion equation may be derived for the case of steady, uniform flow and spatially constant model parameters. Analytical solutions for this simplified form are widely available (Ogata and Banks 1961; Ogata 1964; Thomann and Mueller 1987). The utility of these analytical solutions is twofold: (1) they provide an exact solution when the problem at hand is aptly described by the constant-parameter advection-dispersion equation; and (2) they provide a means to check the accuracy of numerical solutions that are developed for more complex cases.

Published analytical solutions for the constant-parameter advection-dispersion equation generally consider two input-loading scenarios. Under the first scenario, a finite amount of mass is instantaneously released at the upstream boundary of the modeled system. This type of input function is applicable when the solutes of interest are introduced into the system over a short period of time (e.g., a slug injection of dye). In the second input scenario, solutes are continuously released into the system at the upstream boundary.

A special case of this latter scenario is that in which the continuous source is of finite duration. This case is of great importance as it corresponds to many surface-water applications in which solutes are released at a constant, continuous rate for a finite period of time. Examples of these applications include determination of stream hydraulic properties using tracers (Broshears et al. 1993), determination of reaeration coefficients (Rathbun 1979), analysis of combined sewer overflows (Walton and Webb 1994), and assessment of decay mechanisms for aquatic herbicides (O'Loughlin and Bowmer 1975). Unfortunately, most published solutions for the continuous release scenario correspond to the case in which the release continues indefinitely; i.e., there is a continuous source of infinite duration. A review of several standard references for surface-water-quality modeling indicates that the solution for a source of finite duration is either not presented (Fischer et al. 1979; McCutcheon 1989; James 1993; Maidment 1993;

Rutherford 1994) or presented in an abbreviated and approximate form (Thomann and Mueller 1987).

In this paper, an exact analytical solution to the constant-parameter advection-dispersion equation for a continuous source of finite duration is presented. The exact analytical solution is then compared with an approximate analytical solution that has been published elsewhere (O'Loughlin and Bowmer 1975; Rose 1977; Thomann and Mueller 1987).

ANALYTICAL SOLUTIONS

We consider a system in which physical transport is primarily one dimensional; i.e., solute concentrations are horizontally and vertically well mixed such that concentrations vary only in the longitudinal or downstream direction. In addition, a steady, uniform flow field is imposed and the effects of dispersion are spatially constant. Finally, any biogeochemical processes may be described in terms of first-order reactions wherein the transformation rate is proportional to the solute concentration. Given these assumptions, conservation of mass yields the constant-parameter advection-dispersion equation with first-order decay (e.g., Runkel and Bencala 1995):

$$\frac{\partial C}{\partial t} = -U \frac{\partial C}{\partial x} + D \frac{\partial^2 C}{\partial x^2} - \lambda C \tag{1}$$

where C = concentration $[ML^{-3}]$; t = time $[T]$; U = flow velocity $[LT^{-1}]$; x = distance $[L]$; D = dispersion coefficient $[L^2 T^{-1}]$; and λ = first-order rate coefficient $[T^{-1}]$.

Continuous Source of Infinite Duration

Two analytical solutions may be found in the literature for the case of a continuous source of infinite duration. Initial and boundary conditions for this case are given by:

$$C(x, 0) = 0 \text{ for } x \geq 0 \tag{2a}$$

$$C(0, t) = C_0 \text{ for } t \geq 0 \tag{2b}$$

$$C(\infty, t) = 0 \text{ for } t \geq 0 \tag{2c}$$

where C_0 = concentration at the upstream boundary $[ML^{-3}]$. For the case of conservative solutes ($\lambda = 0$), the analytical solution is given by (Ogata and Banks 1961)

$$C(x, t) = \frac{C_0}{2} \left[\text{erfc} \left(\frac{x - Ut}{2\sqrt{Dt}} \right) + \exp \left(\frac{Ux}{D} \right) \text{erfc} \left(\frac{x + Ut}{2\sqrt{Dt}} \right) \right] \tag{3}$$

The analytical solution for nonconservative solutes ($\lambda \neq 0$) is presented by Bear (1972, p. 630) and developed using Laplace transforms by O'Loughlin and Bowmer (1975)

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$$C(x, t) = \frac{C_0}{2} \left\{ \exp\left[\frac{Ux}{2D}(1 - \Gamma)\right] \text{erfc}\left(\frac{x - Ut\Gamma}{2\sqrt{Dt}}\right)\right.$$

$$\left. + \exp\left[\frac{Ux}{2D}(1 + \Gamma)\right] \text{erfc}\left(\frac{x + Ut\Gamma}{2\sqrt{Dt}}\right)\right\} \qquad (4)$$

where

$$\Gamma = \sqrt{1 + 2H} \qquad (5)$$

$$H = 2\lambda D/U^2 \qquad (6)$$

Simplified forms of (3) and (4) are often presented in the literature. Ogata and Banks (1961) state that omission of the second term in (3) results in a maximum error of 3% for values of $D/Ux < 0.002$. Similarly, it can be shown using L'Hospital's theorem that the terms in (4) involving $x + Ut$ are small relative to the other terms (O'Loughlin and Bowmer 1975). Following these ideas, (4) simplifies to

$$C(x, t) = \frac{C_0}{2} \left\{ \exp\left[\frac{Ux}{2D}(1 - \Gamma)\right] \text{erfc}\left(\frac{x - Ut\Gamma}{2\sqrt{Dt}}\right)\right\} \qquad (7)$$

