Analytical Solutions for Determination of Non-Steady-State and Steady-State Capture Zones

by Y. Jeffrey Yang, Richard D. Spencer, and Todd M. Gates

Abstract

This paper presents analytical solutions for determining non-steady-state capture zones produced by a single recovery well and steady-state capture zones produced by multiple recovery wells. Analysis of non-steady-state capture zones is based on the time-dependent location of capture zone stagnation points and the geometric similarity between steady-state and non-steady-state capture zones. The analytical solution of steady-state capture zones is obtained from spatial variations of discharge potential across the capture zone boundary. Both capture zone analyses are based on the assumptions of uniform flow field with a constant hydraulic conductivity, the Dupuit assumption of insignificant vertical flow, a negligible delayed yield, and a fully penetrating well with a constant pumping rate. For a ground water pump-and-treat remediation program, the pumping rate and well location design variables can be adjusted to ensure containment of the ground water contaminant plume.

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Introduction

Pump-and-treat systems are widely used to contain and remediate contaminated ground water. These systems typically use one or more recovery wells to hydraulically control the contaminant plume. Estimation of the capture zone produced by the recovery well(s) is critical and must be incorporated into the design of pump-and-treat remediation systems (Ratzlaff et al. 1992; Nyer and Schafer 1993).

Three approaches are commonly used to determine capture zones (Bair and Roadcap 1992). The first approach uses an analytical flow model based on computation of the drawdown distribution surrounding a recovery well and subsequent superposition of the nonpumping potentiometric surface of a regional flow field. Analytical models are typically based on the Theis equation and incorporate many simplifying assumptions. The second approach employs numerical flow modeling using finite-element, finite-difference, and analytical-element techniques to solve flow equations in a regional flow domain. Numerical flow modeling can incorporate complex boundary conditions and flow system geometry. The third approach uses a semianalytical model that combines analytical and numerical techniques to solve flow equations.

Bair and Roadcap (1992) and Springer and Bair (1992) compared the three modeling approaches for a leaky-confined fractured-carbonate aquifer and a stratified-drift buried-valley aquifer. They demonstrated that the semianalytical and analytical approaches can provide an accurate estimate of capture zones in simple hydrogeologic settings. However, the simplifying assumptions used in these models result in significant errors when the models are applied to complex hydrogeologic settings. Springer and Bair (1992) concluded that application of the analytical and semianalytical models should be limited to confined or unconfined flow systems that have nearly uniform aquifer transmissivity and circumferential recharge to the wells. However, the simplicity afforded by the analytical and semianalytical models is advantageous when addressing relatively simple hydrogeologic settings. As stated by Anderson and Woessner (1992), it is preferable to make a flow model as simple as possible and still preserve the fundamental characteristics of the flow system.

Analytical mathematical solutions have been developed to determine the steady-state capture zone produced by a single recovery well in a uniform confined aquifer based on the potentiometric head and stream function variations (Bear 1979). Most recently, Grubb (1993) obtained a transcendental equation for the non-steady-state capture zone streamline at time t. This equation leads to analytical solutions for the steady-state capture zone produced by a single recovery well in confined, unconfined, or combined confined and unconfined aquifers. However, use of Grubb's (1993) analytical model is limited to estimation of the steady-state capture zone produced by a single recovery well at infinite time. Because remediation is finite and multiple wells are commonly used in pump-and-treat systems, the development of corresponding analytical solutions is warranted. This paper presents analytical solutions for delineating the non-steady-state capture zone produced by a single recovery well and for delineating steady-state capture zones produced by multiple wells.

Steady-State Capture Zone

Capture zone refers to the portion of an aquifer from which all the water will be removed by the recovery well(s). Relative to the well in a regional flow domain, the capture zone streamline has a parabola geometry in plain view, diverges in the upgradient direction, and converges in the downgradient direction (Figure 1). At the stagnation point, vectors of regional ground water flow and flow to the recovery well have equal magnitude but opposite direction.

