Proof in Three Worlds of Mathematics

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Proof in Three Worlds of Mathematics

This presentation considers different modes of proof and the kinds of belief (as 'warrants for truth') that underpin them.

It distinguishes three distinct worlds of mathematical thought:

- *the* ***embodied*** *(based on our perceptions and actions and reflection on them),*
- *the* ***proceptual*** *(using symbols as* ***pro****cess & con****cept*** *in arithmetic and algebra),*
- *the* ***formal*** *(based on formal definitions and formal proof)*

Proof in Three Worlds of Mathematics

Embodied proof *builds from our interaction with the outside world through our senses and develops in sophistication through language and human interaction to include Euclidean Geometry.*

Proceptual proof *refers to the use of symbols in arithmetic and algebra (and the wider theory of procepts in which symbols operate dually as process and concept) where a statement may be proved through calculation and manipulation.*

Formal proof *is built by formal deduction from axioms and concept definitions to construct coherent mathematical systems.*

The presentation will make the case that each of these mathematical worlds of thought develops its own increasingly subtle criteria for belief and truth. This will be used to shed light on the mental constructions required for different modes of proof and on the range of cognitive tasks experienced by individuals in their quest for understanding.

A story:

Once upon a time ...
a Mathematician and a Mathematics Educator were discussing the meaning of mathematics.

Negrepontis explained to me Plato's meaning for 'two'.

'Two stones?'
'Two oxen?' ☹

Two identical things. ☺

But how do you tell them apart.

Use the golden section! So sophisticated!

At this point I asked to say something.

I said, 'Plato was very young when he was born.'

A story:

‘Plato was very young when he was born.’

Negrepontis continued …
Plato took two difference sequences with the same limit.
They are different … but the same.
This is perfection. This is ‘two-ness’.

I reminded him of my one sentence interjection: When Plato was born, he could not talk and lived off his mother’s milk without a philosophical discussion of any kind.

‘How did Plato, who was once a helpless child, develop into the great philosopher whose name has lasted for over two millennia?’

The answer lies in some kind of cognitive development.
*Educators should be interested not just in the **product** of that development, but the **process** of how it occurs and can be encouraged to occur.*

What is proof for a mathematician?

Davis and Hersh caricature 'the ideal mathematician':

> He rests his faith on rigorous proof; he believes that the difference between a correct proof and an incorrect one is an unmistakable and decisive difference. He can think of no condemnation more damning than to say of a student: 'He doesn't even know what a proof is.' Yet he is able to give no coherent explanation of what is meant by rigor, or what is required to make a proof rigorous. (Davis & Hersh, 1981, p. 34)

He is unable to give any explanation to a university's public information officer of details or applications of his work, or say anything understandable to 'the ordinary citizen' (p. 38).

He fares no better with a student, explaining 'proof is what you've been watching me do at the board three times a week for three years' (p. 39) … 'everybody knows what proof is. Just read some books, take courses […], you'll catch on.'*(p. 40.)*

He has difficulties with a philosopher saying, 'I'm not a philosopher, philosophy bores me. You argue, argue, argue and never get anywhere. My job is to prove theorems, not to worry about what they mean.' *(p.41)*.

Proof is sensed by communities of mathematicians who strive for perfection, but have to be satisfied with a compromise.

Proof in Context

Foundations of Mathematics (Ian Stewart, David Tall, 1977).

***Proofs** from explicit hypotheses (and axioms) are written **in a context**.*

Well-established (contextual) truths need not be referenced, to relieve the burden on the reader by concentrating on the important (new) ideas.

In Foundations of Mathematics, the notion of real numbers as a complete ordered field is built up in three stages:

1. *proving sufficient properties directly from the axioms for a field to establish a context for arithmetic.*
2. *Assuming properties of arithmetic as being contextual and focusing on properties of order,*
3. *Assuming properties of arithmetic and order as being contextual and focusing on the property of completeness.*

We attempted to introduce young undergraduates to the style of presentation of the mathematical community.

However, knowing what may be assumed contextually and what needs to be explicitly proved, is highly non-trivial and usually implicit in the minds of most mathematicians.

This is a natural consequence of how the brain focuses on essential ideas and suppresses less significant material.