Because the error introduced by dropping the $x + Ut$ terms is usually less than that associated with experimental data, the simplification yielding (7) is frequently employed. Eq. (7) may be further simplified by noting that for small H (on the order of 0.0025), Γ can be approximated by the first two terms in a binomial expansion [$\Gamma \approx 1 + H$, O'Loughlin and Bowmer (1975)]

$$C(x, t) = \frac{C_0}{2} \left\{ \exp\left(\frac{-\lambda x}{U}\right) \text{erfc}\left(\frac{x - Ut[1 + H]}{2\sqrt{Dt}}\right)\right\} \qquad (8)$$

This second simplification is far less typical than the first, and was introduced by O'Loughlin and Bowmer (1975) in order to derive an expression for the estimation of λ from field data.

Continuous Source of Duration τ

Although these solutions are of interest, a far more useful problem is that in which a continuous source is present for a finite period of time. Letting τ represent the duration of the continuous source [T], initial and boundary conditions are given by

$$C(x, 0) = 0 \text{ for } x \geq 0 \qquad (9a)$$

$$C(0, t) = C_0 \text{ for } \tau \geq t \geq 0 \qquad (9b)$$

$$C(0, t) = 0 \text{ for } t > \tau \qquad (9c)$$

$$C(\infty, t) = 0 \text{ for } t \geq 0 \qquad (9d)$$

An analytical solution for the conditions given in (9) was developed by Rose (1977) in a commentary on the work of O'Loughlin and Bowmer (1975). Rose correctly applied the principle of superposition to (8), yielding an approximate analytical solution. For $t \leq \tau$, the solution is given by (8). For $t > \tau$, the solution is

$$C(x, t) = \frac{C_0}{2} \exp\left(\frac{-\lambda x}{U}\right) \left\{ \text{erfc}\left[\frac{x - Ut(1 + H)}{2\sqrt{Dt}}\right]\right.$$

$$\left. - \text{erfc}\left[\frac{x - U(t - \tau)(1 + H)}{2\sqrt{D(t - \tau)}}\right]\right\} \qquad (10)$$

Recall that the primary purpose of the work presented by O'Loughlin and Bowmer (1975) and Rose (1977) was to estimate λ from field data. As such, application of superposition using the simplified equation was justified, given the stated assumption that terms involving $x + Ut$ are small and the

assumption of small H. This original work has been cited and reproduced in a popular and widely read textbook (Thomann and Mueller 1987). Due to the simplifying assumptions, (10) is only an approximate analytical solution for the problem at hand. An exact analytical solution may be obtained by applying the principle of superposition to the original analytical solution for a continuous source of infinite duration (4). The solution for $t \leq \tau$ is given by (4). For $t > \tau$, superposition yields

$$C(x, t) = \frac{C_0}{2} \left\{ \exp\left[\frac{Ux}{2D}(1 - \Gamma)\right] \left[\text{erfc}\left(\frac{x - Ut\Gamma}{2\sqrt{Dt}}\right)\right.\right.$$

$$\left. - \text{erfc}\left(\frac{x - U(t - \tau)\Gamma}{2\sqrt{D(t - \tau)}}\right)\right] + \exp\left[\frac{Ux}{2D}(1 + \Gamma)\right]$$

$$\left. \cdot \left[\text{erfc}\left(\frac{x + Ut\Gamma}{2\sqrt{Dt}}\right) - \text{erfc}\left(\frac{x + U(t - \tau)\Gamma}{2\sqrt{D(t - \tau)}}\right)\right]\right\} \qquad (11)$$

A similar solution that considers the additional processes of retardation and zero-order production for subsurface applications is given by van Genuchten and Alves (1982).

RESULTS AND CONCLUSIONS

An exact analytical solution to the advection-dispersion equation subject to a continuous load of finite duration is given by (11). Development of the approximate analytical solution given as (10) relies on two assumptions regarding the parameter groups D/Ux and H. The errors associated with the use of the approximate solution are therefore problem specific. Here we examine these errors through a hypothetical example. In the example, a continuous source with a 2-hr duration ($\tau = 2$ hr) is imposed such that the concentration at the upstream boundary is 100 concentration units ($C_0 = 100$). The flow velocity and dispersion coefficient are fixed (at 0.1 m/s and 5.0 m²/s, respectively) and the parameter groups D/Ux and H are allowed to vary as a function of distance (x) and decay rate (λ).