Grubb's (1993) model for confined, unconfined, and combined confined and unconfined aquifers is based on the following assumptions:

1. Infinite, isotropic, and homogeneous aquifer with a horizontal base
2. Dupuit assumption of negligible vertical flow
3. Fully penetrating recovery well with constant pumping rate.

With these assumptions and the x,y-coordinate system shown in Figure 1, the ground water discharge

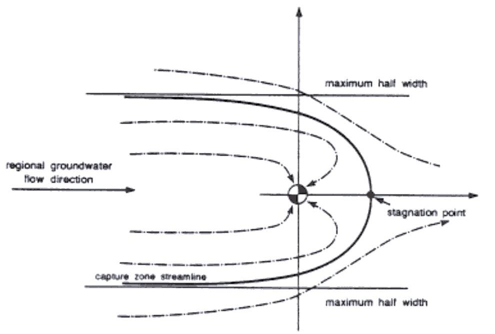

Figure 1. Plain view of capture zone produced by a single recovery well in a regional flow field. Dashed lines with arrows indicate ground water flow streamlines. Note that in the part of the capture zone where x-coordinates are positive, the x-components of regional flow and flow to the well have opposite flow directions.

potential at a point (x,y) is given by:

$$\Phi = -Q_o x + \frac{Q_w}{4\pi}\ln(x^2 + y^2)$$

where Φ is the discharge potential, Q_o is the discharge vector of regional flow, and Q_w is the pumping rate of the recovery well. The first term in the right side of Equation 1 represents the contribution of regional ground water flow to the discharge potential, and the second term describes the influence of pumping at the recovery well.

According to Grubb (1993), Equation 1 leads to analytical solutions for the streamline, stagnation point (X_{stag}), and maximum half width of the capture zone (Y_{div}) at steady state and infinite time:

$$x = \frac{\quad}{\tan(2\pi Q_o y/Q_w)} \qquad (2)$$

$$X_{stag} = \frac{Q_w}{2\pi Q_o} \qquad (3)$$

$$Y_{div} = \pm\frac{Q_w}{2Q_o} \qquad (4)$$

To apply Equations 2, 3, and 4 for a confined or unconfined aquifer, the discharge potential of regional flow (Q_o) is defined based on Strack (1989):

$$\text{confined aquifer:}\quad Q_o = KBi \qquad (5)$$

$$\text{unconfined aquifer:}\quad Q_o = K\frac{(\varphi_2^2 - \varphi_1^2)}{2L} \qquad (6)$$

where B is the aquifer thickness, i is regional flow gradient, K is hydraulic conductivity, φ_1 and φ_2 are the hydraulic heads at locations 1 and 2 along the hydraulic gradient of regional flow, and L is the distance between locations 1 and 2.

Non-Steady-State Capture Zone

Based on variations of the discharge potential and stream function in a flow field with time, Grubb (1993) obtained a flow equation for the non-steady-state capture zone streamline. As stated by Grubb (1993), the transcendental nature of the equation precludes explicit analytical solutions for the non-steady-state capture zone at time t. However, the analytical solution can be obtained from the geometric similarity between steady-state and non-steady-state capture zones using the same assumptions of Grubb (1993) plus the assumption of insignificant delayed yield. As a capture zone expands with time toward its steady-state configuration, the stagnation point and the bounding streamlines move farther away from the recovery well but their parabolic geometry remains unchanged. Therefore, the non-steady-state capture zone can be accounted for in size and geometry by the steady-state capture zone produced by an imaginary well pumping at a smaller rate. The imaginary pumping rate (Q′) can be obtained from Equation 3 provided the location of the non-steady-state stagnation point (Δx, 0) is known:

$$Q' = 2\pi Q_o \Delta x. \tag{7}$$

The location of the non-steady-state stagnation point can be determined based on the variation in discharge potential with time. At y=0, the x-component of ground water flow velocity (v_x) is:

$$v_x = \frac{\partial x}{\partial t} = \frac{1}{bn}\frac{\partial \Phi}{\partial x} \tag{8}$$

where b and n are, respectively, the aquifer thickness and the aquifer porosity at a location (x, 0). On the basis of Equation 1, Equation 8 leads to:

$$\frac{\partial x}{\partial t} = -\frac{Q_o}{bn} + \frac{Q_w}{2\pi bn}\frac{1}{x} \tag{9}$$

