Analytical Model for Estimation of Steady-State Capture Zones of Pumping Wells in Confined and Unconfined Aquifers

by Stuart Grubb[a]

Abstract

The analysis of capture zones of pumping wells is useful for designing pumping systems and wellhead protection programs. Using discharge potentials, equations are derived that can be applied to confined, unconfined, or combined confined and unconfined aquifers. The transient equations are transcendental and cannot be solved explicitly. However, infinite-time (steady-state) equations are presented which can be solved. They define an area in which, theoretically, all the water in the aquifer will eventually reach the pumping well, although the equations do not consider the effects of hydrodynamic dispersion. Equations for calculating the stagnation point, upgradient divide, and dividing streamline within the aquifer and an example problem are presented.

1. Introduction

A capture zone is defined as the area of an aquifer in which all the water will be removed by a pumping well or wells within a certain time period. Capture zone analysis has been recognized as an important consideration in the design of ground-water remediation systems and wellhead protection programs (Javandel and Tsang, 1986; Lee and Wilson, 1988). Bear and Jacobs (1965) investigated the movement of water particles injected into aquifers, and their analytical model is often used for determining capture zones as well. Several standard ground-water texts have simple equations for determining the infinite-time (steady-state) capture zone of a single well in a confined aquifer with uniform regional flow (for example, Bear, 1979; Todd, 1980). Equations can be superimposed to calculate the capture zone of multiple well systems (Javandel and Tsang, 1986), and computer models have been developed for analyzing multiple wells and heterogeneous aquifers (for example, McElwee, 1991). These models include the EPA's wellhead protection area (WHPA) package (EPA, 1990).

This paper presents a model for determining capture zones which is applicable not only to confined aquifers, but to unconfined and combined confined and unconfined aquifers as well. Portions of the model development were presented in Javandel and others (1985) and Bear and Jacobs (1965). These authors used the potential ($K\phi$) and the specific discharge to develop the equations. The primary difference in the model presented here is that the equations are generalized in terms of discharge potential so they can be used for confined aquifers, unconfined aquifers, and combined confined and unconfined aquifers by simply using the appropriate definition of one parameter, the discharge potential. The discharge potential concept was developed over 20 years ago and is fully documented in Strack (1989) and discussed by Marsily (1986), but it is not widely used.

2. Analytical Model

The assumptions for this model are as follows:

- The aquifer is homogeneous, isotropic, and infinite in horizontal extent.
- Uniform flow (steady-state) conditions prevail.
- A confined aquifer has a uniform transmissivity and no leakage through the upper or lower confining layers. An unconfined aquifer has a horizontal lower confining layer with no leakage, rainfall infiltration, or other vertical recharge. The effect of these assumptions is discussed later.
- Because the equations assume steady-state conditions, the storativity of a confined aquifer and the specific yield of an unconfined aquifer have been neglected. Hydrodynamic dispersion is also neglected.
- Dupuit assumption, i.e. vertical gradients are negligible.

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• The well is fully penetrating, is open over the thickness of the confined or unconfined aquifer at the well, and pumps at a constant rate.

Complex potentials are used to describe the distribution of discharge potentials throughout the aquifer. For background on the mathematics of complex potentials, Strack (1989, p. 269) gives a good, concise overview of the theory of complex functions. The complex potential for uniform regional flow in the (x, y) plane is

$$\Omega = -Q_0 z e^{-i\alpha} + C \qquad (1)$$

and the complex potential for a well is (Strack, 1989, p. 279)

$$\Omega = \frac{Q_w}{2\pi} \ln(z - z_w) + C \qquad (2)$$

where Q_0 = discharge vector of uniform flow; z = complex potential = x + iy; z_w = complex potential at the well; α = angle between the x axis and uniform flow; Q_w = discharge from the well; and C = constant which corresponds to the elevation of the bottom of the aquifer. Assume that C = 0. Note that

$$e^{-i\alpha} = \cos\alpha - i\sin\alpha \qquad (3)$$

and

$$Q_0 = Q_{x0} + Q_{y0} = \frac{d\Phi}{dx} + \frac{d\Phi}{dy} \qquad (4)$$

where Q_{x0} = x component of uniform flow; Q_{y0} = y component of uniform flow; and Φ = discharge potential.

