Model Problems

In one space dimension

$$\triangle = \frac{\partial^2}{\partial x^2}$$

In two space dimensions

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Poisson equation

$$\triangle u = f(\vec{x})$$

Heat equation

$$\frac{\partial u}{\partial t} = \triangle u - f(\vec{x})$$

$$u(\vec{x}, 0) = u_0(\vec{x})$$

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = \triangle u$$

$$u(\vec{x}, 0) = u_0(\vec{x})$$

$$\frac{\partial u}{\partial t}(\vec{x}, 0) = 0$$

Finite difference methods

In one dimension

$$a \leq x \leq b$$

$$h = (b - a)/(m + 1)$$

$$x_i = a + ih,\ i = 0, \ldots, m + 1$$

$$\triangle_h u(x) = \frac{u(x + h) - 2u(x) + u(x - h)}{h^2}$$

In two dimensions

$$(x_i, y_j) = (ih, jh)$$

$$\triangle_h u(x,y) = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}$$

$$
\begin{aligned}
P &= (x, y) \\
N &= (x, y + h) \\
E &= (x + h, y) \\
S &= (x, y - h) \\
W &= (x - h, y)
\end{aligned}
$$

$$
\triangle_h u(P) = \frac{u(N) + u(W) + u(E) + u(S) - 4u(P)}{h^2}
$$

Poisson problem

$$\triangle_h u(\vec{x}) = f(\vec{x})$$

If $f(\vec{x})$ is zero,

$$\triangle_h u(x) = 0$$

Heat equation

$$\frac{u(\vec{x}, t+\delta) - u(\vec{x}, t)}{\delta} = \triangle_h u(\vec{x})$$

$$u(\vec{x}, 0) = u_0(\vec{x})$$

$$u(\vec{x}, t+\delta) = u(\vec{x}, t) + \delta \triangle_h u(\vec{x}, t)$$

Wave equation

$$\frac{u(\vec{x}, t+\delta) - 2u(\vec{x}, t) + u(\vec{x}, t-\delta)}{\delta^2} = \triangle_h u(\vec{x}, t)$$

$$\frac{\partial u}{\partial t}(\vec{x}, 0) = 0$$

$$u(\vec{x}, 0) = u_0(\vec{x}), \text{and } u(\vec{x}, \delta) = u_0(\vec{x})$$

$$u(\vec{x}, t+\delta) = 2u(\vec{x}, t) - u(\vec{x}, t-\delta) + \delta^2 \triangle_h u(\vec{x}, t)$$

Matrix Representation

$$\frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

Poisson problem

$$Au = b$$

```
u = A\b
```

```
L =

     0     0     0     0     0     0     0     0     0     0     0
     0     1     5     9    13    17    21    30    39    48     0
     0     2     6    10    14    18    22    31    40    49     0
     0     3     7    11    15    19    23    32    41    50     0
     0     4     8    12    16    20    24    33    42    51     0
     0     0     0     0     0     0    25    34    43    52     0
     0     0     0     0     0     0    26    35    44    53     0
     0     0     0     0     0     0    27    36    45    54     0
     0     0     0     0     0     0    28    37    46    55     0
     0     0     0     0     0     0    29    38    47    56     0
     0     0     0     0     0     0     0     0     0     0     0
```

