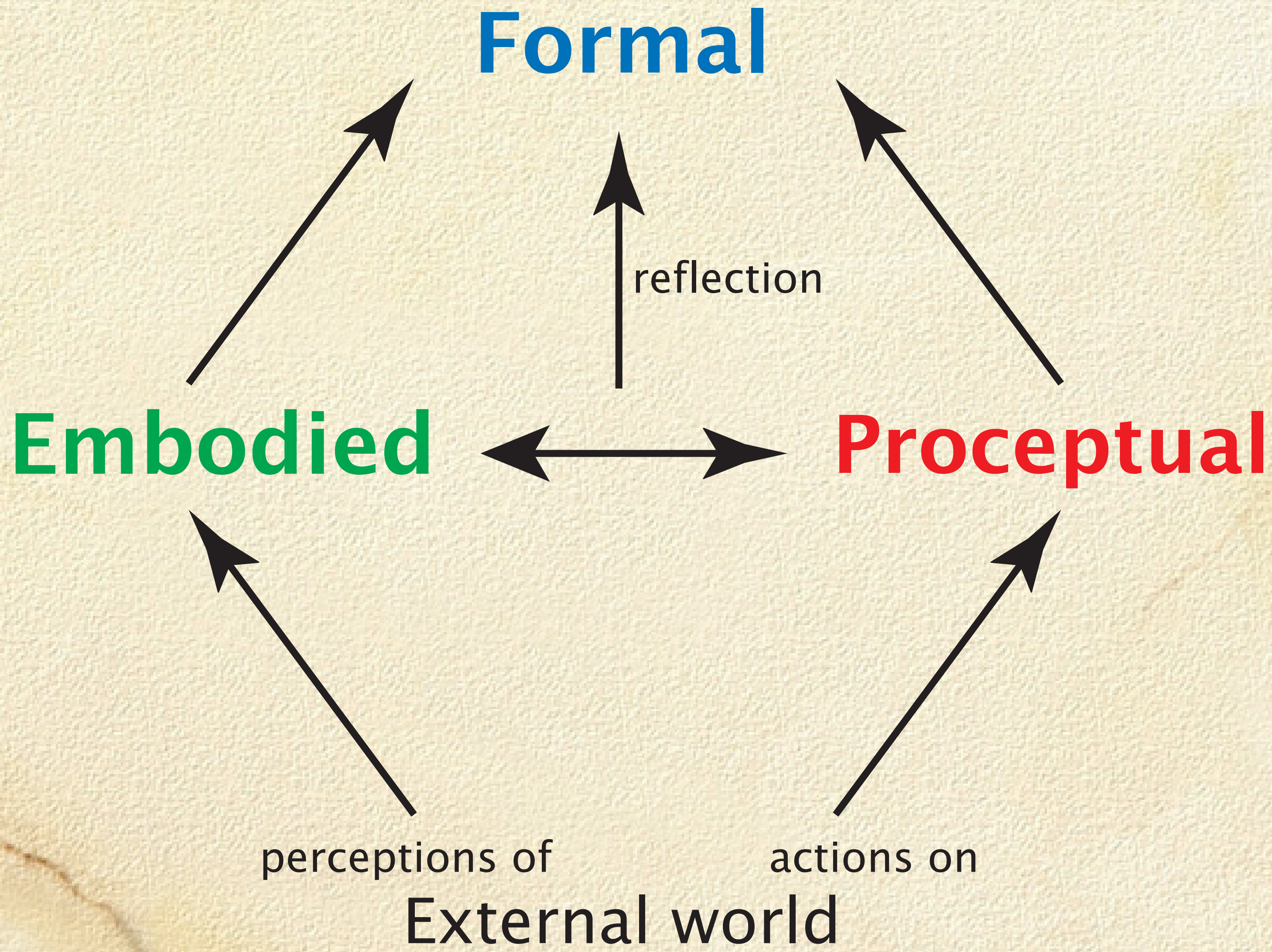
Mathematical Growth



Mathematical Growth

Topics:

1. *The cognitive development of proof*
2. *Symbols as process and concept*
3. *An embodied approach to the calculus*

The Theory of PROCEPTs: Flexible use of symbols as PROcess and conCEPT in Arithmetic, Algebra, Calculus, etc

David Tall
University of Warwick

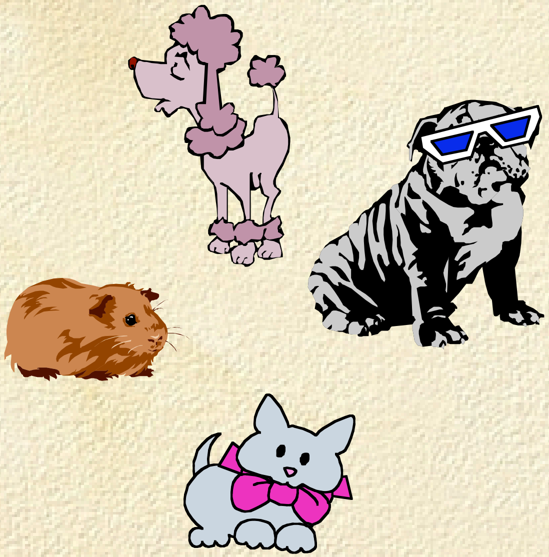
This talk is designed for all who are interested in teaching and using mathematics, from teachers of the young to teachers of mathematics and its applications at university.

It focuses on the role of symbols in mathematics, specifically those which allow the human mind to switch effortlessly from "concepts to think about" to "processes to solve problems". For example, in arithmetic, the symbol 3+4 evokes both the process of addition (initially through counting) and also the concept of sum. Some students develop a flexible manner of relating such symbols with others, for instance, noting that "3+4 is 7", because it is "one less than 4+4, which is 8".

The flexible use of a symbol as either process or concept is called a procept. I hypothesise that those who focus on the limited skill of coping with step-by-step procedures have less chance of long-term success than those who use symbols flexibly as process and concept. I will reveal how procepts in arithmetic, algebra, calculus, and university mathematics operate in subtly different ways, causing obstacles in learning that are not overcome by everyone.

This in turn causes a divergence between those who seek the limited security of step-by-step procedures and those who learn to use symbols in a more flexible, "proceptual" manner. It is my belief that these these theoretical ideas are extremely simple, yet have significant consequences for the learning and teaching of mathematics at all levels.

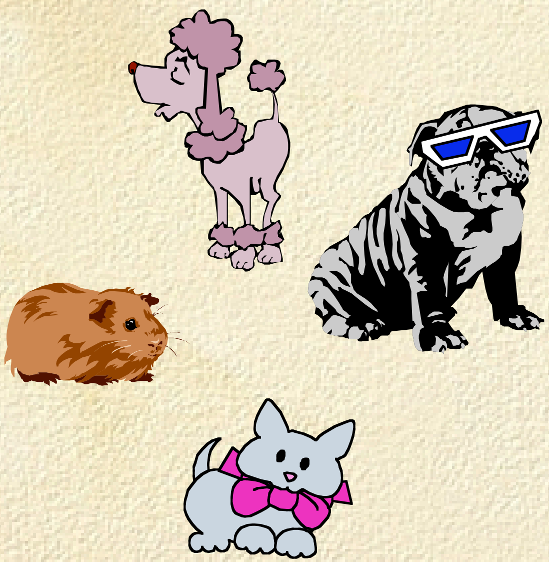
Compression from counting process to number concept.