The discharge potential is defined differently for confined, unconfined, and combined confined and unconfined aquifers as follows (Strack, 1989, p. 49):

Confined aquifer:	$\Phi = Kb\phi$
Unconfined aquifer:	$\Phi = \frac{1}{2}K\phi^2$
Combined confined and unconfined aquifer:	$\Phi = Kb\phi - \frac{1}{2}Kb^2$ for confined part
	$\Phi = \frac{1}{2}K\phi^2$ for unconfined part

where K = hydraulic conductivity; b = confined aquifer thickness; and ϕ = hydraulic head (or phreatic head) above the bottom of the aquifer. Writing equations in terms of discharge potentials is useful because the same equations may be used for all three types of aquifers by simply using the appropriate definition for Φ.

Because the complex potentials and the boundary conditions considered are linear and homogeneous, any linear combination of complex potentials can also be solved according to the principle of superposition. Superimposing (adding) the complex potentials for uniform flow and for flow to the pumping well gives

$$\Omega = \Phi + \Psi_i = -Q_0 z e^{-i\alpha} + \frac{Q_w}{2\pi} \ln(z - z_w) \qquad (5)$$

where Ψ = stream function.

The real and imaginary parts of (5) are

$$\Phi = -Q_0([x - x_w]\cos\alpha + [y - y_w]\sin\alpha) + \frac{Q_w}{4\pi} \ln([x - x_w]^2 + [y - y_w]^2) \qquad (6)$$

$$\Psi = Q_0([x - x_w]\sin\alpha + [y - y_w]\cos\alpha) + \frac{Q_w}{2\pi} \tan^{-1}\left(\frac{y - y_w}{x - x_w}\right) \qquad (7)$$

where x_w, y_w = x and y coordinates of the well.

The velocity components v_x and v_y in the x and y directions, respectively, along a particular streamline are

$$v_x = \frac{dx}{dt} = \frac{1}{Bn}\frac{d\Phi}{dx} = \frac{-Q_0\cos\alpha}{Bn} + \frac{Q_w[x - x_w]}{2\pi Bn([x - x_w]^2 + [y - y_w]^2)} \qquad (8)$$

$$v_y = \frac{dy}{dt} = \frac{1}{Bn}\frac{d\Phi}{dy} = \frac{-Q_0\sin\alpha}{Bn} + \frac{Q_w[y - y_w]}{2\pi Bn([x - x_w]^2 + [y - y_w]^2)} \qquad (9)$$

where n = porosity; t = time since pumping began; and B = aquifer thickness defined for different aquifers as follows

Confined aquifer:	$B = b$	
Unconfined aquifer:	$B = \phi$	
Combined confined and unconfined aquifer:	$B = b$	for confined part
	$B = \phi$	for unconfined part

For this problem assume that the uniform flow is in the direction of the x axis so that $\alpha = 0$. Equation (7) can then be written

$$x - x_w = [y - y_w] \cot \frac{2\pi}{Q_w}(\Psi - Q_0[y - y_w]) \qquad (10)$$

Substituting (10) into (9) yields

$$dt = \frac{2\pi Bn}{Q_w}[y - y_w]\csc^2 \frac{2\pi}{Q_w}(\Psi - Q_0[y - y_w])\, dy \qquad \ldots (11)$$

After integrating,

$$t = \frac{Bn[y - y_w]}{Q_0} \cot \frac{2\pi}{Q_w}(\Psi - Q_0[y - y_w]) + \frac{BnQ_w}{2\pi Q_0^2} \ln \sin \frac{2\pi}{Q_w}(\Psi - Q_0[y - y_w]) + f(\Psi) \qquad (12)$$

where $f(\Psi)$ is a constant dependent on the particular streamline considered. Equation (12) describes the time when water particles starting at a specific (x, y) coordinate along the streamline will reach the pumping well. When pumping first begins, the particles closest to the well will be captured immediately. In other words, $x = x_w, y = y_w, t = 0$ will be a solution to the equation. Therefore

$$f(\Psi) = \frac{-BnQ_w}{2\pi Q_0^2} \ln \sin \frac{2\pi}{Q_w} \Psi \tag{13}$$

Substituting (13) into (12) yields

$$t = \frac{Bn[y - y_w]}{Q_0} \cot \frac{2\pi}{Q_w} (\Psi - Q_0[y - y_w]) + \frac{Bn}{2\pi Q_0^2} \ln \frac{\sin \frac{2\pi}{Q_w} (\Psi - Q_0[y - y_w])}{\sin \frac{2\pi}{Q_w} \Psi} \tag{14}$$