Rearranging Equation 9 yields the following differential equation for the location of the non-steady-state stagnation point at time t:

$$\frac{Q_o}{bn}\partial t = \frac{2\pi Q_o x}{Q_w - 2\pi Q_o x}\partial x. \tag{10}$$

Integration of Equation 10 leads to:

$$\frac{2\pi Q_o^2}{bn} t = Q_w - 2\pi Q_o x - Q_w \ln(Q_w - 2\pi Q_o x) - C \tag{11}$$

where C is the integration constant which is determined using the boundary condition x=0 when t=0:

$$C = Q_w \ln(Q_w) - Q_w. \tag{12}$$

Substituting Equation 12 into Equation 11 yields:

$$\frac{2\pi Q_o^2}{bn} t = -2\pi Q_o x - Q_w \ln\left(1 - \frac{2\pi Q_o x}{Q_w}\right). \tag{13}$$

When x=Δx at the non-steady-state stagnation point, the aquifer thickness (b) equals the full aquifer thickness (B) by definition. Using the two conditions x=Δx and b=B, Equation 13 leads to the analytical solution for the relation between t and the non-steady-state stagnation point location (Δx, 0):

$$t = -\frac{Bn}{Q_o}\Delta x - \frac{BnQ_w}{2\pi Q_o^2}\ln\left(1 - \frac{2\pi Q_o}{Q_w}\Delta x\right). \tag{14}$$

When t is infinite at steady state, the stagnation point location solved from Equation 14 is identical to Equation 3 derived by Grubb (1993).

The non-steady-state capture zone streamline is determined by using Equation 2 and substituting the imaginary pumping rate Q′ for the true pumping rate Q_o at the well. Q′ is solved by using Equation 7, where the non-steady-state stagnation point (Δx, 0) at time t is determined from Equation 14.

Capture Zones Produced by Multiple Wells

It is evident in Equations 2 and 3 that capture zone size is proportional to the pumping rate of the recovery well. Because attainable well yield at a single recovery well is limited, the contaminant plume may not be contained. Consequently, multiple recovery wells are commonly used in remediation programs.

In a uniform flow field containing multiple wells, the principle of superposition is used to determine the discharge potential at a point (x,y):

$$\Phi_{(x,y)} = -Q_o x + \sum\left\{\frac{Q_{wi}}{4\pi}\ln\left[(x-x_{wi})^2 + (y-y_{wi})^2\right]\right\} \tag{15}$$

where i = 1, 2, 3, . . . n, denotes recovery well numbers; x_w, y_w are x,y-coordinates of each well; and Q_w is the pumping rate at well i.

In the portion of the capture zone where x coordinates are positive (Figure 1), the x-component of the ground water flow vector has an opposite sign across the capture zone streamline. The ground water within the capture zone flows toward recovery wells, whereas ground water outside the capture zone participates in regional ground water flow. Correspondingly, the discharge potential surface slopes toward different directions across the bounding capture zone streamline. The first partial derivative of the discharge potential against x, namely $\partial\Phi_{(x,y)}/\partial x$ representing the x-component of ground water flow, is zero at the bounding steady-state capture zone streamline and has opposite signs outside and inside the capture zone. From Equation 15, the derivative $\partial\Phi_{(x,y)}/\partial x$ is:

$$\partial\Phi_{(x,y)}/\partial x = -Q_o + \sum\left\{\frac{Q_{wi}}{2\pi}\frac{x-x_{wi}}{(x-x_{wi})^2 + (y-y_{wi})^2}\right\}. \tag{16}$$

