Lecture 28: Sturm-Liouville Boundary Value Problems

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In this lecture we abstract the eigenvalue problems that we have found so useful thus far for solving the PDEs to a general class of boundary value problems that share a common set of properties. The so-called *Sturm-Liouville Problems* define a class of eigenvalue problems, which include many of the previous problems as special cases. The $S-L$ Problem helps to identify those assumptions that are needed to define an eigenvalue problems with the properties that we require.

Key Concepts: Eigenvalue Problems, Sturm-Liouville Boundary Value Problems; Robin Boundary conditions.

Reference Section: Boyce and Di Prima Section 11.1 and 11.2

28 Boundary value problems and Sturm-Liouville theory:

28.1 Eigenvalue problem summary

- We have seen how useful eigenfunctions are in the solution of various PDEs.
- The eigenvalue problems we have encountered thus far have been relatively simple

I: The Dirichlet Problem:

$$\left.\begin{array}{c} X''+\lambda^2 X=0 \\ X(0)=0=X(L) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \lambda_n=\frac{n\pi}{L},\ n=1,2,\ldots \\ X_n(x)=\sin\left(\frac{n\pi x}{L}\right) \end{array}\right.$$

II: The Neumann Problem:

$$\left.\begin{array}{c} X''+\lambda^2 X=0 \\ X'(0)=0=X'(L) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \lambda_n=\frac{n\pi}{L},\ n=0,1,2,\ldots \\ X_n(x)=\cos\left(\frac{n\pi x}{L}\right) \end{array}\right.$$

III: The Periodic Boundary Value Problem:

$$\left.\begin{array}{c} X''+\lambda^2 X=0 \\ X(-L)=0=X(L) \\ X'(-L)=0=X'(L) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \lambda_n=\frac{n\pi}{L},\ n=0,1,2,\ldots \\ X_n(x)\in\left\{1,\cos\left(\frac{n\pi x}{L}\right),\sin\left(\frac{n\pi x}{L}\right)\right\} \end{array}\right.$$

IV: Mixed Boundary Value Problem A:

$$\left.\begin{array}{c} X''+\lambda^2 X=0 \\ X(0)=0=X'(L) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \lambda_k=\frac{(2k+1)\pi}{2L},\ k=0,1,2,\ldots \\ X_n(x)=\sin\left(\frac{(2k+1)\pi}{2L}x\right) \end{array}\right.$$

V: Mixed Boundary Value Problem B:

$$\left.\begin{array}{c} X''+\lambda^2 X=0 \\ X'(0)=0=X(L) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \lambda_k=\frac{(2k+1)\pi}{2L},\ k=0,1,2,\ldots \\ X_n(x)=\cos\left(\frac{(2k+1)\pi}{2L}x\right) \end{array}\right.$$

28.2 The regular Sturm-Liouville problem:

Consider the the following two-point boundary value problem

$$\begin{aligned} &\big(p(x)y'\big)' - q(x)y + \lambda r(x)y = 0 \quad 0 < x < \ell \\ &\alpha_1 y(0) + \alpha_2 y'(0) = 0 \quad \beta_1 y(\ell) + \beta_2 y'(\ell) = 0 \end{aligned} \tag{28.1}$$

where p, p', q and r are continuous on $0 \le x \le \ell$ and $p(x) \ge 0$ and $r(x) > 0$ on $0 \le x \le \ell$.

We define the Sturm-Liouville eigenvalue problem as:

$$\left.\begin{array}{l} \mathcal{L}y = \lambda r y \quad \text{where} \quad \mathcal{L}y = -(py')' + qy \\ \alpha_1 y(0) + \alpha_2 y'(0) = 0 \quad \text{and} \quad \beta_1 y(\ell) + \beta_2 y'(\ell) = 0 \\ p(x) > 0 \text{ and } r(x) > 0. \end{array}\right\} \quad \text{SL} \tag{28.2}$$

Remark 1 Note:

(1) If $p = 1$, $q = 0$, $r = 1$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 1$, $\beta_2 = 0$ we obtain Problem (I) above whereas if $p = 1$, $q = 0$, $r = 1$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 0$, $\beta_2 = 1$, we obtain Problem (II) above. Notice that the boundary conditions for these two problems are specified at separate points and are called *separated BC*. The periodic BC $X(0) = X(2\pi)$ are not separated so that Problem (III) is not technically a SL Problem.

(2) If $p > 0$ and $r > 0$ and $\ell < \infty$ then the SL Problem is said to be regular. If $p(x)$ or $r(x)$ is zero for some x or the domain is $[0, \infty)$ then the problem is singular.