Substituting (7) into (14) yields

$$t = \frac{Bn[x - x_w]}{Q_0} - \frac{BnQ_w}{2Q_0^2} \ln \frac{\sin\left(\frac{2\pi}{Q_w} Q_0[y - y_w] + \theta\right)}{\sin \theta} \tag{15}$$

where $\theta = \tan^{-1}([y - y_w]/[x - x_w])$.

Three dimensionless parameters may be introduced:

$$\bar{x} = \frac{2\pi Q_0}{Q_w} [x - x_w]; \quad \bar{y} = \frac{2\pi Q_0}{Q_w} [y - y_w]; \quad \bar{t} = \frac{2\pi Q_0^2}{BnQ_w} t \quad \ldots\ldots (16)$$

Substituting (16) into (15) yields

$$\bar{t} = \bar{x} + \ln \frac{\sin\theta}{\sin(\bar{y} + \theta)} \tag{17}$$

or

$$e^{\bar{x} - \bar{t}} = \sin\bar{y} \frac{\bar{x}}{\bar{y}} + \cos\bar{y} \tag{18}$$

Bear and Jacobs (1965) provide additional analysis of equation (18) and its implications for ground-water transport in confined aquifers. Unfortunately, equation (18) is transcendental and cannot be solved explicitly for either x or y. Iterative solutions have been developed for solving special cases of the equation (for example, McElwee, 1991). These solutions are valid for unconfined aquifers as well if the dimensionless parameters introduced in equation (16) are used in equation (18).

3. Single Well in Uniform Flow at Infinite Time (Steady State)

A quick and simple analysis which is useful for many hydrogeologic projects is determining the capture zone of a single well in uniform flow at infinite time, or steady state. This will define an area in which all the water in the aquifer will reach the well if the well pumps for a sufficiently long time. At infinite time equation (18) can be simplified considerably and solved for x. The equations below give three critical parameters, the stagnation point, the upgradient divide, and the equation for the dividing streamline.

For simplicity, consider $x_w = 0$, $y_w = 0$, and $\alpha = 0$ as shown in Figure 1. The stagnation point is where $v_x = v_y = 0$. From equation (9) it is clear that $v_y = 0$ when $y = y_w = 0$. Substituting into equation (8) and solving for x yields

$$x_{STAG} = \frac{Q_w}{2\pi Q_0} \tag{19}$$

where x_{STAG} is the distance from the well to the downgradient stagnation point. As $\bar{t} \to \infty$, the equation for a streamline [equation (18)] becomes

$$\bar{x} = \frac{-\bar{y}}{\tan\bar{y}} \tag{20}$$

As $x \to \infty$ then $\tan y \to 0$, and $y \to N\pi$ where N = integer. Therefore, by equation (16), as $x \to \infty$

$$\bar{y} \to \frac{NQ_w}{2Q_0} \tag{21}$$

Substituting equation (21) into equation (7) with $x \to \infty$ yields

$$\Psi = \frac{NQ_w}{2} \tag{22}$$

The dividing streamline will approach the stagnation point. Substituting equation (22) and the coordinates of the stagnation point into equation (7) yields N = 1. Therefore, as $x \to \infty$, the dividing streamline will approach the line

$$y_{DIV} = \pm \frac{Q_w}{2Q_0} \tag{23}$$

which represents half the width of the capture zone far upgradient of the well. Considering that N = 1, substituting equation (16) into equation (20) yields the equation for the dividing streamline

$$x = \frac{y}{\tan\left(\frac{2\pi Q_0}{Q_w} y\right)} \tag{24}$$

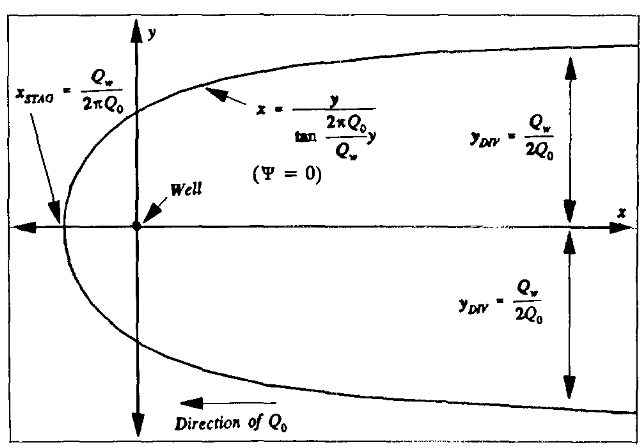

Fig. 1. Stagnation point, upgradient divide, and dividing streamline at infinite time (steady state).