$$h^2 \triangle_h u(43) = u(34) + u(42) + u(44) + u(52) - 4u(43)$$

$$a_{43,34} = a_{43,42} = a_{43,44} = a_{43,52} = 1, \text{ and } a_{43,43} = -4$$

```
A =

    -4   1   1   0   0   0   0   0   0   0   0   0   0   0   0   0
     1  -4   0   1   0   0   0   0   0   0   0   0   0   0   0   0
     1   0  -4   1   1   0   0   0   0   0   0   0   0   0   0   0
     0   1   1  -4   0   1   0   0   0   0   0   0   0   0   0   0
     0   0   1   0  -4   1   1   0   0   0   0   0   0   0   0   0
     0   0   0   1   1  -4   0   1   0   0   0   0   0   0   0   0
     0   0   0   0   1   0  -4   1   0   0   0   1   0   0   0   0
     0   0   0   0   0   1   1  -4   1   0   0   0   1   0   0   0
     0   0   0   0   0   0   0   1  -4   1   0   0   0   1   0   0
     0   0   0   0   0   0   0   0   1  -4   1   0   0   0   1   0
     0   0   0   0   0   0   0   0   0   1  -4   0   0   0   0   1
     0   0   0   0   0   0   1   0   0   0   0  -4   1   0   0   0
     0   0   0   0   0   0   0   1   0   0   0   1  -4   1   0   0
     0   0   0   0   0   0   0   0   1   0   0   0   1  -4   1   0
     0   0   0   0   0   0   0   0   0   1   0   0   0   1  -4   1
     0   0   0   0   0   0   0   0   0   0   1   0   0   0   1  -4
```

Numerical Stability

$$u^{(k+1)} = u^{(k)} + \sigma A u^{(k)}$$

$$\sigma = \frac{\delta}{h^2}$$

$$u^{(k+1)} = M u^{(k)}$$

$$M = I + \sigma A$$

For the heat equation, in one dimension

$$\sigma \le \frac{1}{2}$$

In two dimensions

$$\sigma \le \frac{1}{4}$$

$$u^{(k+1)} = 2u^{(k)} - u^{(k-1)} + \sigma A u^{(k)}$$

$$\sigma = \frac{\delta^2}{h^2}$$

For the wave equation in one dimension

$$\sigma \leq 1$$

In two dimensions

$$\sigma \leq \frac{1}{2}$$

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = \triangle u$$

$$u(\vec{x}, t) = \cos\left(\sqrt{\lambda}\, t\right)\; v(\vec{x})$$

$$\triangle v + \lambda v = 0$$

In one dimension

$$v_k(x) = \sin(kx)$$

$$v_k(\pi) = 0$$

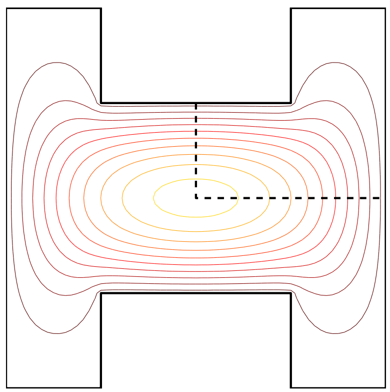
k must be an integer

$$\lambda_k = k^2.$$

$$u_0(x) = \sum_k a_k \sin(kx)$$

$$\begin{aligned} u(x,t) &= \sum_k a_k \cos(kt) \sin(kx) \\ &= \sum_k a_k \cos(\sqrt{\lambda_k}\, t)\ v_k(x) \end{aligned}$$

H-shaped domain



L-shaped membrane

Finite difference methods

```
m = 200
h = 1/m
A = delsq(numgrid('L',2*m+1))/h^2

size(A) = 119201-by-119201
nnz(A) = 594409

lambda = eigs(A,6,0)
```

```
lambda =
   9.64147
  15.19694
  19.73880
  29.52033
  31.91583
  41.47510
```

The exact values are

```
   9.63972
  15.19725
  19.73921
  29.52148
  31.91264
  41.47451
```

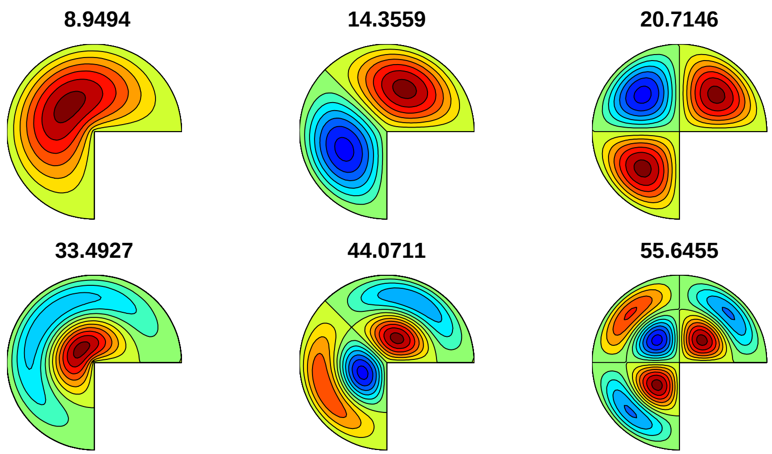
Circular sector with angle π/α and radius R.