First I must learn to count.

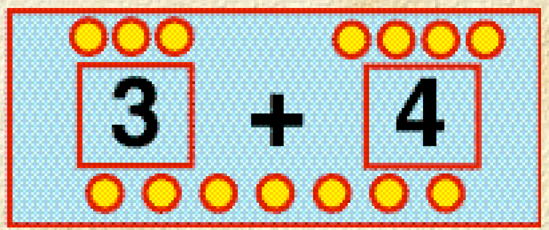
and realize that the total is independent of the order of counting

Compression from counting process to number concept

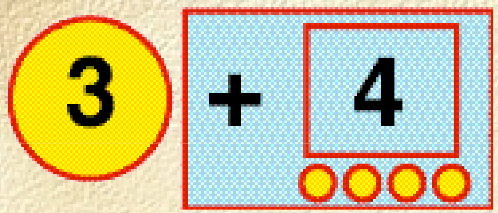


- There are “one, two, three, four.”
- There are “[one, two, three,] … four.”
- There are “ … four.”
- There are “ four.”

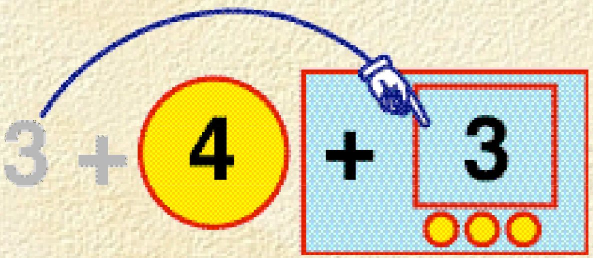
Compression of addition into sum



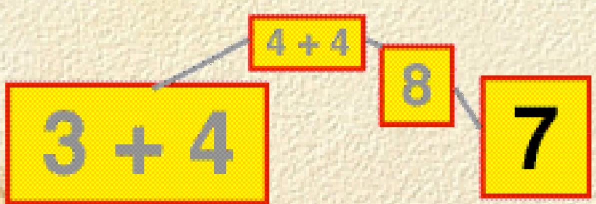
Count 3, count 4,
put together and ***count all****.*



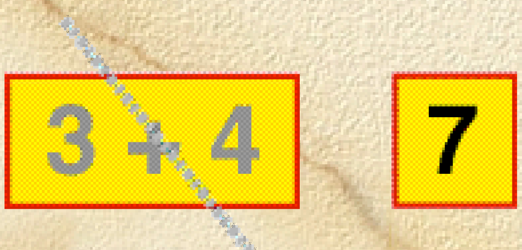
Start at 3, ***count on*** *4.*



Count on from largest
Start at 4, count on 3.



Derived fact
3+4 is one less than 4+4



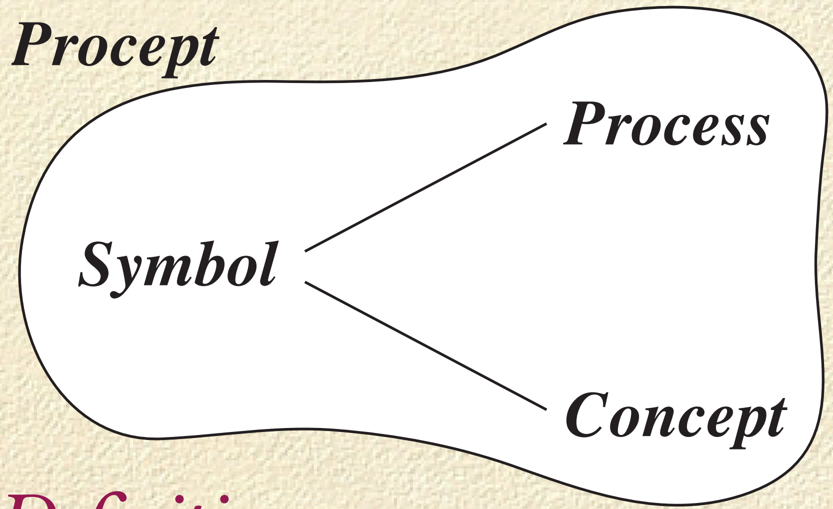
Known fact
3+4 is 7

Symbol as process and concept

Symbol	Process	Concept
$3+2$	addition	sum
$^{-}3$	subtract 3	negative 3
$3/4$	division (sharing)	fraction
$3+2x$	evaluation	expression
$v=s/t$	ratio	rate
$\lim \Sigma 1/n^2$	tend to limit	limit value
dy/dx	differentiation	derivative
$\int f(x)\,dx$	integration	integral

Symbol as process and concept

Symbol	Process	Concept
$f(x)$	evaluation	function f
$\mathbf{v}$	translation	vector
$\sigma \in S_n$	permuting $\{1,2,\ldots,n\}$	element of S_n



Preliminary Definition:
*The combination of **pro**cess and con**cept** represented by the same symbolism is defined to be a **procept**.*

I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is $^{134}/_{29}$ (and so forth). What a tremendous labor-saving device! To me, '134 divided by 29' meant a certain tedious chore, while $^{134}/_{29}$ was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, *a*/*b* and *a* divided by *b* are just synonyms. To him it was just a small variation in notation.

William P. Thurston, Fields Medallist, 1990

More sophisticated definition

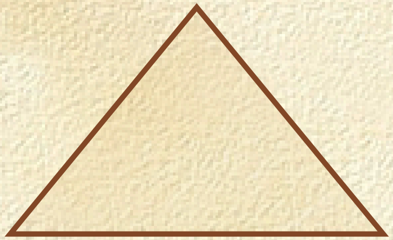
An ***elementary procept*** is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object.

A ***procept*** consists of a collection of elementary procepts which have the same object.