When $\partial\Phi_{(x,y)}/\partial x$ equals zero, Equation 16 is reduced to yield the following equation for determining the composite bounding steady-state capture zone streamline:

$$Q_o = \sum \left\{ \frac{(x-x_{wi})\, Q_{wi}}{2\pi[(x-x_{wi})^2 + (y-y_{wi})^2]} \right\} \qquad (17)$$

A Hypothetical Example

A contaminant plume is present in an unconfined aquifer at a petroleum distribution facility and extends off-site in a downgradient direction (Figure 2). The aquifer has a thickness of 20 feet, a hydraulic conductivity of 10 feet/day, and a porosity of 0.3. Underlying the aquifer is a thick and laterally continuous clay layer at an elevation of approximately 130 feet above mean sea level. Ground water elevations at two monitoring wells along the ground water streamline are 150 feet and 148 feet, respectively. The distance between the two monitoring wells is 200 feet. Based on this information, the following parameter values are used for capture zone calculation:

hydraulic conductivity:

$k = 10$ feet/day

porosity:

$n = 0.3$

natural ground water gradient:

$$i = \frac{150 \text{ feet} - 148 \text{ feet}}{200 \text{ feet}} = 0.01$$

potentiometric head at monitoring well #1

$\varphi_1 = 150 \text{ feet} - 130 \text{ feet} = 20 \text{ feet}$

potentiometric head at monitoring well #2:

$\varphi_2 = 148 \text{ feet} - 130 \text{ feet} = 18 \text{ feet}$

regional flow discharge potential:

$$Q_o = \frac{10 * (20^2 - 18^2)}{2 * 200} = 1.9 \text{ ft}^2/\text{day}.$$

An early design of the remediation program uses a single 4-inch-diameter, fully penetrating recovery well at the eastern edge of the property (Figure 3). A pumping rate (Q_w) of 10 gpm is assumed for the well. Under these conditions, the steady-state capture zone has a stagnation point and a maximum half width as follows:

$$X_{stag} = \frac{Q_w}{2\pi Q_o} = \frac{10*24*60}{2*3.14*1.9*7.48} = 161 \text{ feet}$$

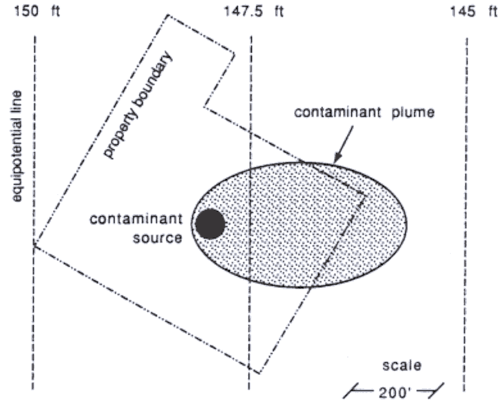

Figure 2. Contaminant plume at a hypothetical petroleum distribution facility. Potentiometric surface is in feet above the mean sea level.

$$Y_{div} = \frac{Q_w}{2Q_o} = \frac{15*24*60}{2*1.9*7.48} = 507 \text{ feet.}$$

At 50, 100, and 1000 days after the initiation of pumping, the stagnation point location is determined using Equation 14:

$\Delta x_{50} = 62$ feet

$\Delta x_{100} = 81$ feet

$\Delta x_{1000} = 152$ feet.

Based on Equation 7, the corresponding imaginary pumping rates necessary to produce the above stagnation point locations are:

$Q'_{50} = 734 \text{ ft}^3/\text{day} = 3.8$ gpm

$Q'_{100} = 967 \text{ ft}^3/\text{day} = 5.0$ gpm

$Q'_{1000} = 1820 \text{ ft}^3/\text{day} = 9.5$ gpm.