$$v(r,\theta) = J_\alpha(\sqrt{\lambda}\, r)\, \sin\,(\alpha\, \theta)$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \lambda v = 0$$

$$v(r,0) = 0, \text{and } v(r,\pi/\alpha) = 0$$

$$J_\alpha(\sqrt{\lambda} R) = 0$$


8.9494
14.3559
20.7146
33.4927
44.0711
55.6455

L-shaped membrane

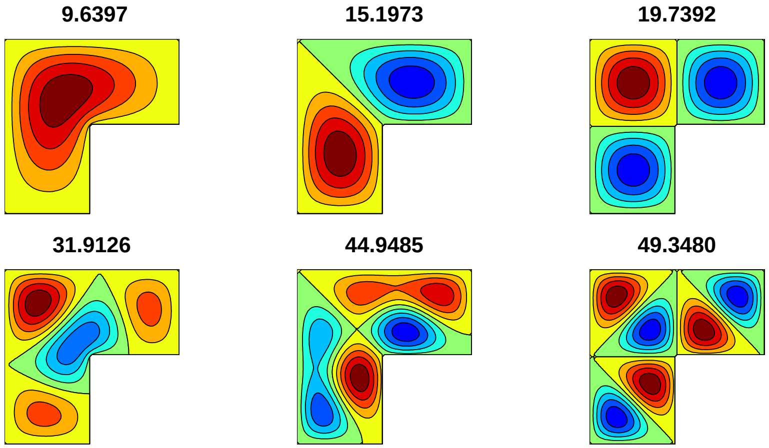
$$v(r,\theta) = \sum_j c_j J_{\alpha_j}(\sqrt{\lambda}\, r) \sin(\alpha_j\, \theta)$$

$$\alpha_j = \frac{2j}{3}$$

$$A_{i,j}(\lambda) = J_{\alpha_j}(\sqrt{\lambda}\, r_i) \sin(\alpha_j\, \theta_i), \;\; i = 1, \ldots, m, \;\; j = 1, \ldots, n$$

$\sigma_n(A(\lambda))$ = smallest singular value of $A(\lambda)$.

$$\lambda_k = k\text{-th minimizer}(\sigma_n(A(\lambda)))$$


9.6397
15.1973
19.7392
31.9126
44.9485
49.3480

Symmetric about the center line.

$\alpha_j = \frac{2j}{3}$, j odd and not a multiple of 3.

Antisymmetric about the center line.

$\alpha_j = \frac{2j}{3}$, j even and not a multiple of 3.

Eigenfunction of the square.

$\alpha_j = \frac{2j}{3}$, j a multiple of 3.

Heat Equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Explicit Finite Difference Scheme

$$\begin{array}{ccccc} & & n & & \\ & & | & & \\ w & - & p & - & e \\ & & | & & \\ & & s & & \end{array}$$

$$\sigma = \frac{\delta t}{(\delta x)^2}$$

$$u_p^{k+1} = (1 - 4\sigma)u_p^k + \sigma(u_e^k + u_w^k + u_s^k + u_n^k)$$

ADI – Alternating Directions Implicit

$$-\sigma u_w^{k+\frac{1}{2}} + (1+2\sigma)u_p^{k+\frac{1}{2}} - \sigma u_e^{k+\frac{1}{2}} =$$

$$\sigma u_n^k + (1-2\sigma)u_p^k + \sigma u_s^k$$

$$-\sigma u_n^{k+1} + (1+2\sigma)u_p^{k+1} - \sigma u_s^{k+1} =$$

$$\sigma u_w^{k+\frac{1}{2}} + (1-2\sigma)u_p^{k+\frac{1}{2}} + \sigma u_e^{k+\frac{1}{2}}$$