Gray & Tall, JRME, 1994

e.g. 6, 4+2, 3×2, 7–1 *are different symbols for the same procept.*

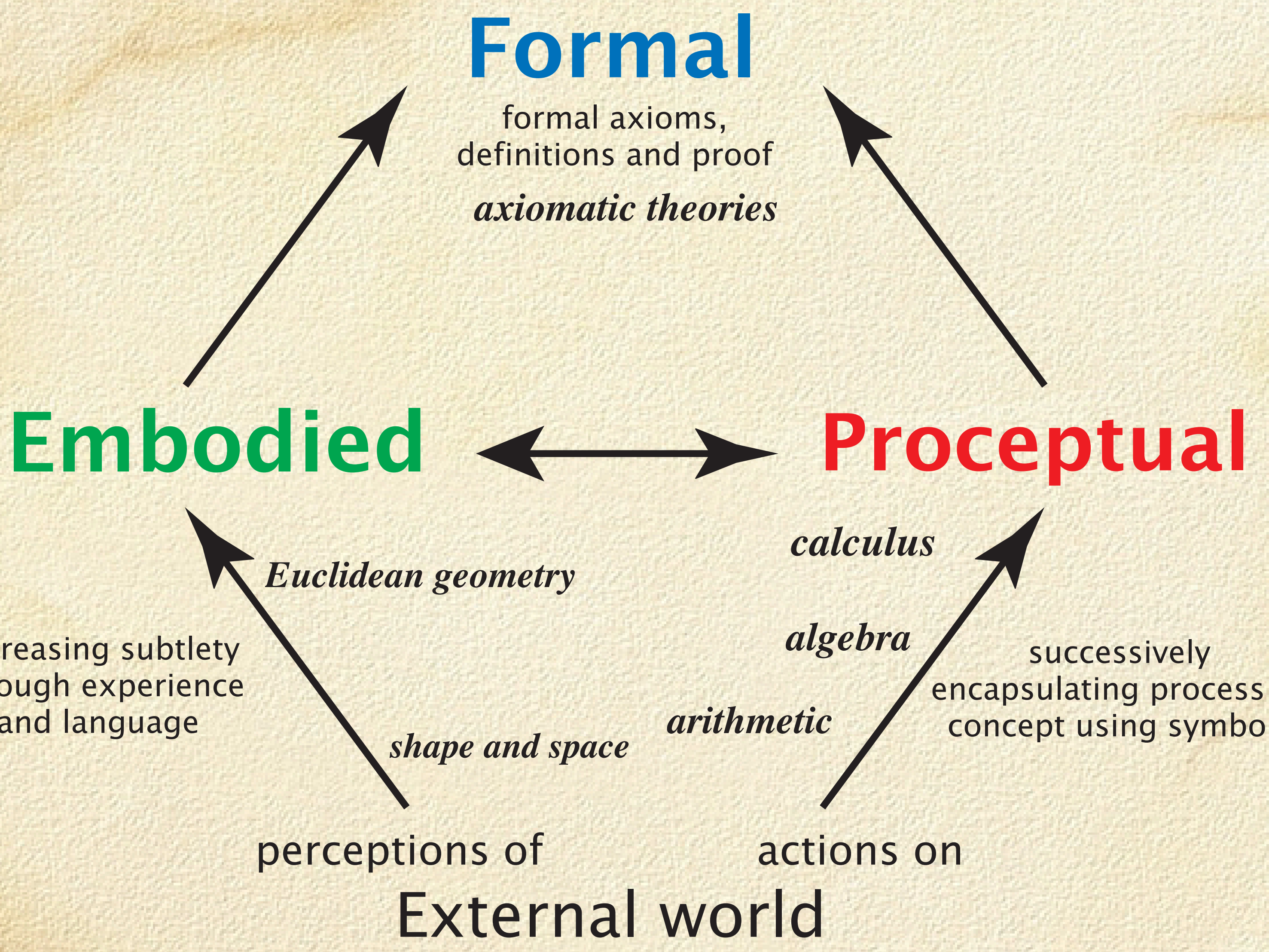
But not all mathematical concepts are procepts …



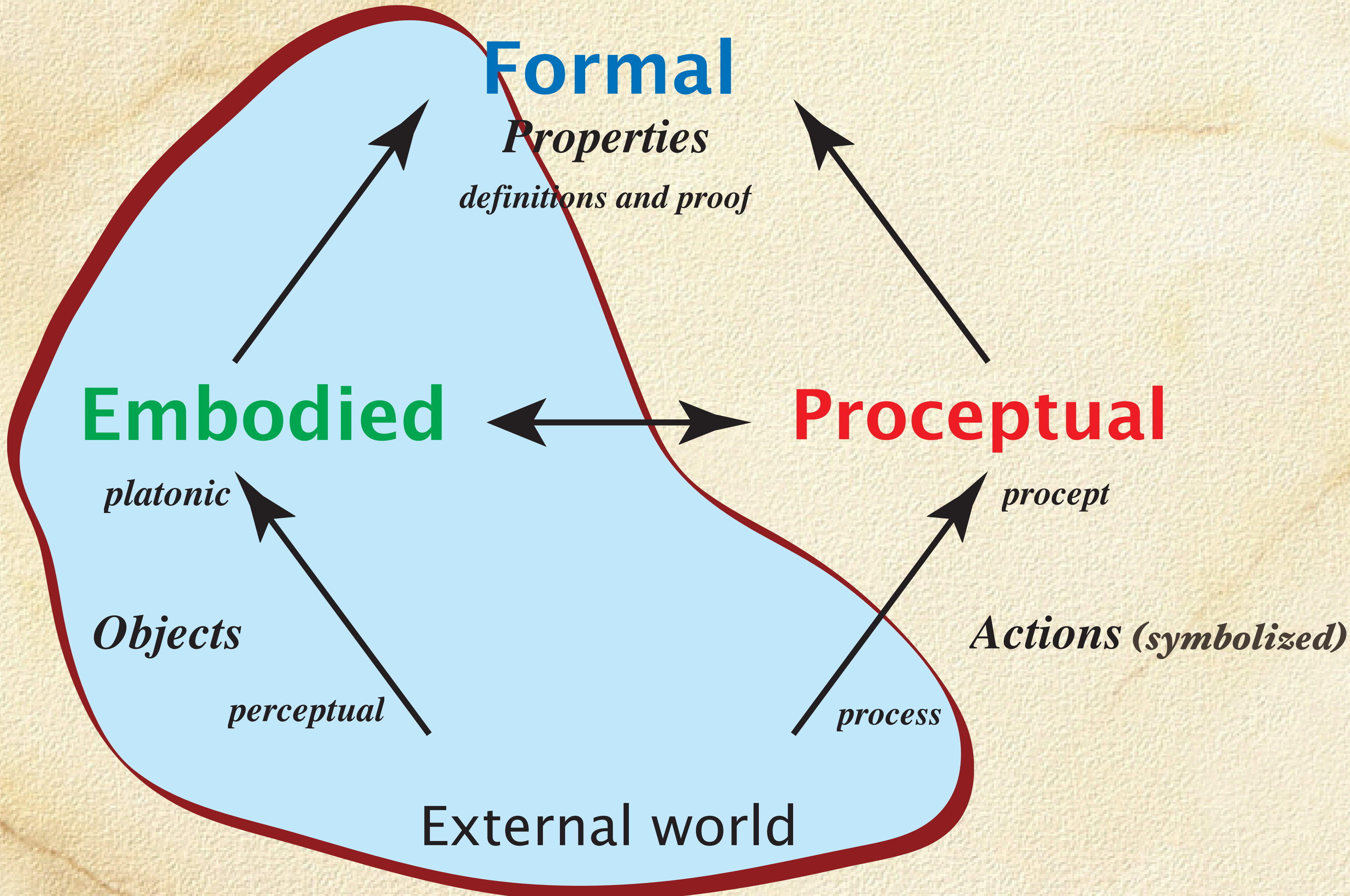
e.g. a triangle is an object in its own right …