The stagnation point, upgradient divide, and dividing streamline are shown on Figure 1. Because the direction of uniform flow for this problem is aligned with the x axis, $d\Phi/dy = 0$ and

$$Q_0 = \frac{d\Phi}{dx} \cong \frac{\Phi_1 - \Phi_2}{L} \tag{25}$$

where Φ_1 and Φ_2 = downgradient and upgradient discharge potentials, respectively, along a streamline before pumping begins; and L = distance between the locations where Φ_1 and Φ_2 were measured. The equations for the stagnation point, upgradient divide, and dividing streamline can be simplified into more common terms by substituting the above definition for Q_0 and the appropriate definitions for Φ. For a confined aquifer

$$x_{STAG} = \frac{Q_w}{2\pi Ti} \tag{26}$$

where i = natural hydraulic gradient = $d\phi/dx$ and T = aquifer transmissivity = Kb,

$$y_{DIV} = \pm \frac{Q_w}{2Ti} \tag{27}$$

and the dividing streamline is

$$x = \frac{y}{\tan\left(\frac{2\pi Ti}{Q_w} y\right)} \tag{28}$$

For an unconfined aquifer

$$x_{STAG} = \frac{Q_w L}{\pi K (\phi_1^2 - \phi_2^2)} \tag{29}$$

$$y_{DIV} = \pm \frac{Q_w L}{K (\phi_1^2 - \phi_2^2)} \tag{30}$$

and the dividing streamline is

$$x = \frac{y}{\tan\left[\frac{\pi K (\phi_1^2 - \phi_2^2)}{Q_w L} y\right]} \tag{31}$$

Equations (19), (23), and (24) can also be applied to combined confined and unconfined aquifers. To calculate Q_0 for this scenario, substitute the appropriate definition for Φ into equation (25) based on whether Φ was measured in the confined or unconfined part of the aquifer. For example, if Φ_1 is measured in the unconfined portion of the aquifer and Φ_2 is measured in the confined portion of the aquifer, then $\Phi_1 = \frac{1}{2}K\phi_1^2$ and $\Phi_2 = Kb\phi_2 - \frac{1}{2}Kb^2$. Substituting into equation (25) yields

$$Q_0 = \frac{K(\phi_1^2 - 2b\phi_2 + b^2)}{2L} \tag{32}$$

Note also that equation (6) may be used to obtain values of Φ throughout the aquifer by substituting the appropriate x and y coordinates. The effect of several pumping (or injection) wells on the value of Φ at any point in the aquifer may also be determined by using equation (6) and the principle of superposition. A separate equation for Φ is written for each well being considered based on its x_w, y_w, and Q_w. The separate equations are then added to yield one equation for Φ for any point in the aquifer.

4. Example Problem

The data for this example problem were adapted from a site in Wisconsin which formerly had a leaking underground storage tank. The leak had been detected shortly after it occurred, and a pumping well was to be installed to contain the spread of petroleum hydrocarbon contamination in the aquifer. The project hydrogeologist needed to determine the capture zone of the well as part of the pumping system design and evaluation.

In this example, the problem will be solved assuming the aquifer is confined [using equations (26)-(28)] and unconfined [using equations (29)-(31)], and the results will be compared. A site map is shown on Figure 2. Note that the x-axis has been aligned with the ground-water flow direction.

The aquifer and well characteristics are: Hydraulic conductivity (K) (determined from aquifer tests): 72 ft/day; Elevation of the lower confining layer: 1618.00 ft; Elevation of the upper confining layer (confined aquifer only): 1629.00 ft; Measured ground-water elevations in piezometers: P-1 = 1630.50 ft and P-2 = 1629.50 ft; Distance between P-1 and P-2 (L): 235 ft; Pumping rate (Q_w): 963 ft^3/day (5 gpm); ϕ_1 = 1630.50 ft − 1618.00 ft = 12.50 ft; and ϕ_2 = 1629.50 ft − 1618.00 ft = 11.50 ft. A cross section of the aquifer is shown on Figure 3.

