Eigenvalue and *Eigenvector*

$$Ax = \lambda x$$

Singular value and *Singular vectors*

$$\begin{aligned} Av &= \sigma u \\ A^H u &= \sigma v \end{aligned}$$

Eigenvalue

$$(A - \lambda I)x = 0, \;\; x \neq 0$$

Characteristic polynomial of A.

$$\det(A - \lambda I) = 0$$

$$AX = X\Lambda$$

$$A = X\Lambda X^{-1}$$

$$A^p = X\Lambda^p X^{-1}$$

Similarity transformation

$$B = T^{-1}AT$$

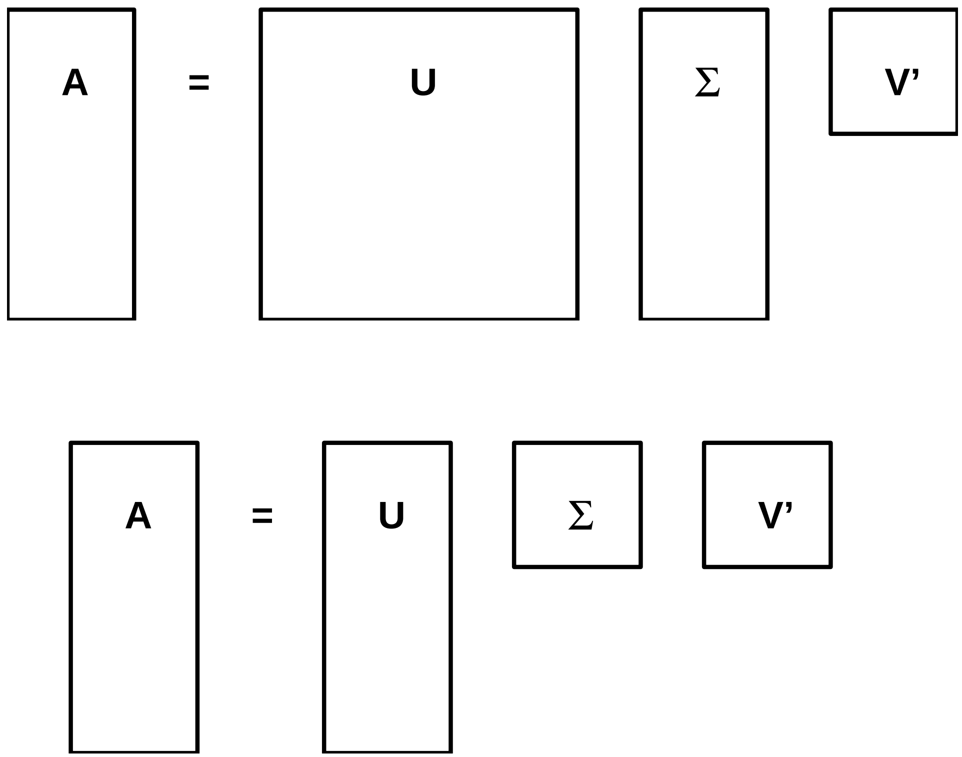
Singular value

$$AV = U\Sigma$$
$$A^H U = V\Sigma^H$$

SVD

$$A = U\Sigma V^H$$

Economy-sized SVD



```
A = gallery(3)
```

$$A = \begin{pmatrix} -149 & -50 & -154 \\ 537 & 180 & 546 \\ -27 & -9 & -25 \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \lambda^3 - 6\lambda^2 + 11\lambda - 6 \\ &= (\lambda - 1)(\lambda - 2)(\lambda - 3) \end{aligned}$$

$\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & -4 & 7 \\ -3 & 9 & -49 \\ 0 & 1 & 9 \end{pmatrix}$$

$$X^{-1} = \begin{pmatrix} 130 & 43 & 133 \\ 27 & 9 & 28 \\ -3 & -1 & -3 \end{pmatrix}$$

$$A = X \Lambda X^{-1}$$

$$\sigma^6 - 668737\sigma^4 + 4096316\sigma^2 - 36 = 0$$

Does not factor nicely.

```
[U,S,V] = svd(A)

U =
   -0.2691   -0.6798    0.6822
    0.9620   -0.1557    0.2243
   -0.0463    0.7167    0.6959

S =
  817.7597         0         0
         0    2.4750         0
         0         0    0.0030

V =
    0.6823   -0.6671    0.2990
    0.2287   -0.1937   -0.9540
    0.6944    0.7193    0.0204
```