Forgetting how we learnt

Children forget:

After the formalization had been taught, or three months later, the practical or pre-formalization work which led up to it was often forgotten or not seen as significant. (The Learning of Mathematics 8–13, Johnson (ed.), 1989, p. 219)

Mathematicians forget:

After mastering mathematical concepts, even after great effort, it becomes very hard to put oneself back into the frame of mind of someone to whom they are mysterious. (Proof and Progress, Thurston, 1994, p. 947)

Educators forget:

One finally masters an activity so perfectly that the question of how and why students don't understand them is not asked anymore, cannot be asked anymore and is not even understood anymore as a meaningful and relevant question. (Didactic phenomenology of mathematical structures, Freudenthal, 1983, p. 469)

We must seek to:

- ***revisit the notion of proof****, to seek its special characteristics, dependent in part on the community of mathematicians,*
- ***investigate the cognitive growth of proof*** *as the individual matures to find out how we might hope to encourage our students to grasp its essential nature.*

Three worlds of mathematics

Over recent years my work coalesced into 3 distinct threads:

1. *Relating to our* ***sensory perceptions of and physical actions on the real world and our reflections on these****, which lead to conceptions of properties of objects, then relationships between properties. (I began to realise this leads naturally to the individual (re-)inventing Platonism.)*
2. *Those relating to our use of* ***symbolism*** *in arithmetic, algebra and more general analytic forms that enabled us to calculate and manipulate to get answers.*
3. *The* ***formal axiomatic approach*** *of mathematicians that is the final bastion of presentation of coherent theories and logical proof using axioms and definitions expressed in quantified set-theoretical statements, manipulated using the laws of logic.*