A vector space V   … is a defined concept …

Three different worlds of mathematics



Three different worlds of mathematics



In this presentation I will focus on

The cognitive development

of symbols

as process and concept

Compression from procedure to procept

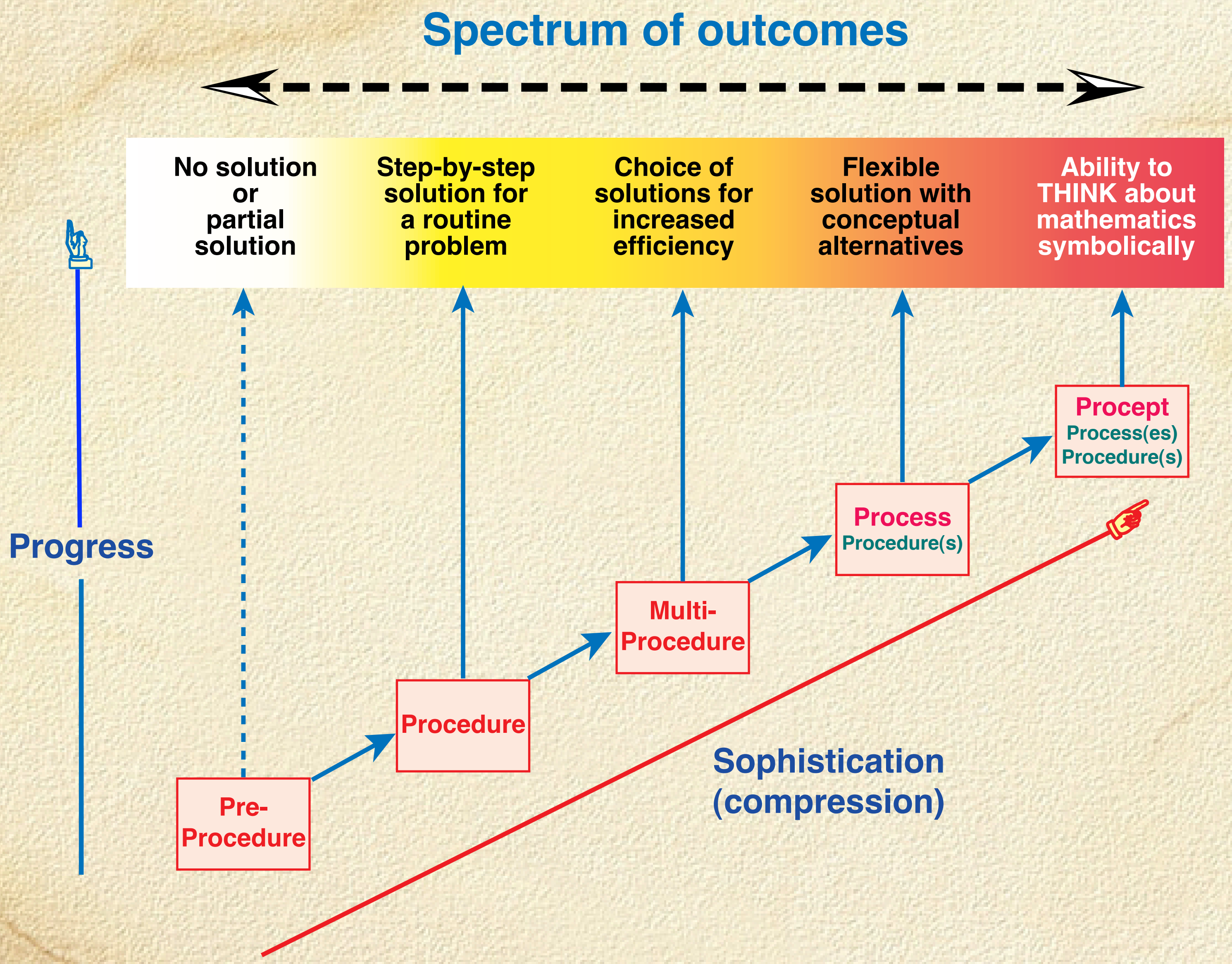
- ***procedure***: finite sequence of decisions and actions is built up into a coherent sequence.
- ***process***: increasingly efficient ways become available to achieve the same result, now seen as a whole.
- ***procept***: where the symbols are conceived flexibly as processes to do and concepts to think about.

Our small conscious focus of attention has the following effects:

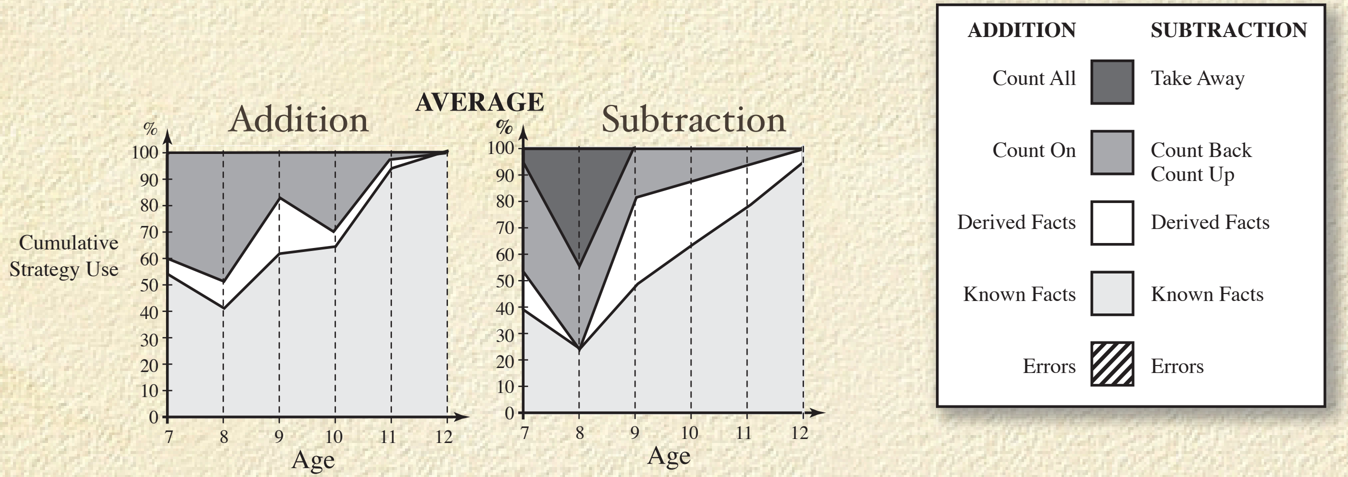
***Procedures** occur in time and take up mental space.*
*Procedures are more primitive ways of thinking which may give short-term success but prove **less effective for long-term thinking** about mathematics.*

***Procepts** can be conceived and manipulated as mental concepts.*
*Procepts are **easier** to manipulate for those who are flexible thinkers.*

