Solving Linear Systems

$$Ax = b$$

$$7x = 21$$

$$x = \frac{21}{7} = 3$$

$$x = 7^{-1} \times 21 = .142857 \times 21 = 2.99997$$

The MATLAB Backslash Operator

$$AX = B$$

$$X = A\backslash B$$

$$XA = B$$

$$X = B/A$$

A 3-by-3 Example

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{aligned} 10x_1 - 7x_2 \phantom{{}+5x_3} &= 7 \\ -3x_1 + 2x_2 + 6x_3 &= 4 \\ 5x_1 - x_2 + 5x_3 &= 6 \end{aligned}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ 0 & -0.1 & 6 \\ 0 & 2.5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 6.1 \\ 2.5 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 2.5 \\ 6.1 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 2.5 \\ 6.2 \end{pmatrix}$$

$$6.2x_3 = 6.2$$

$$2.5x_2 + (5)(1) = 2.5.$$

$$10x_1 + (-7)(-1) = 7$$

$$x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.3 & -0.04 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$LU = PA$$

Permutation and Triangular Matrices

permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$Px = b$$

$$x = P^T b$$

upper triangular matrix

$$U = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

unit lower triangular matrix

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 5 & 1 & 0 \\ 4 & 6 & 7 & 1 \end{pmatrix}$$

LU Factorization

$$U = M_{n-1}P_{n-1}\cdots M_2P_2M_1P_1A$$

$$L_1L_2\cdots L_{n-1}U = P_{n-1}\cdots P_2P_1A$$

$$L = L_1L_2\cdots L_{n-1}$$
$$P = P_{n-1}\cdots P_2P_1$$

$$LU = PA$$

$$A = \begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ -0.5 & 0 & 1 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.04 & 1 \end{pmatrix},$$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.3 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.04 & 1 \end{pmatrix},$$

$$Ax = b$$

$$Ly = Pb$$
$$Ux = y$$

Why Is Pivoting Necessary?

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3.901 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 6.001 \\ 2.5 \end{pmatrix}$$

$$(5 + (2.5 \cdot 10^3)(6))x_3 = (2.5 + (2.5 \cdot 10^3)(6.001))$$

$$(5 + 1.5000 \cdot 10^4)x_3 = (2.\mathit{5} + 1.5002\mathit{5} \cdot 10^4)$$

$$1.5005 \cdot 10^4 x_3 = 1.5004 \cdot 10^4$$

$$x_3 = \frac{1.5004 \cdot 10^4}{1.5005 \cdot 10^4} = 0.99993$$

$$-0.001x_2 + (6)(0.99993) = 6.001$$

$$x_2 = \frac{1.5 \cdot 10^{-3}}{-1.0 \cdot 10^{-3}} = -1.5$$

$$10x_1 + (-7)(-1.5) = 7$$

$$x_1 = -0.35$$

Effect of Roundoff Errors

error

$$e = x - x_*$$

residual

$$r = b - Ax_*$$

$$\begin{pmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.217 \\ 0.254 \end{pmatrix}$$

$$\frac{0.780}{0.913} = 0.854 \quad \text{(to three places)}$$

$$\begin{pmatrix} 0.913 & 0.659 \\ 0 & 0.001 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.254 \\ 0.001 \end{pmatrix}$$