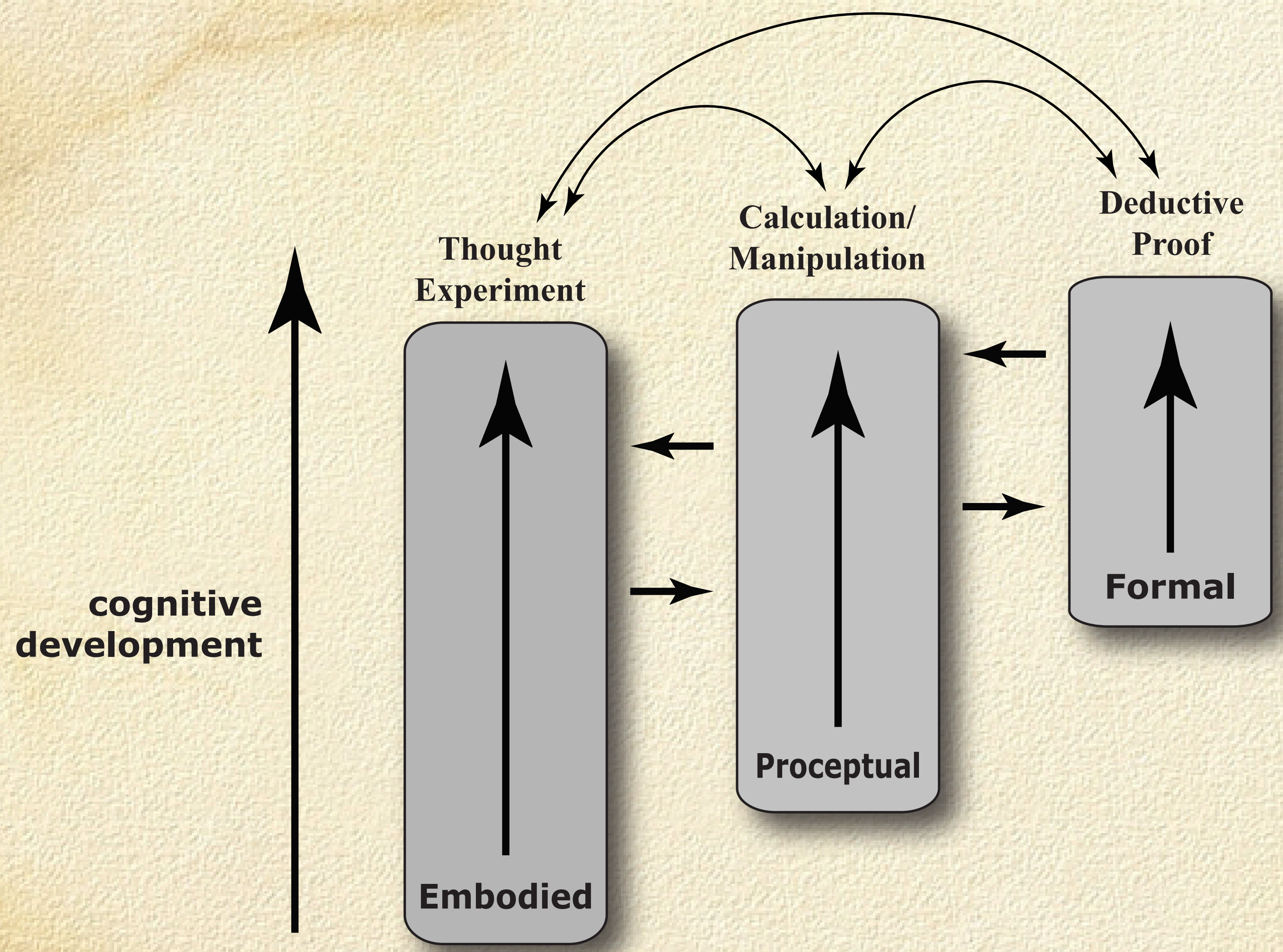
Three worlds of mathematics

I see these worlds having different ways of operating, developing different standards of validity and truth.

1. *The embodied world of perception and action, including reflection on perception and action, which develops into a more sophisticated Platonic framework,*
2. *The proceptual world of symbols, such as those in arithmetic, algebra and calculus that act as both processes to do (e.g. 4+3 as a process of addition) and concepts to think above (e.g. 4+3 as the concept of sum) as formulated in the theory of procepts (Gray and Tall, 1994).*
3. *The formal world of definitions and proof leading to the construction of axiomatic theories, (Tall, 1991).*

For a more details, consult Tall 2002, Watson et al, 2002, available as downloads from www.davidtall.com/papers.

Cognitive development of the three worlds



Cognitive Growth in the Embodied World:

- *Perception of the properties of physical objects.*
- *Language allows descriptions to identify them – that is a circle 'because it is round', that is a square' because it has 4 equal sides and its angles are right-angle.'*
- ***Descriptions** become more precise until the purpose is shifted to definitions of objects.*
- *We can now test if an object is what we claim it is by checking it satisfies the definition.*
- *We can perform **thought experiments**.*
- ***Definitions** lead to implication: if this is true, then that is true …*
- *and on, naturally to Euclidean proof (van Hiele, 1986).*

Cognitive Growth in the Proceptual World:

In the proceptual world of symbols, each concept starts out as an embodied process, eg counting, and uses symbols that are thought of as (number) concepts. The process of addition 3+4 becomes the concept of sum, the process of repeated addition (multiplication) becomes the concept of product.

Algebra develops, in part, as generalized arithmetic, where a symbol such as 2n–1 represents a process of evaluation, (double n and take away one). This process (of evaluation) is encapsulated as a concept (of expression), the expression 2n–1, which can now be manipulated by algebraic operations.

Cognitive Growth in the Formal World:

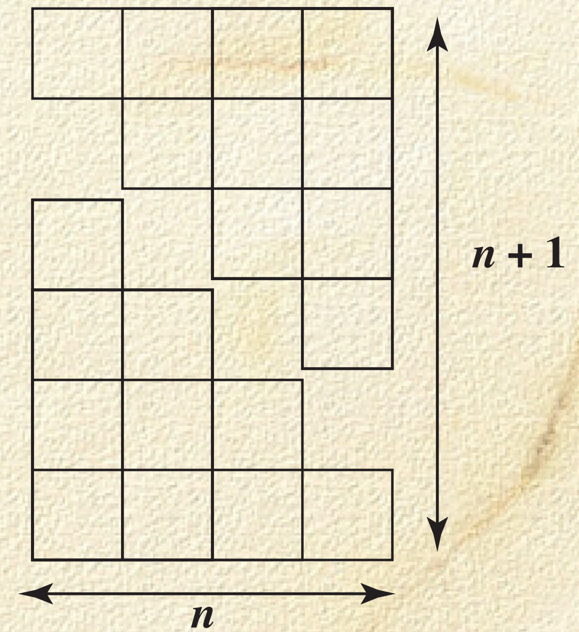
*The formal world uses such experiences from both these worlds to build the world of **formal definition** and **proof**.*
Here the construction of meaning is reversed, we specify properties and build concepts from them (in the embodied world we 'have' concepts and we tease out their properties from them).

concept → definition	becomes	definition → concept

Validity in Different worlds of Mathematics

Each world develops distinct notions of validity.
Mathematical 'Warrants for Truth'
A warrant is 'that which secures knowledge.' (Rodd, 2000).
Example:
Different warrants for the truth that the sum of the first n whole numbers is:

- a picture of the sum of the first n numbers as a staircase then putting two together as a rectangle of area n by n+1.