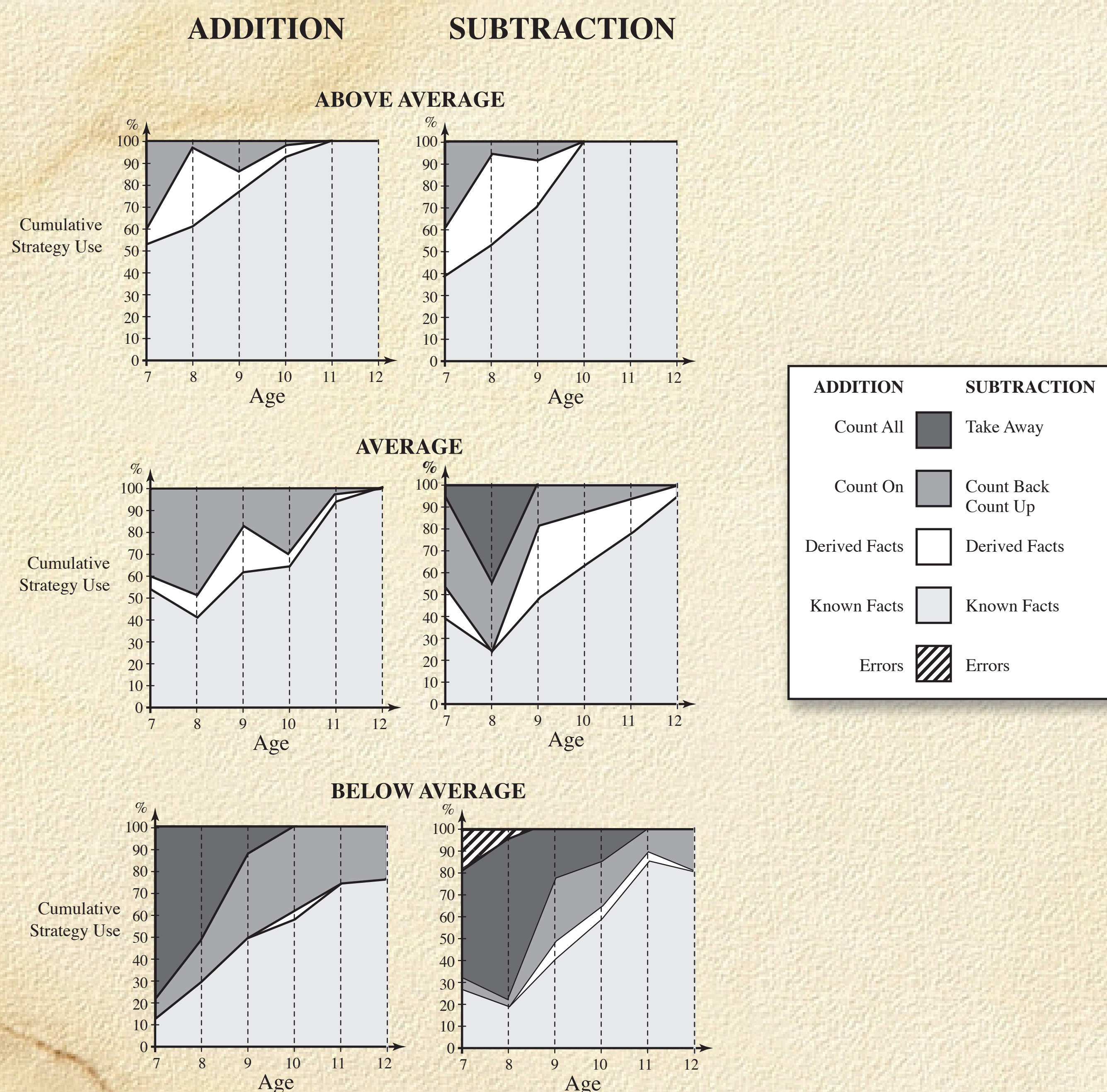
Compression from procedure to procept



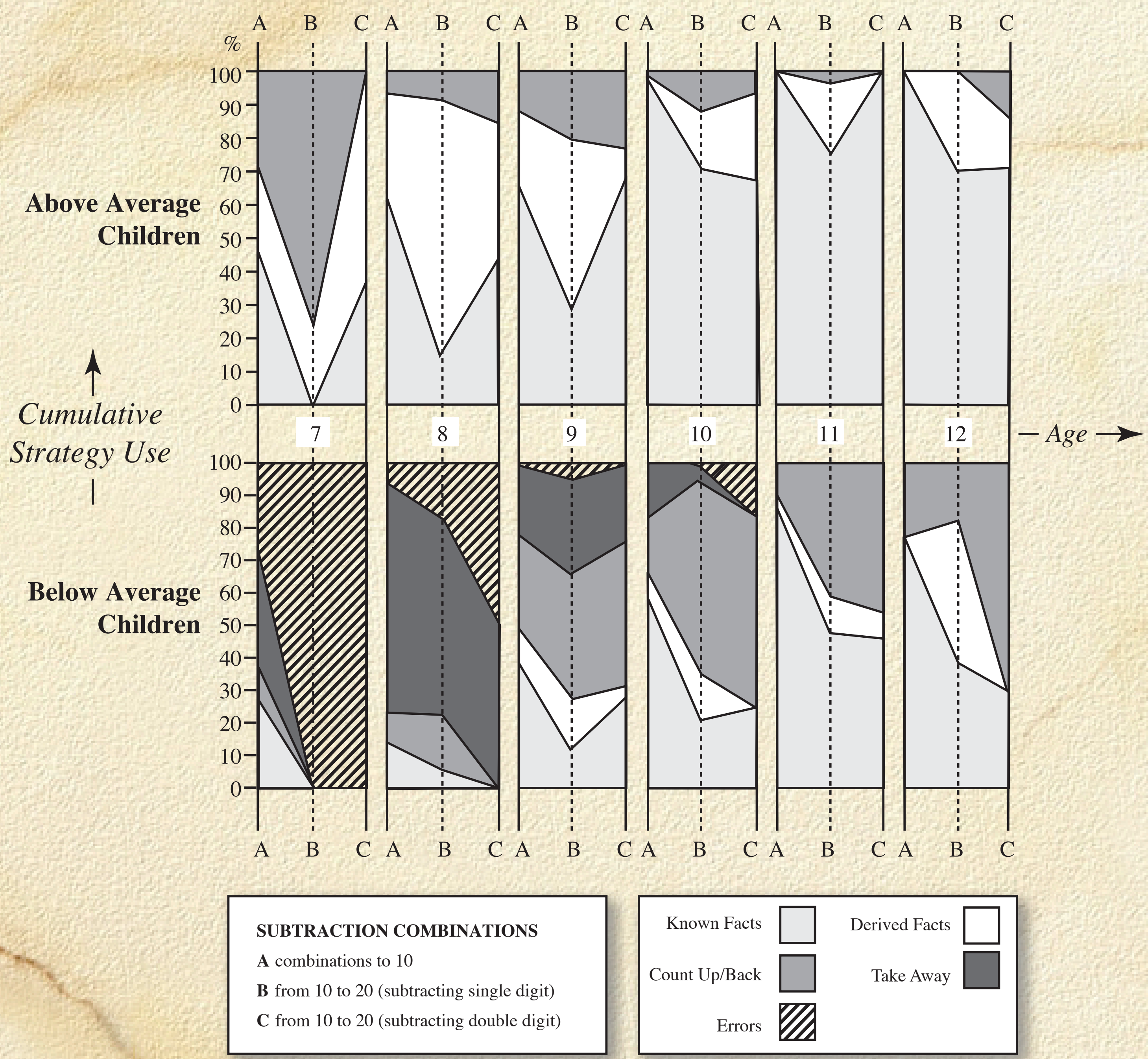
Compression of number facts to 10

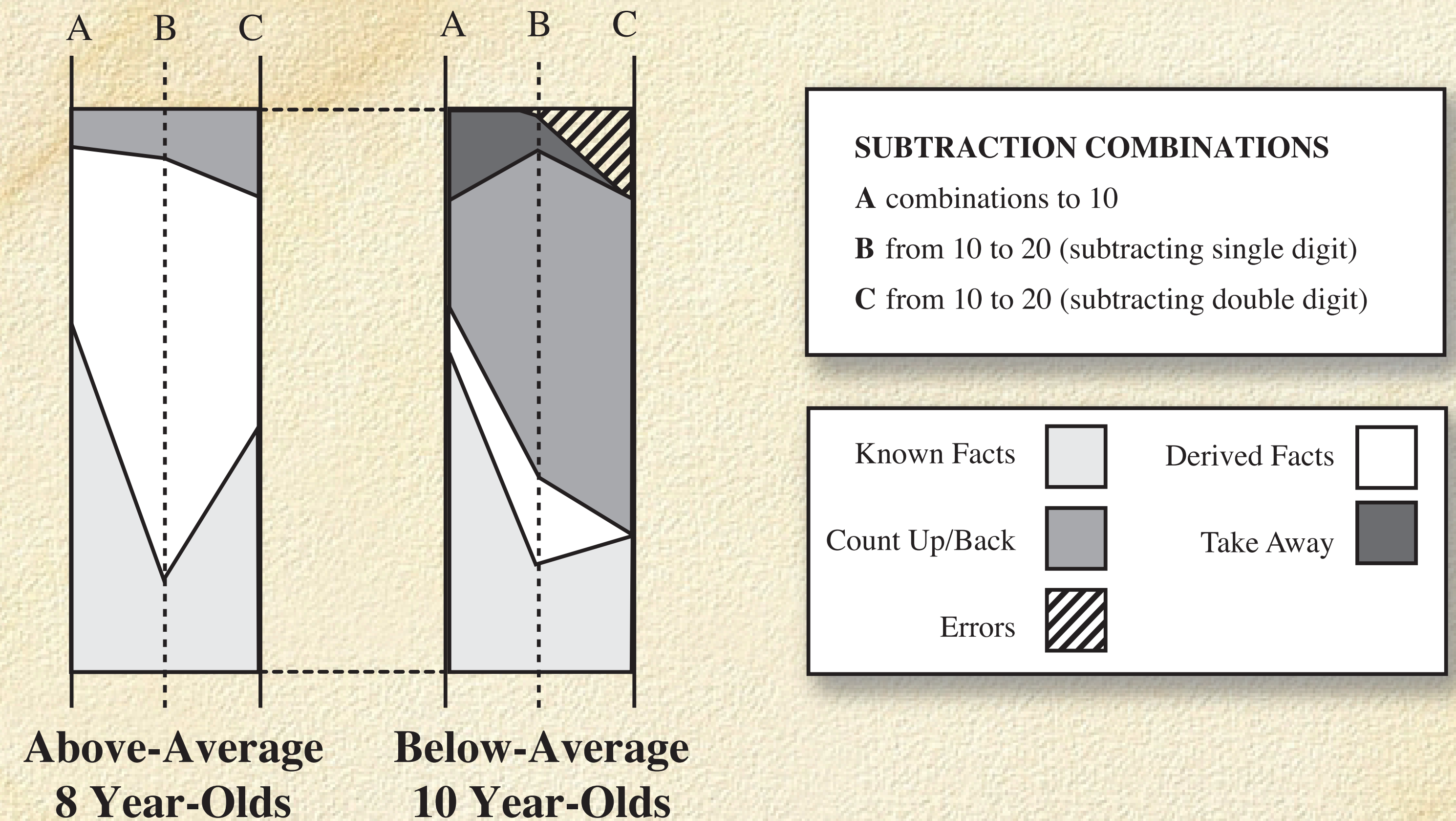


Each vertical line is the combined total of 4 children, aged 7, 8, 9, 10, 11, 12.


ADDITION
SUBTRACTION
ABOVE AVERAGE
AVERAGE
BELOW AVERAGE
Cumulative Strategy Use
%
100
90
80
70
60
50
40
30
20
10
0
7
8
9
10
11
12
Age
ADDITION
SUBTRACTION
Count All
Take Away
Count On
Count Back
Count Up
Derived Facts
Derived Facts
Known Facts
Known Facts
Errors
Errors

Development of number knowledge up to twenty.



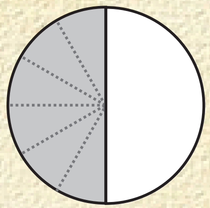


Below-average students are not simply slower doing the same thing, they do *different* things.

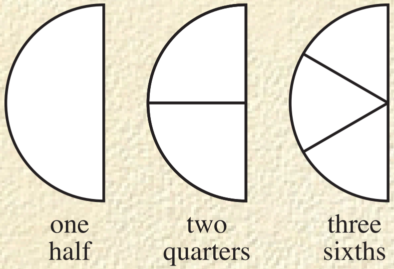
Embodied Meaning for Compression

The case of Fractions

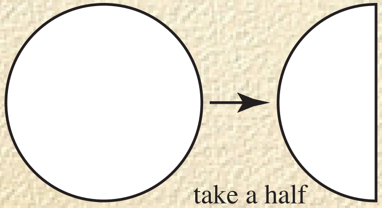
Embodied World



Think of the effect
as a **prototype**



Different actions that
have the **same effect**



Action on object(s)

Cognitive Compression

Proceptual World

$^1/_2$

Procept

a manipulable symbol
that can be changed into
equivalent forms and
operated with and on