$$x_2 = \frac{0.001}{0.001} = 1.00 \quad \text{(exactly)},$$

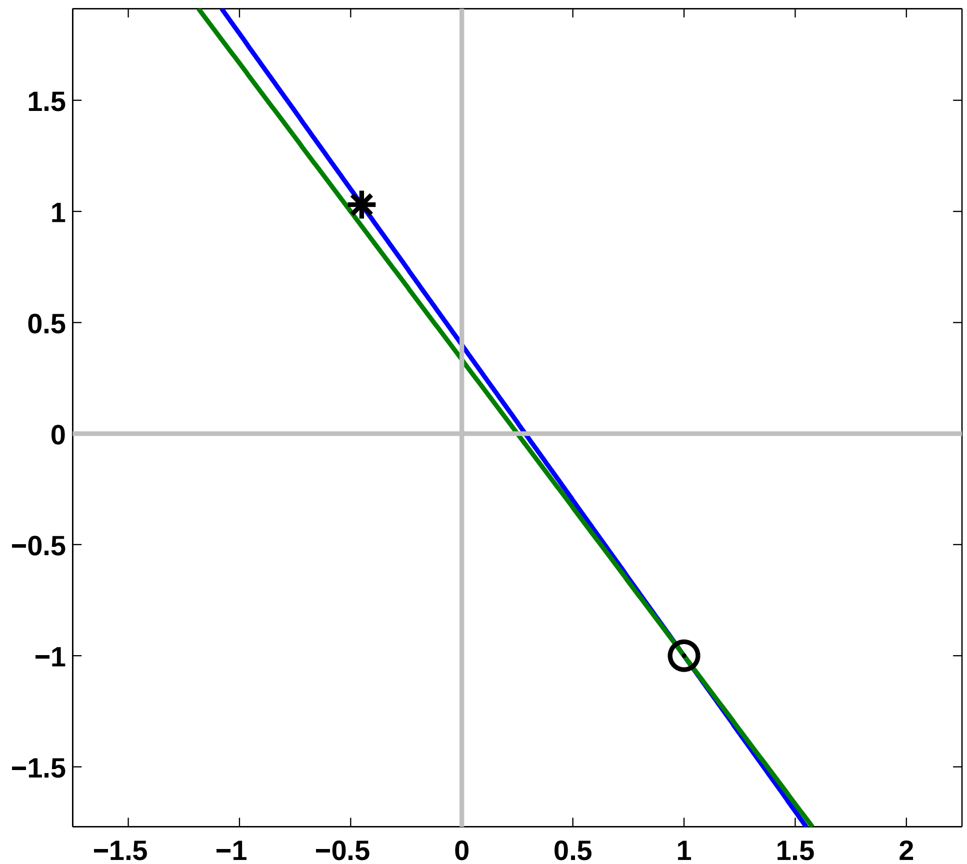
$$\begin{aligned} x_1 &= \frac{0.254 - 0.659x_2}{0.913} \\ &= -0.443 \quad \text{(to three places)}. \end{aligned}$$

$$x_* = \begin{pmatrix} -0.443 \\ 1.000 \end{pmatrix}$$

$$\begin{aligned} r = b - Ax_* &= \begin{pmatrix} 0.217 - ((0.780)(-0.443) + (0.563)(1.00)) \\ 0.254 - ((0.913)(-0.443) + (0.659)(1.00)) \end{pmatrix} \\ &= \begin{pmatrix} -0.000460 \\ -0.000541 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 0.913000 & 0.659000 \\ 0 & 0.000001 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.254000 \\ -0.000001 \end{pmatrix}$$

$$\begin{aligned} x_2 &= \frac{-0.000001}{0.000001} = -1.00000, \\ x_1 &= \frac{0.254 - 0.659 x_2}{0.913} \\ &= 1.00000, \end{aligned}$$


1.5
1
0.5
0
−0.5
−1
−1.5
−1.5
−1
−0.5
0
0.5
1
1.5
2

Gaussian elimination with partial pivoting is guaranteed to produce small residuals.

Norms and Condition Numbers

$$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$$

$$\begin{aligned}
\|x\|_1 &= \sum_{i=1}^{n} |x_i| \\
\|x\|_2 &= (\sum_{i=1}^{n} |x_i|^2)^{1/2} \\
\|x\|_\infty &= \max_i |x_i|
\end{aligned}$$

$$\begin{aligned}
\|x\| &> 0 \ \text{ if } x \neq 0 \\
\|0\| &= 0 \\
\|cx\| &= |c|\|x\| \text{ for all scalars } c \\
\|x+y\| &\leq \|x\| + \|y\|, \ \text{(the \textit{triangle inequality})}
\end{aligned}$$

$$M = \max \frac{\|Ax\|}{\|x\|}$$

$$m = \min \frac{\|Ax\|}{\|x\|}$$

$$\kappa(A) = \frac{\max \frac{\|Ax\|}{\|x\|}}{\min \frac{\|Ax\|}{\|x\|}}$$

$$\kappa(A) = \|A\|\|A^{-1}\|$$

$$Ax = b$$

$$A(x + \delta x) = b + \delta b$$

$$\|b\| \leq M\|x\|$$

$$\|\delta b\| \geq m\|\delta x\|$$

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A)\frac{\|\delta b\|}{\|b\|}$$