- the arithmetic proof of Gauss:

add columns

$$\begin{array}{rrrrr} 1 + & 2 + \ldots + & 99 + & 100 & \\ 100 + & 99 + \ldots + & 2 + & 1 & \\ \hline 101 + & 101 + \ldots + & 101 + & 101 & = 100 \times 101 \end{array}$$

- proof by induction

Validity in Different worlds of Mathematics

These examples offer three ways of convincing:

*(a) the pattern in a picture (**embodied**),*

*(b) the pattern in an arithmetic calculation (**proceptual**),*

*(c) an induction proof (**formal**).*

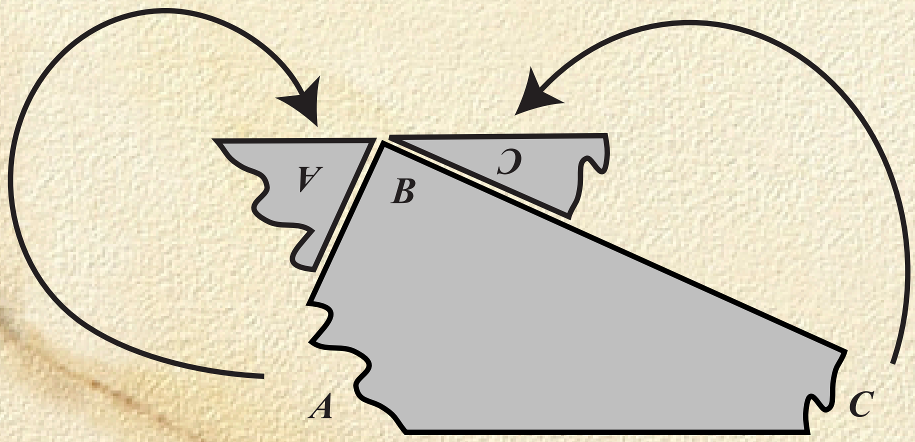
Question: which of these are 'proofs'? (and what does 'proof' mean???)

Different sequences of development for proof

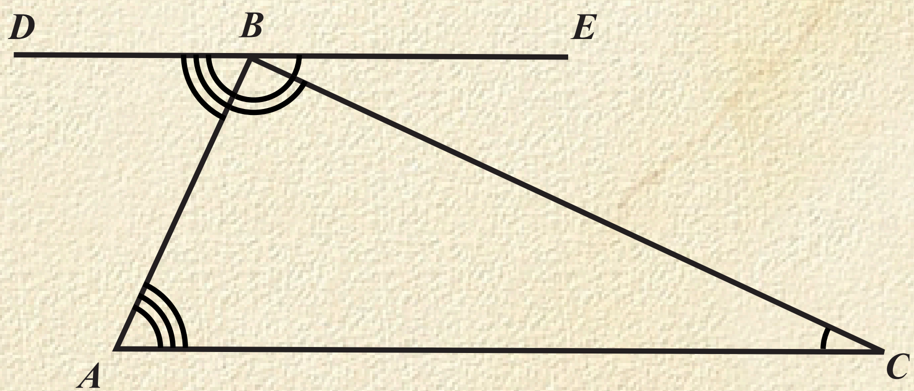
It is not just that each world has its own warrant for truth, each world develops its warrants in an increasingly sophisticated way…

The embodied world encourages thought experiments with the physical world that may give later proofs in Euclidean geometry.

The 'warrant' that 'the angles of a triangle add up to 180°.' Cut a triangle out of paper, and tear off its corners. Put the three corners together, they form a straight line!



physical demonstration



Euclidean Proof

Different sequences of development for proof

THE DAILY PLANET
Extra! Extra! Read All About It!
Homo sapiens *invents* Platonism

The embodied world is a natural environment to grow from physical experiment on figures through the use of language to build deductive relationships in Euclidean geometry.