Characteristic polynomial

```
A = diag(1:20)
```

$\det(A - \lambda I) =$

$$\lambda^{20} - 210\lambda^{19} + 20615\lambda^{18} - 1256850\lambda^{17} + 53327946\lambda^{16}$$

$$-1672280820\lambda^{15} + 40171771630\lambda^{14} - 756111184500\lambda^{13}$$

$$+11310276995381\lambda^{12} - 135585182899530\lambda^{11}$$

$$+1307535010540395\lambda^{10} - 10142299865511450\lambda^{9}$$

$$+63030812099294896\lambda^{8} - 311333643161390640\lambda^{7}$$

$$+1206647803780373360\lambda^{6} - 3599979517947607200\lambda^{5}$$

$$+8037811822645051776\lambda^{4} - 12870931245150988800\lambda^{3}$$

$$+13803759753640704000\lambda^{2} - 8752948036761600000\lambda$$

$$+2432902008176640000$$

1.00000000000000
2.00000000000096
2.99999999986640
4.00000000495944
4.99999991473414
6.00000084571661
6.99999455544845
8.00002443256894
8.99992001186835
10.00019696490537
10.99962843024064
12.00054374363591
12.99938073455790
14.00054798867380
14.99962658217055
16.00019208303847
16.99992773461773
18.00001875170604
18.99999699774389
20.00000022354640

Eigenvalue Sensitivity and Accuracy

$$A = X \Lambda X^{-1}$$

$$\Lambda = X^{-1} A X$$

$$\Lambda + \delta \Lambda = X^{-1} (A + \delta A) X$$

$$\delta \Lambda = X^{-1} \delta A X$$

$$\| \delta \Lambda \| \leq \| X^{-1} \| \| X \| \| \delta A \| = \kappa(X) \| \delta A \|$$

The sensitivity of the eigenvalues is estimated by the condition number of the matrix of eigenvectors.

```
A = gallery(3)
[X,lambda] = eig(A);
condest(X)

  1.2002e+003
```

Left eigenvectors

$$y^H A = \lambda y^H$$

$$Ax = \lambda x$$

$$\dot{A}x + A\dot{x} = \dot{\lambda}x + \lambda\dot{x}$$

$$y^H \dot{A}x + y^H A\dot{x} = y^H \dot{\lambda}x + y^H \lambda\dot{x}$$

$$\dot{\lambda} = \frac{y^H \dot{A}x}{y^H x}$$

$$|\dot{\lambda}| \le \frac{\|y\|\|x\|}{y^H x}\|\dot{A}\|$$

Eigenvalue condition

$$\kappa(\lambda, A) = \frac{\|y\|\|x\|}{y^H x}$$

$$|\dot{\lambda}| \leq \kappa(\lambda, A)\|\dot{A}\|$$

$$\kappa(\lambda, A) \geq 1$$

$$Y^H = X^{-1}$$

$$Y^H A = \Lambda Y^H$$

$$Y^H X = I$$

$$\kappa(\lambda, A) = y^H x = 1$$

$$\kappa(\lambda, A) = \|y\|\|x\|$$

$$\|x\| \leq \|X\|, \quad \|y\| \leq \|X^{-1}\|$$

$$\kappa(\lambda, A) \leq \kappa(X)$$

```
A = gallery(3)
lambda = eig(A)
kappa = condeig(A)

lambda =
    1.0000
    2.0000
    3.0000

kappa =
  603.6390
  395.2366
  219.2920
```

```
lambda = eig(A + 1.e-6*randn(3,3))

   1.00011344999452
   1.99992040276116
   2.99996856435075

lambda - (1:3)'

  1.0e-003 *
   0.11344999451923
  -0.07959723883699
  -0.03143564924635

delta*condeig(A)

  1.0e-003 *
   0.60363896495665
   0.39523663799014
   0.21929204271846
```

If A is real and symmetric, or complex and Hermitian

$$y^H x = \|y\| \|x\|$$

$$\kappa(\lambda, A) = 1$$

Multiple eigenvalue

$$p(\lambda) = \mathsf{det}(A - \lambda I) = (\lambda - \lambda_k)^m q(\lambda)$$

$$p(\lambda) = O(\delta)$$

$$(\lambda - \lambda_k)^m = O(\delta)/q(\lambda)$$

$$\lambda = \lambda_k + O(\delta^{1/m})$$

16-by-16

$$A = \begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & \ddots & \ddots & \\ & & & 2 & 1 \\ \delta & & & & 2 \end{pmatrix}$$