$^1/_2$ $^2/_4$ $^3/_6$

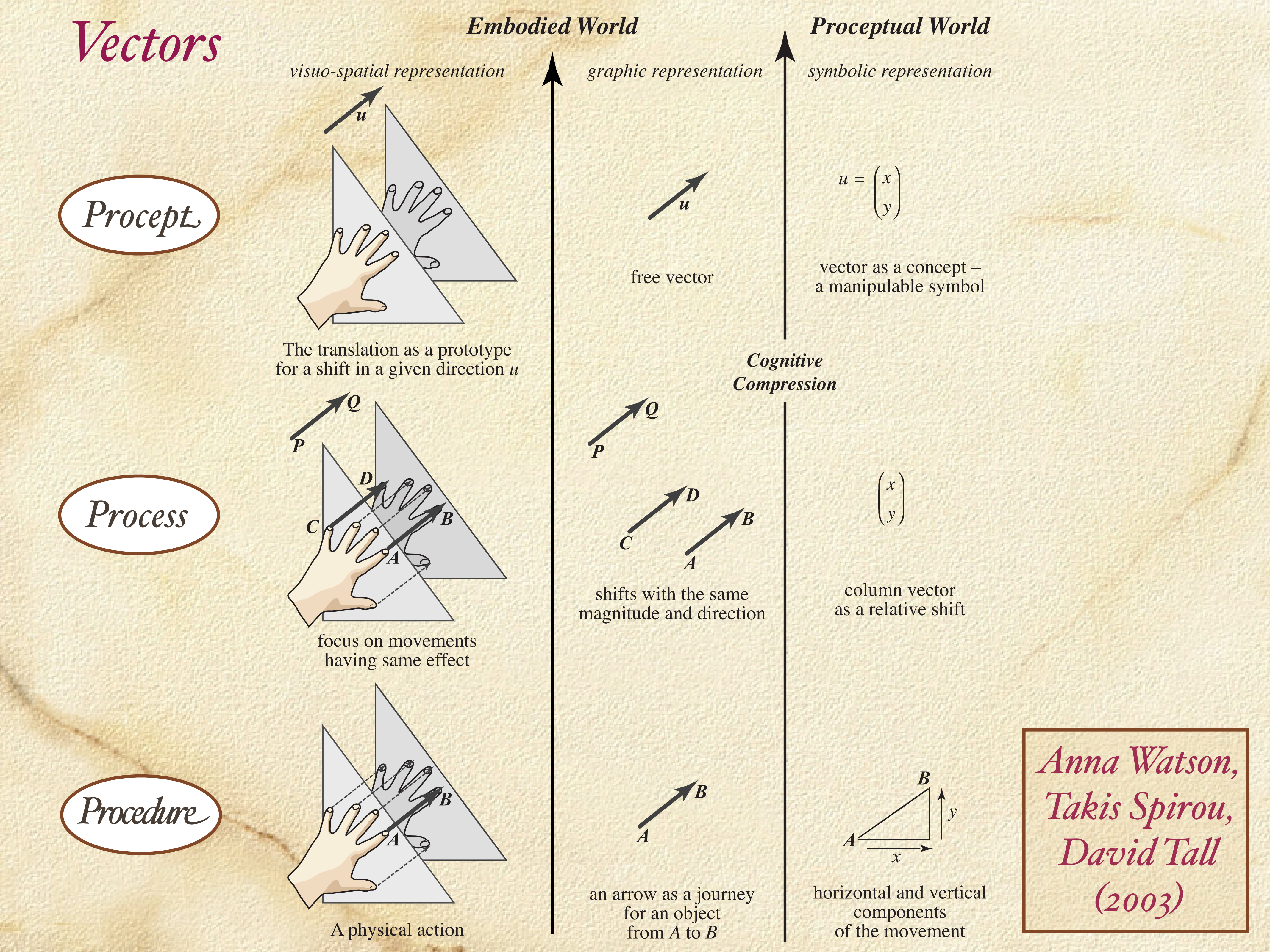
Process

equivalent fractions that
have the same effect

$^1/_2$

Procedure

share into two equal parts



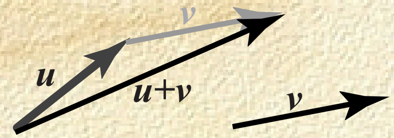
Vectors
Embodied World
Proceptual World
visuo-spatial representation
graphic representation
symbolic representation
Procept
Process
Procedure
u
The translation as a prototype for a shift in a given direction u
u
free vector
$u = \begin{pmatrix} x \\ y \end{pmatrix}$
vector as a concept – a manipulable symbol
Cognitive Compression
P
Q
C
D
A
B
focus on movements having same effect
P
Q
C
D
A
B
shifts with the same magnitude and direction
$\begin{pmatrix} x \\ y \end{pmatrix}$
column vector as a relative shift
A
B
A physical action
A
B
an arrow as a journey for an object from A to B
A
B
x
y
horizontal and vertical components of the movement
Anna Watson, Takis Spirou, David Tall (2003)

Vector Addition

Graphical representation

Symbolic representation

Procept

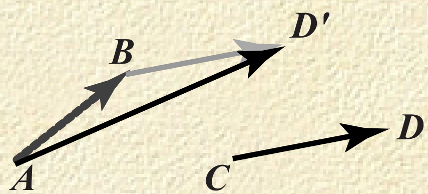


adding ***u*** and ***v***
using triangle (or parallelogram)

$$u = \begin{pmatrix} x \\ y \end{pmatrix} \quad v = \begin{pmatrix} r \\ s \end{pmatrix} \quad u + v = \begin{pmatrix} x + r \\ y + s \end{pmatrix}$$

Adding vectors

Process

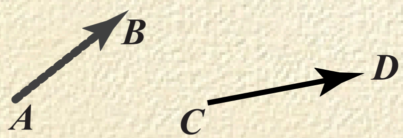


AB+CD = AD'
using triangle (or parallelogram)

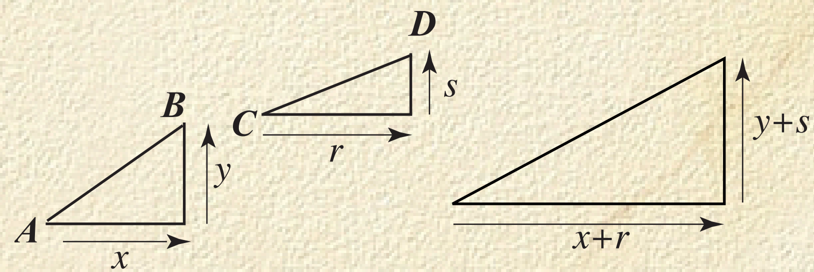
$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} x + r \\ y + s \end{pmatrix}$$

adding vectors
by adding components

Procedure



can only add arrows as journeys
if ***B*** is coincident with ***C***



adding components (eg by counting squares)