Eleanor Rosch and colleagues (1976) revealed information about 'basic categories':

- A 'basic category' is the first level identified by children and the quickest to be identified by adults.
- The names of basic level items are usually short and easily reproduced,
- it is the level at which knowledge is easiest organised.
- it is also the highest level for which category members have similar shapes,
- for which similar motor activities are used to interact with them,
- has a single mental image that can be used to picture them.

Example: basic categories are 'dog', 'ball', 'apple', developed before sub-ordinate categories such as 'corgi' or 'alsatian' for dog, or super-ordinate categories such as 'animal' or 'vertebrate'.

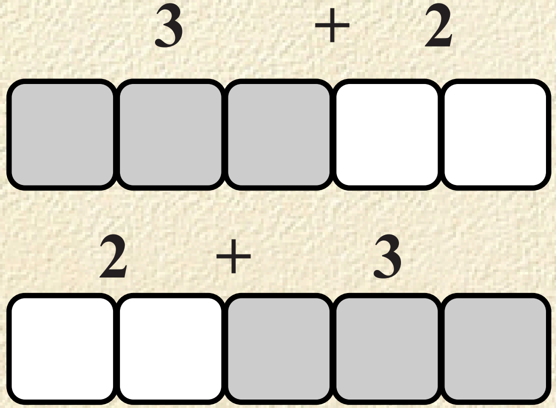
Mathematicians do what children do. They take basic categories of geometric figures as a starting point, such as triangle, circle, point, line, and build up super-ordinate and sub-ordinate relationships between them.

Q. E. D. !

W^5

Warrants for truth in arithmetic and algebra

Initially, operations are on embodied objects and these are the source of generic examples in arithmetic and algebra.
e.g., we can see two and three make the same total as three and two by holding up the requisite number of objects and reversing the order. This 'order irrelevancy principle', later called 'the commutative law', arises through experience and observation. It is not a 'law' imposed on the real world, it is an observation of what happens in the real world.



It does not matter if we draw 3+2 or 5+7 or 12+9, all follow the same pattern.
A generic example gives a warrant for truth of the notion of order irrelevance of addition.

Warrants for truth in arithmetic and algebra

A foundational problem in cognitive development:
In the proceptual world, truth can be tested by computation and manipulation. Our experiences of generic examples from the embodied beginnings of arithmetic give us warrants for truth of the general laws such as the 'commutativity of addition' or the 'distributivity of multiplication over addition.

$$x + y = y + x \qquad\qquad x(y+z) = xy + xz$$

Children are assumed to 'know' them to be true from experience of arithmetic and assume they will also naturally hold in algebra.
Other 'truths', such as

$$(a-b)(a+b) = a^2 - b^2$$

we 'show to be true' by operating on 'facts' already 'known' to be true. For instance:

$$(a-b)(a+b) = (a-b)a + (a-b)b = a^2 - ba + ab - b^2 = a^2 - b^2$$

But what warrants do we give the child to show which of these is 'really true' (whatever that means) and which 'need' proof?

Do (can?) some/all children understand Algebraic Proof?

Collis showed some children are happy that sums of small numbers are commutative and agree that $x + y = y + x$

But when confronted with a sum involving large numbers, they feel the need to calculate, and lose confidence that such a large sum is commutative.

We cannot assume that children 'know' the basic rules, nor that they have the sophistication to know which rules are 'known' and which 'need to be proved.'

Peacock's Folly

In the nineteenth century, Peacock pronounced his law of 'algebraic permanence', that laws that hold in one mathematical system would naturally hold in a larger system. He used this law to justify carrying the rules of arithmetic over to the rules of algebra. He was, in a word, ***wrong.***

The reverse is true, in any extension of a given system, there are always rules from the old system that no longer hold in the bigger system as the bigger system has more structure in it.

- *In the natural numbers, there is always a 'next' number, in rational numbers there is not;*
- *The real numbers are ordered, the complex numbers are not;*

Tall et al (2001) show many discontinuities in the expansion of number systems in secondary schools, from whole numbers to integers, from integers to rational numbers, the use of whole number powers, fractional powers, negative powers, infinite decimals, infinite limit processes and so on.

How, in such a system is a child to maintain his or her bearings to be sure that what 'is true' remains true?

The Formal World

Statements like

$$x + y = y + x \qquad x(y + z) = xy + xz$$

are no longer true because of one's prior experience, they are true because they are asserted to be true as axioms in an axiomatic mathematical structure, say in a field or ring.