$$\kappa(A) \geq 1$$

$$\kappa(P) = 1$$

$$\kappa(cA) = \kappa(A)$$

$$\kappa(D) = \frac{\max |d_{ii}|}{\min |d_{ii}|}$$

$$
\begin{aligned}
A &= \begin{pmatrix} 4.1 & 2.8 \\ 9.7 & 6.6 \end{pmatrix} \\
b &= \begin{pmatrix} 4.1 \\ 9.7 \end{pmatrix} \\
x &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{aligned}
$$

$$\|b\| = 13.8, \quad \|x\| = 1$$

$$\tilde{b} = \begin{pmatrix} 4.11 \\ 9.70 \end{pmatrix}$$

$$\tilde{x} = \begin{pmatrix} 0.34 \\ 0.97 \end{pmatrix}$$

$$\begin{aligned} \|\delta b\| &= 0.01 \\ \|\delta x\| &= 1.63 \end{aligned}$$

$$
\begin{aligned}
\frac{\|\delta b\|}{\|b\|} &= 0.0007246 \\
\frac{\|\delta x\|}{\|x\|} &= 1.63
\end{aligned}
$$

$$\kappa(A) \geq \frac{1.63}{0.0007246} = 2249.4$$

$$\kappa(A) = 2249.4$$

$$\frac{\|b - Ax_*\|}{\|A\|\|x_*\|} \leq \rho\epsilon,$$

$$\frac{\|x - x_*\|}{\|x_*\|} \leq \rho\kappa(A)\epsilon$$

$$\|A\| = \max \frac{\|Ax\|}{\|x\|}$$

$$\begin{aligned}
\|A\|_1 &= \max_j \sum_i |a_{i,j}| \\
\|A\|_\infty &= \max_i \sum_j |a_{i,j}|
\end{aligned}$$

$$(A+E)x_* = b$$

$$\frac{\|E\|}{\|A\|} = \rho\epsilon$$

$$b - Ax_* = Ex_*$$

$$\|b - Ax_*\| = \|Ex_*\| \leq \|E\|\|x_*\|$$

$$\frac{\|b - Ax_*\|}{\|A\|\|x_*\|} \leq \rho\epsilon$$

$$x - x_* = A^{-1}(b - Ax_*)$$

$$\|x - x_*\| \leq \|A^{-1}\|\|E\|\|x_*\|$$

$$\frac{\|x - x_*\|}{\|x_*\|} \leq \rho\|A\|\|A^{-1}\|\epsilon$$

$$\frac{\|x - x_*\|}{\|x_*\|} \leq \rho\kappa(A)\epsilon$$

Sparse Matrices and Band Matrices

$$\begin{pmatrix} b_1 & c_1 & & & & \\ a_1 & b_2 & c_2 & & & \\ & a_2 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-2} & b_{n-1} & c_{n-1} \\ & & & & a_{n-1} & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ d_n \end{pmatrix}$$

PageRank and Markov Chains

$$g_{ij} = 1, \text{ if node}_j \rightarrow \text{node}_i$$

$$c_j = \sum_i g_{ij}, \quad r_i = \sum_j g_{ij}$$

$$a_{ij} = pg_{ij}/c_j + \delta, \text{ where } \delta = (1-p)/n.$$

$$x = Ax$$

$$\sum_i x_i = 1$$

$$(I - A)x = 0$$

$$x = Ax$$

$$x = \left(p\tilde{G} + \delta e e^T\right) x$$

$$e^T x = 1$$

$$(I - p\tilde{G})x = \delta e$$

```
[U,G] = surfer('http: ... ');
c = sum(G)
D = spdiags(1./c',0,n,n)
p = .85
delta = (1-p)/n
e = ones(n,1)
I = speye(n,n)
x = (I - p*G*D)\(delta*e)
```

Power method

```
while termination_test
   x = A*x;
end
```

Inverse iteration

```
x = (I - A)\e
x = x/sum(x)
```