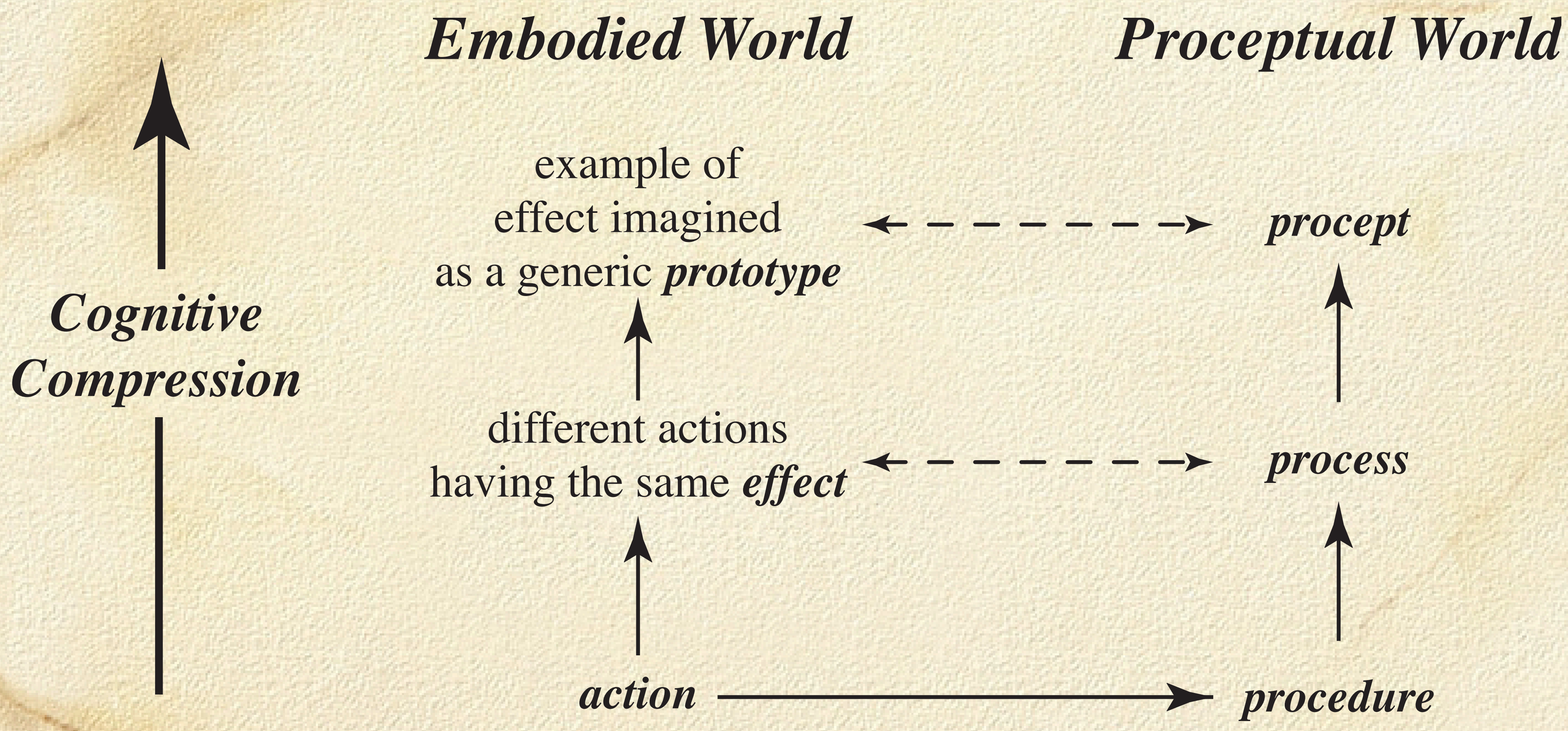
add arrows

Cognitive Development

$$\underline{\quad x \quad}\underline{\quad r \quad}$$
$$x + r$$

add numbers

Embodied Meaning for Compression



Long-term Shift from Embodied to Proceptual

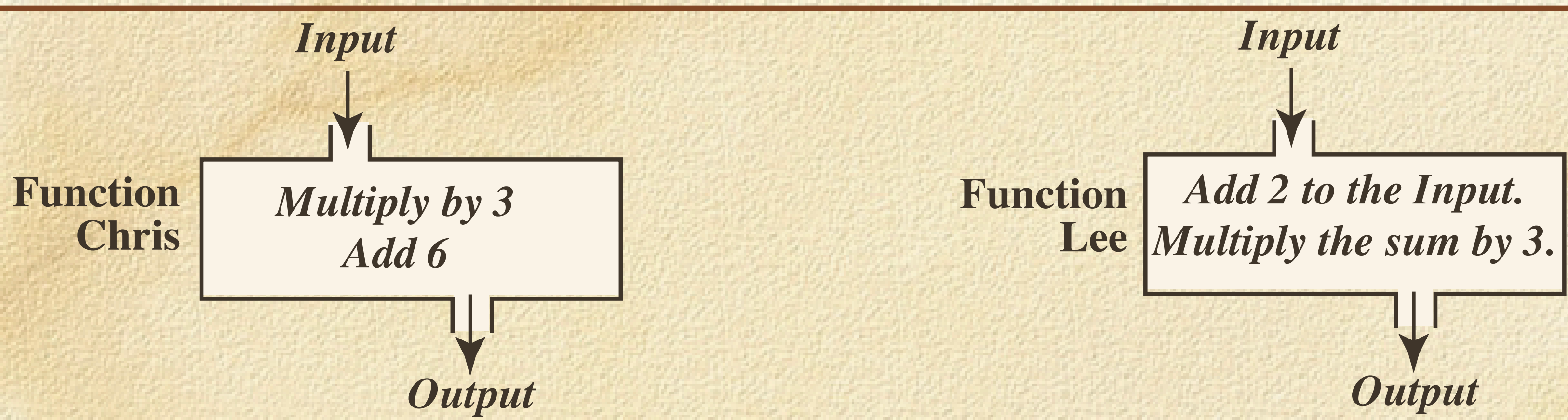
- Embodied 'effect' can give initial meaning to shift from procedure to process.
- For long-term development, there is a general shift from embodiment to symbolism.
- Different embodiments can complicate matters for less successful students.
- Long-term most successful students rely less on embodiment and more on symbols.
- Successful students retaining links with embodiments must cope with different embodied meanings.

Eddie Gray, Demetra Pitta, Marcia Pinto, David Tall (1999), Knowledge Construction and diverging thinking in elementary and advanced mathematics, *Educational Studies in Mathematics*, 38 (1–3), 111–133.

The Transition from Arithmetic to Algebra

- Procepts in arithmetic all have an *operational* procedure to produce an ‘answer’: 3+2 is 5.
- Procepts in algebra have a *potential* operation of evaluation: $3+2x$ has no answer unless x is known.
- Consequence: algebra is a mystery to many.
- Children may interpret expressions in various ways eg $3b$ might involve a code (a=1, b=2, ...) so it is 6, or it might mean ‘3 bananas’, or, if b=5, $3b$ might be 35 ...
- Even when it is given a satisfactory meaning, many consider it a *procedure*, not a *concept*.

The Transition from Arithmetic to Algebra



What are the outputs of these two function boxes and are they the same?

Student	Chris	Lee	Are they equal?	
A	$3x+6$	$3(x+2)$	Yes, if I distribute the 3 in Lee, I get the same function as Chris	Procept
B	$x3+6$	$(x+2)3$	Yes, but different processes	Process
C	$3x+6$	$x+2(3\times)$	No, you come up with the same answer, but they are different processes	Procedure

Phil Demarois, PhD Thesis, Warwick 1998

Stacey (1993) asked 255 children aged 13 or 14 in a mixed abilty year group to translate
'z is equal to the sum of 3 and y'
into mathematical symbols.

28% *made the direct translation* **z=3+y** — *'assignment' order*

29% *wrote either* **3+y=z** *or* **y+3=z** — *'procedural' order*

29% *conjoined '3 and y' as* **'3y'**

14% *other errors or no response*

(29% + 14%) — *43% errors*

Possibilities:

Nearly ***half*** *do not understand the symbolism*

Half *those giving the* ***correct answer*** *may think of it as a* ***procedure***

Which of these equations is easier to solve:

... and why ?

$$3x+2 = 14 \qquad ..... (A)$$

*Can be considered as a **process**, which can be undone.*
14 take off 2, gives 12, divide by 3 gives x=4.

$$3x+2 = 4x-2 \qquad ..... (B)$$

*Two different **processes** must be considered*
*as the same **concept** (written in two different ways).*

Add 2 to each side to give $3x+4 = 4x$
Subtract 3 x from each side: $4 = x$.

*(A) can be solved as a **procedure***
*(B) requires manipulating symbols as **concepts***

Symbols as process or concept

- *The symbols in* ***arithmetic*** *are* ***operational procepts*** *(with answers)*
- *The symbols in* ***algebra*** *are* ***potential procepts*** *(with evaluation only possible if variables are given numerical values).*
- *Students who operate only with procedures have difficulty with algebra.*
- *Students who can see algebraic symbols flexibly as process or concept can solve more subtle problems.*

Limits as potentially infinite processes

There is a great deal of research into the conceptual difficulties involved with limits.

- *A limit has a potentially infinite process of computation that may not even have a finite algorithm to obtain the limit value.*
- *Many (most) students believe a limit goes on forever and the limit is not attained.*
- *For example,*
  - 1/9 = 0.1+0.01+0.001+ … *is true*
  - *but* 0.1+0.01+0.001+ … = 1/9 *is false.*

as a continuing process of division

because the process goes on forever

Lan Li, MSc Thesis, Warwick, 1992

Derivatives as operational procepts

- Symbolic derivatives have an operational procedure of calculation using the symbolic rules of differentiation e.g. $D(uv) = vD(u)+uD(v)$.
- The limit process is a potentially infinite procept.
- Symbolic differentiation is operational

Consequences:
The limit concept is the wrong place to start calculus.
Students prefer the algorithms of differentiation
(if they can do them!)
I prefer an embodied approach to calculus (to give meaning)

Summary: Different kinds of Procepts

- ***arithmetic: operational procepts*** *with computational processes and manipulable number concepts*
- ***algebra: potential procepts*** *with potential processes of evaluation and expressions as manipulable concepts*
- ***limits: potentially infinite procepts*** *with infinite processes and (unattainable) concepts*
- ***symbolic calculus: operational procepts*** *with prescribed processes of differentiation and computable concepts.*

Formal mathematics with formal definitions and proof is different again, and occupies a distinct world of its own.