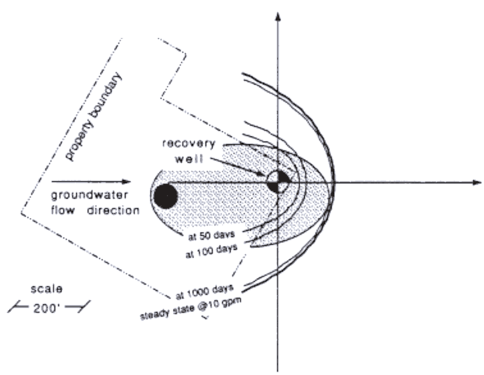

Figure 3. Calculated steady-state capture zone and non-steady-state capture zones produced by a single recovery well with a pumping rate of 10 gpm.

Figure 3 shows the non-steady-state capture zones corresponding to the imaginary pumping rates. Also shown in Figure 3 is the steady-state capture zone produced by the recovery well at the assumed 10 gpm pumping rate.

It is apparent from Figure 3 that the steady-state capture zone encompasses the entire ground water contaminant plume. However, continuous downgradient migration of the plume prior to establishing the steady-state capture zone may result in failure of the intended containment. At 1000 days after the initiation of pumping, the non-steady-state capture zone stagnation point will be located 152 feet away from the recovery well. During this time period, the portion of the contaminant plume outside the non-steady-state capture zone advances downgradient in response to the regional ground water flow with a velocity less than 0.03 feet/day. Because of this, the leading edge of the contaminant plume may migrate beyond the limit of the steady-state capture zone before the capture zone is established.

Additionally, a critical factor in evaluating the early design is the yield (Q) obtainable from the 4-inch-diameter recovery well. Based on the Cooper-Jacob nonequilibrium equation (Driscoll 1984), the maximum probable specific capacity for the well is 0.69 gpm/ft at 1000 days after the initiation of pumping. Considering the effect of dewatering and assuming 100 percent drawdown in the recovery well, the maximum well yield will be approximately 7 gpm. Therefore, it is unlikely that the assumed 10 gpm pumping rate can be sustained during establishment of the steady-state capture zone.

To overcome the deficiency of a single recovery well, a multiple recovery well system was considered. Three recovery wells are located at the eastern boundary of the property (Figure 4). Each well is fully penetrating and has a sustainable pumping rate of 6 gpm. Using the x,y-coordinate system (Figure 4), the following parameter values are used to calculate the steady-state capture zone with Equation 17:

$\Delta x_1 = 0$
$\Delta y_1 = 0$
$\Delta x_2 = 57$ feet
$\Delta y_2 = 101.5$ feet
$\Delta x_3 = 18.7$ feet
$\Delta y_3 = 208.5$ feet
and $Q_{wi} = 6$ gpm $i=1, 2, 3$.

Figure 4 shows the calculated steady-state composite bounding capture zone produced by the multiple well system. The recovery well system is likely to provide the desired containment of the ground water contaminant plume.

Conclusions

Analytical solutions have been obtained for a non-steady-state capture zone produced by a single recovery well and for the steady-state capture zones produced by multiple wells in a uniform regional flow field. In

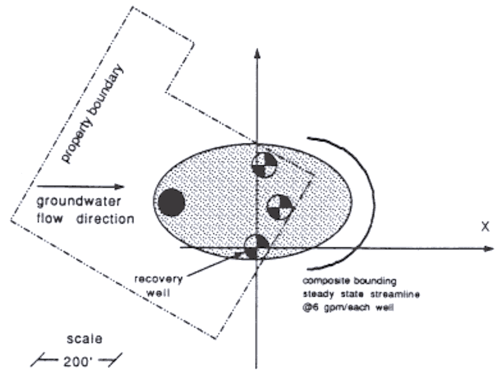

Figure 4. Steady-state composite bounding capture zone produced by three recovery wells, each with a pumping rate of 6 gpm.

designing the remediation program, the stagnation point of the capture zone for a finite remediation duration should be located beyond the contaminant plume to accommodate plume migration as the capture zone expands with time. To meet this design requirement, multiple recovery wells may be necessary, particularly in aquifers where well yield is a significant limitation.

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