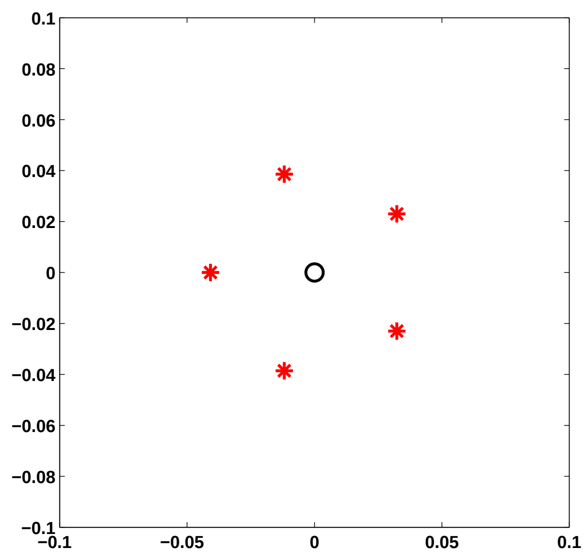
$$(\lambda - 2)^{16} = \delta$$

$$(10^{-16})^{1/16} = 0.1$$

```
A = gallery(5)

A =
        -9          11         -21          63        -252
        70         -69         141        -421        1684
      -575         575       -1149        3451      -13801
      3891       -3891        7782      -23345       93365
      1024       -1024        2048       -6144       24572

lambda = eig(A)

lambda =
  -0.0408
  -0.0119 + 0.0386i
  -0.0119 - 0.0386i
   0.0323 + 0.0230i
   0.0323 - 0.0230i
```

$$\lambda^5 = 0$$

```
A = gallery(5)
e = eig(A)
plot(real(e),imag(e),'r*',0,0,'ko')
axis(.1*[-1 1 -1 1]), axis square
```




```
e = eig(A + eps*randn(5,5).*A)
```

Singular Value Sensitivity and Accuracy

$$\Sigma + \delta\Sigma = U^H(A + \delta A)V$$

$$\|\delta\Sigma\| = \|\delta A\|$$

```
A = gallery(5)
format long e
svd(A)

 1.010353607103610e+005
 1.679457384066496e+000
 1.462838728086172e+000
 1.080169069985612e+000
 4.988578262459575e-014
```

```
while 1
   clc
   svd(A+eps*randn(5,5).*A)
   pause(.25)
end
```

```
 1.010353607103610e+005
 1.67945738406****e+000
 1.46283872808****e+000
 1.08016906998****e+000
 *.***************-0**
```

Jordan form

$$A = XJX^{-1}$$

Schur form

$$B = T^H A T$$

```
A = gallery(3)
[T,B] = schur(A)

A =
  -149   -50  -154
   537   180   546
   -27    -9   -25
T =
    0.3162   -0.6529    0.6882
   -0.9487   -0.2176    0.2294
    0.0000    0.7255    0.6882
B =
    1.0000   -7.1119 -815.8706
         0    2.0000  -55.0236
         0         0    3.0000
```

```
n = size(A,1)
I = eye(n,n)

s = A(n,n);  [Q,R] = qr(A - s*I);  A = R*Q + s*I
```

$$A - sI = QR$$

$$RQ + sI = Q^T(A - sI)Q + sI = Q^T AQ$$

```
A = gallery(3)

    -149      -50     -154
     537      180      546
     -27       -9      -25

     28.8263 -259.8671  773.9292
      1.0353   -8.6686   33.1759
     -0.5973    5.5786  -14.1578

     2.7137  -10.5427 -814.0932
    -0.0767    1.4719  -76.5847
     0.0006   -0.0039    1.8144

     3.0716   -7.6952  802.1201
     0.0193    0.9284  158.9556
    -0.0000    0.0000    2.0000
```