Results for the hypothetical example are shown graphically in Fig. 1. Figs. 1(a) and 1(b) show the results for a conservative solute ($\lambda = 0$) at 100 and 2,000 m, respectively. Errors (the discrepancy between the exact and approximate analytical solution) in this case are due entirely to the initial assumption that the term involving $x + Ut$ is small relative to the other term in (3). As suggested by Ogata and Banks (1961), the error should decrease as D/Ux becomes small. This is verified by comparing errors at the first location [Fig. 1(a), $x = 100$ m] with those at the second location [Fig. 1(b), $x = 2,000$ m], and noting that the errors decrease as D/Ux decreases from 0.5 to 0.025. Figs. 1(c) and 1(d) show the results for a nonconservative solute ($\lambda = 1 \times 10^{-4}$) at 100 and 2,000 m. For a nonconservative solute, errors are due to both the initial assumption that the $x + Ut$ terms are small and the additional assumption that H is small. Comparison of Figs. 1(a) and 1(c) indicates that the error introduced by the additional assumption is negligible for short distances (small values of x). In contrast, the assumption of small H introduces considerable error at longer distances, as suggested by a comparison of Figs. 1(b) and 1(d).

The errors depicted in Fig. 1 are quantified in Table 1 where the maximum error is given as a percentage of the peak concentration. In general, the errors associated with the use of the approximate analytical solution decrease with increasing distance for a conservative solute (D/Ux decreases with x) and increase with increasing λ for a nonconservative solute. An exception to this trend is noted at $x = 100$ m, where the percent error is relatively insensitive to the specified value of λ. This exception is due to the short transport time that limits the effect of decay.
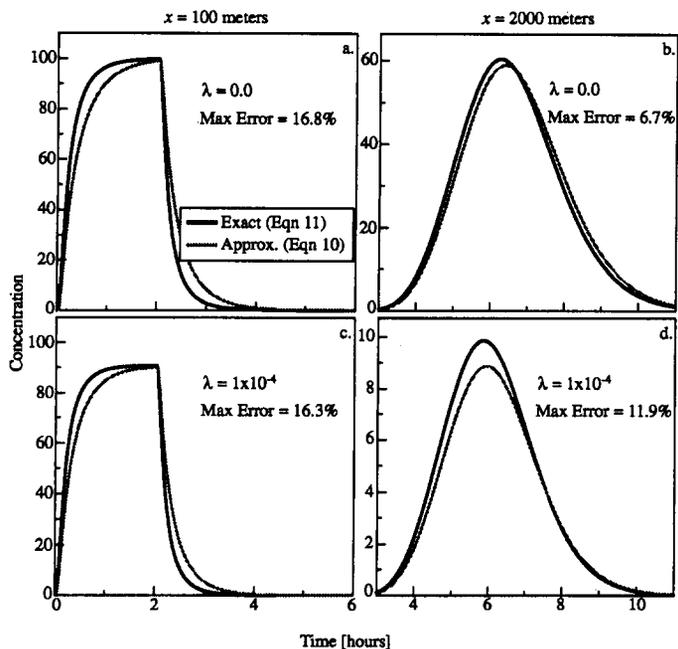
FIG. 1. Exact and Approximate Analytical Solutions: (a) Conservative Transport ($\lambda = 0$) at 100 m; (b) Conservative Transport ($\lambda = 0$) at 2,000 m; (c) Nonconservative Transport ($\lambda = 1 \times 10^{-4}$) at 100 m; (d) Nonconservative Transport ($\lambda = 1 \times 10^{-4}$) at 2,000 m

TABLE 1. Maximum Error as Percentage of Peak Concentration

Decay rate (/s) (1)	Distance from Source		
	x = 100 m (D/Ux = 0.5) (2)	x = 1,000 m (D/Ux = 0.05) (3)	x = 2,000 m (D/Ux = 0.025) (4)
$\lambda = 0.0$ ($H = 0.00$)	16.8	7.8	6.7
$\lambda = 5 \times 10^{-5}$ ($H = 0.05$)	16.5	7.9	7.7
$\lambda = 1 \times 10^{-4}$ ($H = 0.10$)	16.3	9.3	11.9
$\lambda = 2 \times 10^{-4}$ ($H = 0.20$)	16.0	17.1	28.9

Given the power and availability of today's computing resources, there is little need for approximate analytical solutions such as that given by (10). Indeed the use of (10) when testing a numerical model may lead to incorrect conclusions regarding the accuracy of the numerical methods under examination. The exact analytical solution (11) is therefore a more appropriate tool for model verification. In addition, the results presented herein indicate that substantial errors may arise through the use of (10). Therefore, the exact analytical solution given as (11) should be used for solute-transport problems that require a high level of accuracy.

APPENDIX I. REFERENCES

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APPENDIX II. NOTATION

The following symbols are used in this paper:

C = solute concentration $[ML^{-3}]$;
C_0 = concentration upstream boundary $[ML^{-3}]$;
D = dispersion coefficient $[L^2 T^{-1}]$;
U = flow velocity $[LT^{-1}]$;
t = time $[T]$;
x = distance $[L]$;
λ = first-order rate coefficient $[T^{-1}]$; and
τ = duration of continuous source $[T]$.