The symbol **+** need no longer carry any meaning of addition or involve any actual process of computation.

What matters is that the structure obeys the given axioms related to the symbols. Having proved a result by logical deduction from the axioms, one knows that it is then true of all the examples for which the axioms are satisfied.

The Formal World

There are several serious problems:

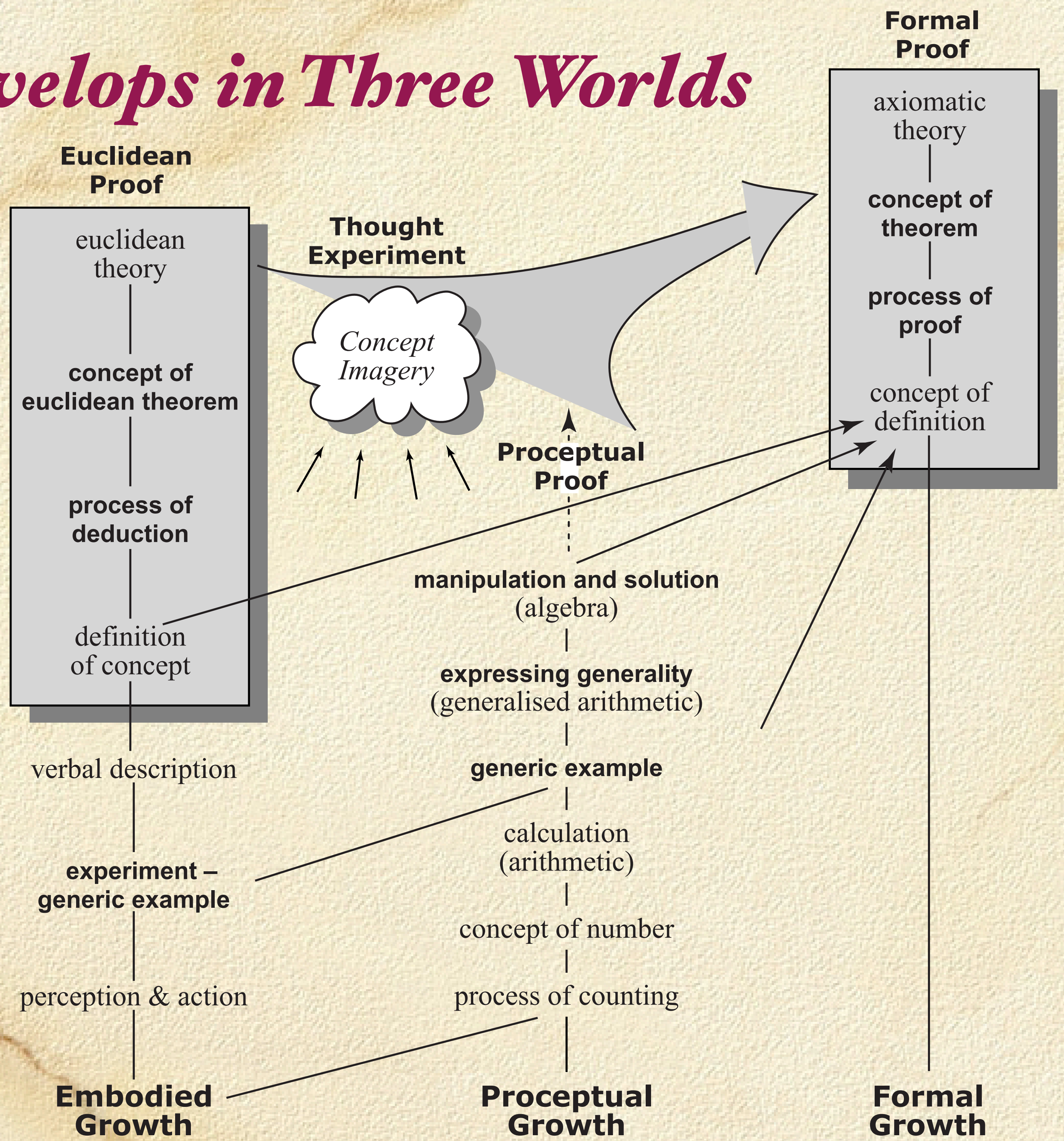
1. The reversal of meaning of a definition, where a definition now constructs a concept, whereas in the other two worlds, a definition is constructed from a concept.

concept → definition becomes *definition → concept*

2. There are serious difficulties with quantified statements in formal proof, as compared with the generic examples of thought experiment.