Principal Component Analysis

$$A = U\Sigma V^T$$

$$A = E_1 + E_2 + \ldots + E_p,\ \ p = \mathrm{min}(m, n)$$

$$E_k = \sigma_k u_k v_k^T$$

$$E_j E_k^T = 0,\ \ j \neq k$$

$$\|E_k\| = \sigma_k$$

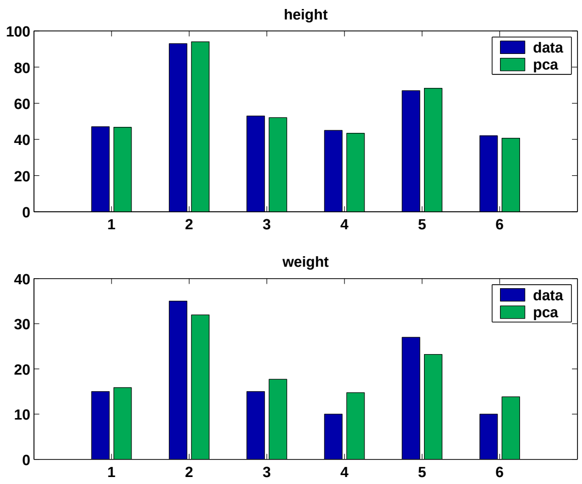
$$A_r = E_1 + E_2 + \ldots + E_r$$

$$\|A - A_r\| = \sigma_{r+1}$$

$$A^T A V = V \Sigma^2$$

$$U \Sigma = A V$$

height	weight
47	15
93	35
53	15
45	10
67	27
42	10



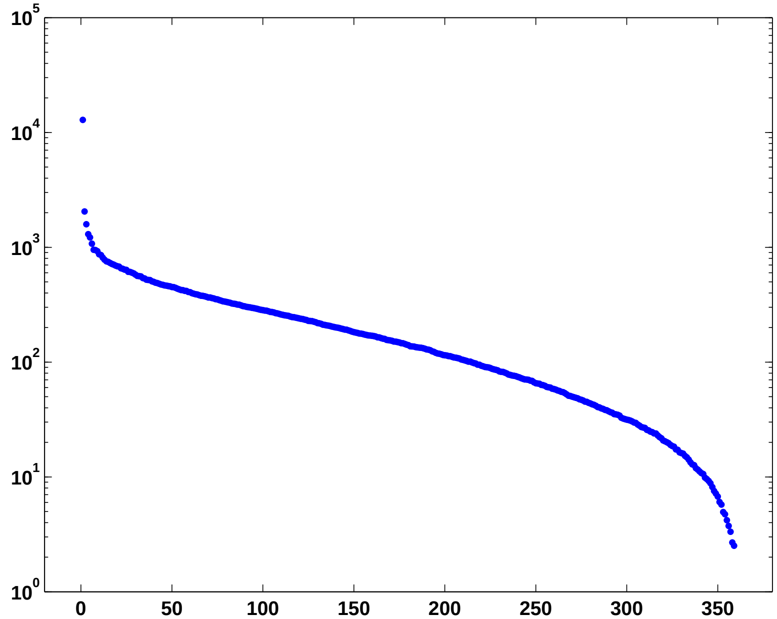
```
[U,S,V] = svd(A,0), sigma = diag(S)

U =
    0.3153    0.1056
    0.6349   -0.3656
    0.3516    0.3259
    0.2929    0.5722
    0.4611   -0.4562
    0.2748    0.4620

V =
    0.9468    0.3219
    0.3219   -0.9468

sigma =
  156.4358
    8.7658
```

```
E1 = sigma(1)*U(:,1)*V(:,1)'

E1 =
   46.7021   15.8762
   94.0315   31.9657
   52.0806   17.7046
   43.3857   14.7488
   68.2871   23.2139
   40.6964   13.8346

size = sigma(1)*U(:,1)

size =
   49.3269
   99.3163
   55.0076
   45.8240
   72.1250
   42.9837
```

height ≈ size*V(1,1)

weight ≈ size*V(2,1)