For the confined aquifer:

$$b = 1629.00\text{ ft} - 1618.00\text{ ft} = 11.00\text{ ft}$$

$$T = Kb = 790\text{ ft}^2/\text{day}$$

$$i = \frac{\Phi_1 - \Phi_2}{L} = 0.00425$$

$$x_{STAG} = \frac{Q_w}{2\pi Ti} = 46\text{ ft}$$

$$y_{DIV} = \pm \frac{Q_w}{2Ti} = \pm 140\text{ ft}$$

and the dividing streamline is

$$x = \frac{y}{\tan\left(\frac{2\pi Ti}{Q_w} y\right)} = \frac{y}{\tan 0.022y}$$

For the unconfined aquifer:

$$x_{STAG} = \frac{Q_w L}{\pi K (\phi_1^2 - \phi_2^2)} = 42\text{ ft}$$

$$y_{DIV} = \pm \frac{Q_w L}{K (\phi_1^2 - \phi_2^2)} = \pm 130\text{ ft}$$

and the dividing streamline is

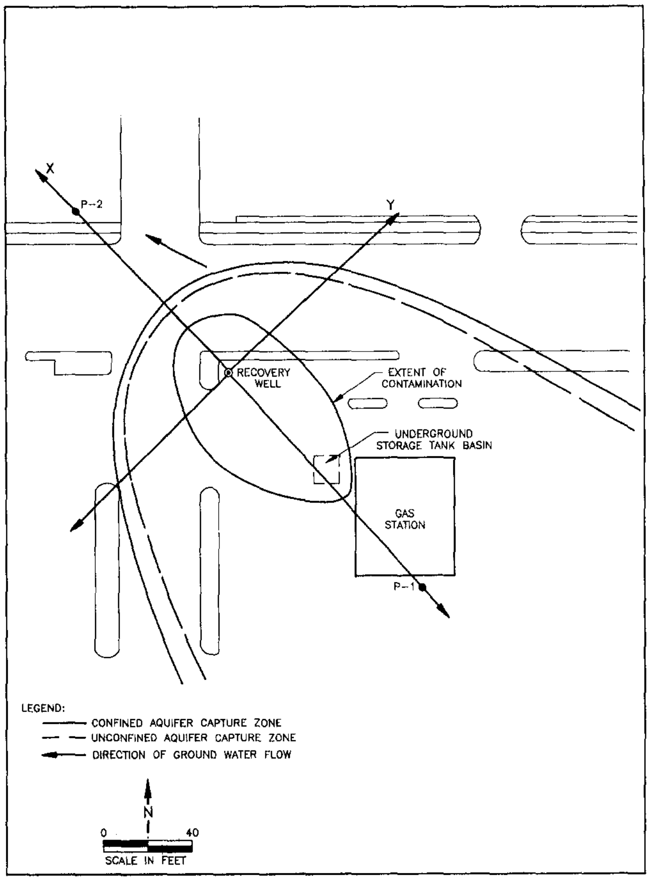

Fig. 2. Site map for the example problem showing the calculated capture zones.

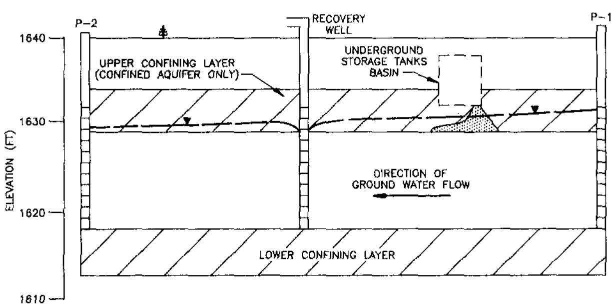

Fig. 3. Cross section of the aquifer along the x-axis.

$$x = \frac{y}{\tan\left[\frac{\pi K(\phi_1^2 - \phi_2^2)}{Q_w L}\, y\right]} = \frac{y}{\tan 0.024y}$$

The results of the analyses are shown on Figure 2.