3. The proof is written without mentioning 'contextual' truths, requiring real sophistication to understand precisely what should be put in and what left out.

Proof develops in Three Worlds



The Formal World for Students

There is a parallel between the development of Euclidean proof in the embodied world and formal proof in the formal world. Indeed, in line with van Hiele theory, we see the definition leading to the process of proof, encapsulated as the concept of proof and then the concepts are organised in a deductive sequence to give the whole theory. This parallel was noted by Chin (2002) in his PhD thesis. There is a sequence of sophistication consisting of:

Definition
Process of proof
Concept of proof
Axiomatic theory,

In van Hiele theory, Guttierez et al (1991) found that there was significant overlap of van Hiele levels. Exactly the same happens here. The categories overlap and the next one starts before the previous one is established. This suggests that:

However we attempt to grasp the growth of proof with highly detailed levels, we will fail to describe what is going on in the student's head.

The Formal World for Mathematians

Proof is not a purely formal art, whatever is claimed.
All new theorems must come from somewhere.
Mathematicians prove new theorems, not because they find them by formal proof, but because they have intuitions that suggest certain theorems might be true, and then they set out to prove them.
Mathematicians are happier when they have problems to solve, the more intractable, the more beguiling. Consider Fermat's Last Theorem, (e.g. Stewart & Tall, 2002).
Consider John Nash, featured in the book and film, A Beautiful Mind.
Mathematicians at Princeton despised people who focused only on proof. The 'big men' were those with ideas that were intriguing but no-one had proved them.
Unproven hypotheses are the stuff of mathematics. Once proved they are not interesting.
Lefschetz the head of department at Princeton wrote a famous book on Topology, which I read as a graduate at Oxford. He wrote it on sabbatical, and it had many mistakes, for there were no students to correct him!
John Nash valued 'good theorems' that others said were important but were not proved.
He set such theorems as exercises for his undergraduates on their examinations! A fresh mind might find a solution.

Proof is not achieved by formal means.
So, how is it achieved?

Creating New Theorems

Embodied ideas, sense of pattern in both embodiment and symbol use, all contribute to the ***concept image****, ('the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes'. Tall & Vinner (1981), p.152).*