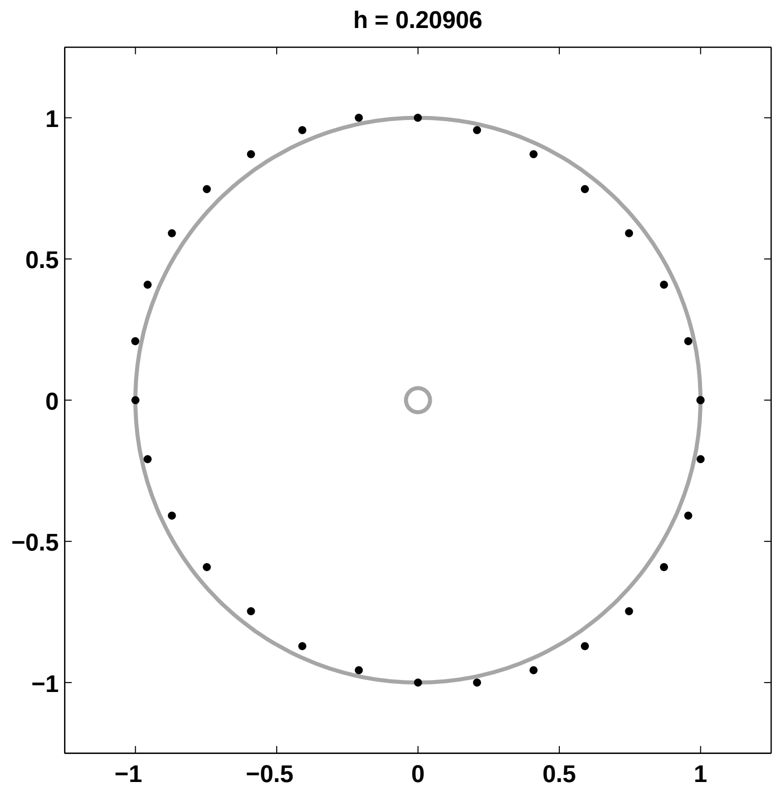
```
load detail
subplot(2,2,1)
image(X)
colormap(gray(64))
axis image, axis off
r = rank(X)
title(['rank = ' int2str(r)])

[U,S,V] = svd(X,0);
sigma = diag(S);
semilogy(sigma,'.')
```



10^5
10^4
10^3
10^2
10^1
10^0
0
50
100
150
200
250
300
350

Circle Generator

```
   x = 32768
   y = 0
L: load y
   shift right 5 bits
   add x
   store in x
   change sign
   shift right 5 bits
   add y
   store in y
   plot x y
   go to L
```


h = 0.20906
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1

```
h = 1/32;
x = 1;
y = 0;
while 1
   x = x + h*y;
   y = y - h*x;
   plot(x,y,'.')
   drawnow
end
```

$$
\begin{aligned}
x_{n+1} &= x_n + h y_n \\
y_{n+1} &= y_n - h x_{n+1}
\end{aligned}
$$

$$
\begin{aligned}
x_{n+1} &= x_n + h y_n \\
y_{n+1} &= -h x_n + (1 - h^2) y_n
\end{aligned}
$$

$$A = \begin{pmatrix} 1 & h \\ -h & 1 - h^2 \end{pmatrix}$$

$$x_{n+1} = Ax_n$$

$$x_n = A^n x_0$$

```
[X,Lambda] = eig(A)
```

$AX = X\Lambda$

If X^{-1} exists

$A = X\Lambda X^{-1}$

$A^n = X\Lambda^n X^{-1}$

$|\lambda_k| \leq 1$

```
h = 2*rand, A = [1 h; -h 1-h^2], lambda = eig(A), abs(lambda)
```

Use up arrow to iterate.

For any h in the interval $0 < h < 2$, the eigenvalues of the circle generator matrix A are complex numbers with absolute value 1.

```
syms h
A = [1 h; -h 1-h^2]
lambda = eig(A)

A =
[      1,      h]
[     -h, 1-h^2]

lambda =
[ 1-1/2*h^2+1/2*(-4*h^2+h^4)^(1/2)]
[ 1-1/2*h^2-1/2*(-4*h^2+h^4)^(1/2)]
```

```
d = det(A)
```

or

```
d = simple(prod(lambda))

d =
1
```

$$\lambda = 1 - h^2/2 \pm h\sqrt{-1 + h^2/4}$$

$$\cos\theta = 1 - h^2/2$$

$$\sin\theta = h\sqrt{1 - h^2/4}$$

$$\lambda = \cos\theta \pm i\sin\theta$$

```
theta = acos(1-h^2/2);
Lambda = [cos(theta)-i*sin(theta); cos(theta)+i*sin(theta)]
diff = simple(lambda-Lambda)

Lambda =
[ 1-1/2*h^2-1/2*i*(4*h^2-h^4)^(1/2)]
[ 1-1/2*h^2+1/2*i*(4*h^2-h^4)^(1/2)]

diff =
[ 0]
[ 0]
```

If $|h| < 2$,

$$\lambda = e^{\pm i\theta}$$

$$A^n = X \begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix} X^{-1}$$

$$\dot{x} = Qx$$

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} x(0)$$

$$\begin{pmatrix} \cos h & \sin h \\ -\sin h & \cos h \end{pmatrix}$$

generates perfect circles

$$A = \begin{pmatrix} 1 & h \\ -h & 1 - h^2 \end{pmatrix}$$