5. Limitations of the Model

The steady-state equations presented in Section 1 neglect the influence of storativity and specific yield. The significance of this assumption decreases as pumping continues, and by definition storativity and specific yield = 0 at infinite time (steady state). Bear and Jacobs (1965) present a discussion of the effect of neglecting storativity for a confined aquifer with an injection well. They state that the actual front of the water injected from the well will lag behind the calculated front due to the storativity of the well and the aquifer. Similarly, in a pumping situation the actual capture zone will be somewhat smaller than the calculated capture zone due to the water being removed from storage.

The influence of water naturally added to or subtracted from the aquifer system other than regional uniform flow (leakage and infiltration) is not included in the equations. For unconfined aquifers, this may be a good assumption in urban areas or other areas where drainage systems prevent rainfall infiltration. If the addition of water to the aquifer through leakage and infiltration were considered in the equations, the result would be a smaller calculated capture zone.

The model is based on the Dupuit assumption, i.e., vertical gradients are negligible. For this reason, the model may not be accurate in areas of aquifer recharge or discharge, including the area near a well.

Hydrodynamic dispersion is commonly neglected from capture zone analyses. If dispersion were included in the analysis, there would not be a sharp capture zone boundary but rather a wide boundary with width proportional to the dispersion coefficient. Within the boundary only some fraction of the water particles would be captured by the well after a given time.

While the capture zone equations are clearly useful for solving problems related to contaminant transport or wellhead protection, it should be noted that the equations consider only advective flow. The solution to a contaminant transport problem must also incorporate the effects of dispersion, diffusion, sorption, degradation, and retardation.

6. Conclusions

Despite the assumptions and simplifications necessary to derive these equations, the equations can provide useful information for designing pumping systems or wellhead protection programs. Although they do not consider hydrodynamic dispersion, equations (26) through (31) are particularly useful for a quick analysis of critical properties of an aquifer and pumping system. While the many assumptions greatly restrict its applicability, users of the model should find many hydrogeologic problems of limited scope which could benefit from this analysis. The model presented in Section 2 is developed in terms of discharge potentials, which makes the equations applicable to confined, unconfined, and combined confined and unconfined aquifers. Previously derived capture zone equations (and computer programs) could also be modified and written in terms of discharge potentials to make them applicable to both confined and unconfined aquifers.

Computer Programs

A computer program is available which will solve and graph the capture zone equations in this paper. Included on the same computer diskette are spreadsheets for Lotus 1-2-3 and Quattro Pro which solve and graph these equations and other equations commonly used for well design and groundwater modeling. To order these programs, send a check or money order for $20 to Grubb Environmental Services, 2233 15th Avenue, North St. Paul, MN 55109. Please indicate whether you prefer 5.25-inch or 3.5-inch IBM formatted diskettes.

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Nomenclature

- B aquifer thickness [L];
- b confined aquifer thickness [L];
- C constant which corresponds to the elevation of the bottom of the aquifer;
- i natural ground-water gradient [L/L];
- K hydraulic conductivity [L/T];
- L distance between locations where Φ_1 and Φ_2 were measured [L];
- n porosity;
- N integer constant;
- Q_0 discharge vector of uniform flow (L^2/T];
- Q_w discharge from the well [L^3/T];
- Q_{x0} x component of uniform flow [L^2/T];
- Q_{y0} y component of uniform flow [L^2/T];
- t time since pumping began [T];
- T aquifer transmissivity [L^2/T];
- v_x velocity component in the x direction [L/T];
- v_y velocity component in the y direction [L/T];
- x_{STAG} distance from the well to the downgradient stagnation point [L];
- x_w x coordinate of the well [L];
- y_{DIV} y coordinate of the dividing streamline far upgradient of the well [L];
- y_w y coordinate of the well [L];
- z complex potential $x + iy$;
- z_w complex potential at the well;
- α angle between the x axis and uniform flow;
- θ $\tan^{-1}([y - y_w]/[x - x_w])$;
- Φ discharge potential [L^3/T];
- Φ_1 downgradient discharge potential [L^3/T];
- Φ_2 upgradient discharge potential [L^3/T];
- ϕ hydraulic head (or phreatic head) above the bottom of the aquifer [L]; and
- Ψ stream function [L^3/T].

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