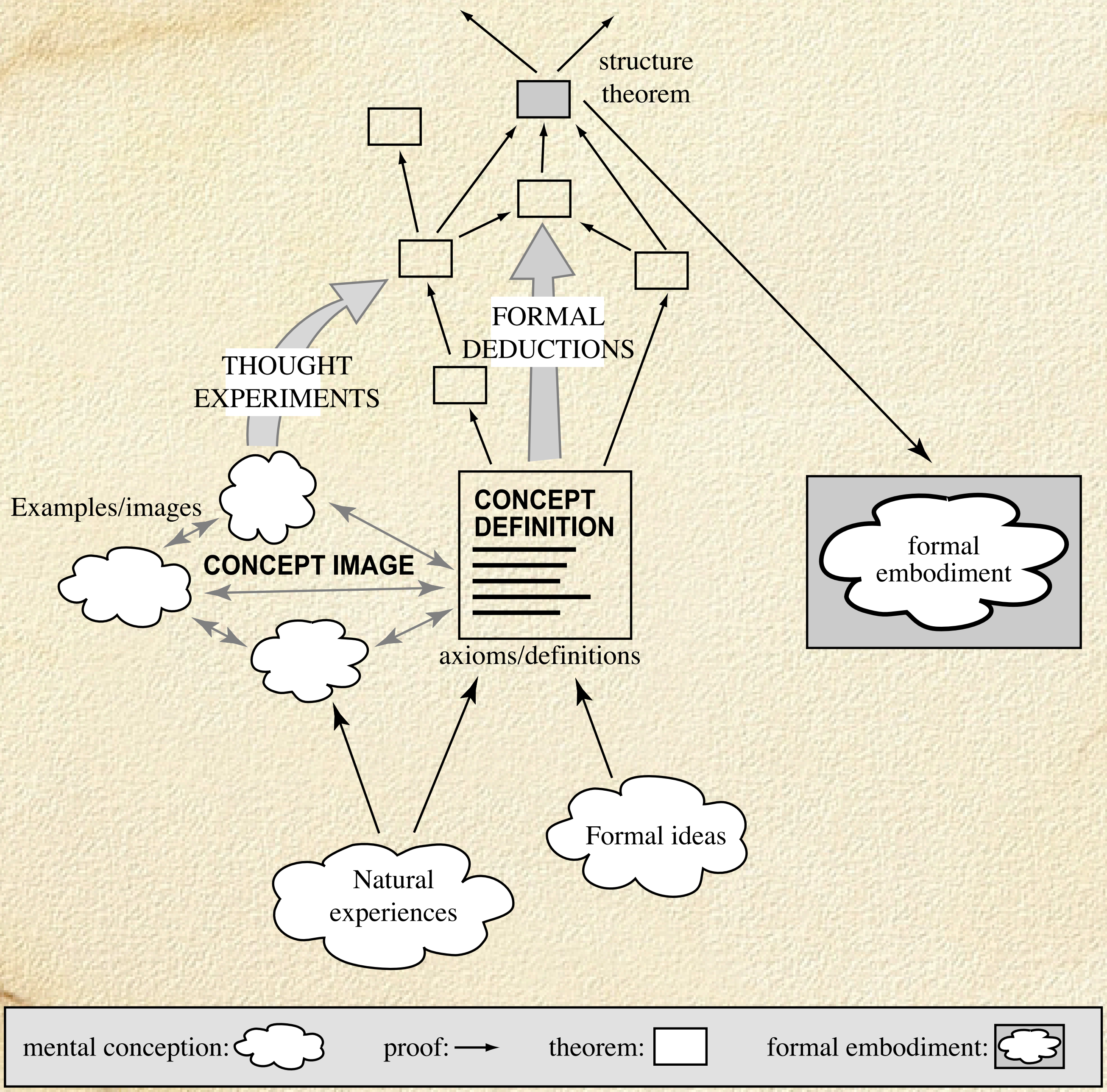
The concept image is used to imagine thought experiments, to conceive of possible definitions and possible theorems that might arise from those definitions. It gives a possible route from intuition to the formulation of theorems that might subsequently be given a formal proof.

Pinto and Tall (2001, 2002) show such routes are available to undergraduates.

Some students take a formal route *to learning about proof, accepting the definitions, committing them to memory and working with the logical relationships to build up a knowledge of formal mathematics.*

Others take a natural route*, building from their concept images and reconstructing them in increasingly sophisticated ways to convert embodied images into a meaningful version of the definition. (Marcia Pinto, PhD, Warwick 2000).*

Concept images, formal proof & embodiment



Concept images, formal proof & embodiment

The links between concept imagery and formal proof go in both directions.
Concept imagery *can* ***suggest theorems*** *to* ***prove formally****.*
Structure theorems *formulate the structure of an axiomatic system that* ***may be used as a more sophisticated image*** *for new thought experiments.*

Examples of Structure Theorems

A finite dimensional vector space over a field F defined axiomatically has the structure of n-tuples in the field F.

A group defined axiomatically must, by Cayley's Theorem, have the structure of a (sub-)group of permutations of a set.

A complete ordered field has the structure of the real number line.

Embodiment and formal proof

Lakoff and his colleagues (Lakoff & Johnson, 1999, Lakoff & Nunez, 2000) claim that all mathematics comes from embodiment. We have seen it certainly underpins conceptual growth.

However, formal mathematics attempts to base its deductions on more than just imagistic thought experiments, so that ***theorems proved work*** *not only in a single embodiment (such as that of the geometric figures in Euclidean geometry) but* ***in any structure that obeys the axioms****.*

This is the power of formal proof, over and above the power of insight that may come from a particular embodiment.

It is the fundamental reason why formal proof is the foundation of the work of research mathematicians.

Summary

- Formal proof is the pinnacle of mathematical development, and requires the implicit understanding of context shared by mathematicians to sustain it.
- To reach such a refined level requires a long and complex cognitive development.
- Important aspects of cognitive development of mathematics can be formulated in terms of three distinct worlds: the embodied, the proceptual and the formal.
- As development progresses, different warrants for truth in different worlds of mathematics develop different meanings through different ways of building knowledge.
- Without the biological development of the human brain we would not have the power of mathematics
- We would do well to remember this and work to understand the nature of mathematical growth and how we may use this knowledge in the education of our children.