(3) There is no loss of generality in the so-called self-adjoint form $\mathcal{L}y = -(py')' + qy$ since it is possible to convert a general 2nd order eigenvalue problem

$$-P(x)y'' - Q(x)y' + R(x)y = \lambda y \tag{28.3}$$

to self-adjoint form by multiplying by a suitable integrating factor $\mu(x)$

$$-\mu(x)P(x)y'' - \mu Q(x)y' + \mu(x)R(x)y = \lambda\mu(x)y \tag{28.4}$$

but expanding the differential operator we obtain

$$\mathcal{L}y = -py'' - p'y' + qy = \lambda r y. \tag{28.5}$$

Thus comparing (28.5) and (28.4) we can make the following identifications: $p = \mu P$ and $p' = \mu Q \Rightarrow p' = \mu' P + \mu P' = \mu Q$ which is a linear 1st order ODE for μ with integrating factor $exp(\int \frac{P'}{P} - \frac{Q}{P}\,dx)$

$$\mu' + \left(\frac{P'}{P} - \frac{Q}{P}\right)\mu = 0 \Rightarrow \left[Pe^{-\int \frac{Q}{P}\,dx}\mu\right]' = 0 \quad \Rightarrow \boxed{\mu = \frac{e^{\int \frac{Q}{P}\,dx}}{P}}. \tag{28.6}$$

Example 28.1 *Reducing a boundary value problem to SL form:*

$$\phi'' + x\phi' + \lambda\phi = 0 \tag{28.7}$$

$$\phi(0) = 0 = \phi(1) \tag{28.8}$$

We bring (28.7) into SL form by multiplying by the integrating factor

$$\begin{aligned}
&\mu = \frac{1}{P} e^{\int \frac{Q}{P}\,dx} = e^{\int x\,dx} = e^{x^2/2}, \quad P(x) = 1, \quad Q(x) = x, \quad R(x) = 1. \\
&e^{x^2/2}\phi'' + e^{x^2/2} x\phi' + \lambda e^{x^2/2}\phi = 0 \\
&\qquad -\left(e^{x^2/2}\phi'\right)' = \lambda e^{x^2/2}\phi \\
&p(x) = e^{x^2/2} \quad r(x) = e^{x^2/2}
\end{aligned} \tag{28.9}$$

Example 28.2 *Convert the equation $-y'' + x^4 y' = \lambda y$ to SL form*

$$P = 1, \quad Q = -x^4, \quad \mu = e^{-\int x^4\,dx} = e^{-x^5/5} \tag{28.10}$$

$$\textit{Therefore } - e^{-x^5/5}y'' + e^{-x^5/5}x^4 y' = \lambda e^{-x^5/5} \tag{28.11}$$

$$-\left(e^{-x^5/5}y'\right)' = \lambda e^{-x^5/5}y. \tag{28.12}$$

28.3 Properties of SL Problems

(1) **Eigenvalues**:

- **(a)** The eigenvalues λ are all real.
- **(b)** There are an ∞ # of eigenvalues λ_j with $\lambda_1 < \lambda_2 < \ldots < \lambda_j \to \infty$ as $j \to \infty$.
- **(c)** $\lambda_j > 0$ provided $\frac{\alpha_1}{\alpha_2} < 0$, $\frac{\beta_1}{\beta_2} > 0$ $q(x) > 0$.

(2) **Eigenfunctions**: For each λ_j there is an eigenfunction $\phi_j(x)$ that is unique up to a multiplicative const. and which satisfy:

- **(a)** $\phi_j(x)$ are real and can be normalized so that $\int_0^\ell r(x)\phi_j^2(x)\,dx = 1$.
- **(b)** The eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function $r(x)$:

$$\int_0^\ell r(x)\phi_j(x)\phi_k(x)\,dx = 0 \quad j \neq k. \tag{28.13}$$

- **(c)** $\phi_j(x)$ has exactly $j - 1$ zeros on $(0, \ell)$.

(3) **Expansion Property**: $\{\phi_j(x)\}$ are complete if $f(x)$ is piecewise smooth then

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} c_n \phi_n(x) \\
\text{where} \qquad c_n &= \frac{\int_0^\ell r(x)f(x)\phi_n(x)\,dx}{\int_0^\ell r(x)\phi_n^2(x)\,dx}
\end{aligned} \tag{28.14}$$

Example 28.3 Robin Boundary Conditions:

$$\begin{aligned} X'' + \lambda X &= 0, \qquad \lambda = \mu^2 \\ X'(0) &= h_1 X(0), \quad X'(\ell) = -h_2 X(\ell) \end{aligned} \tag{28.15}$$

where $h_1 \geq 0$ *and* $h_2 \geq 0$.

$$X(x) = A\cos\mu x + B\sin\mu x \tag{28.16}$$

$$X'(x) = -A\mu\sin\mu x + B\mu\cos\mu x \tag{28.17}$$

BC 1: $X'(0) = B\mu = h_1 X(0) = h_1 A \quad \Rightarrow A = B\mu/h_1$.

BC 2: $X'(\ell) = -A\mu\sin(\mu\ell) + B\mu\cos(\mu\ell) = -h_2 X(\ell) = -h_2[A\cos\mu\ell + B\sin\mu\ell]$

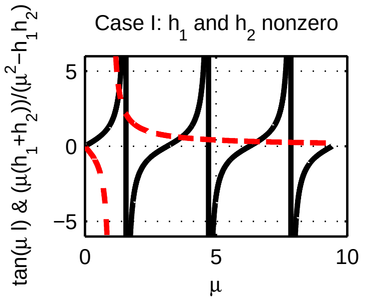
$$\Rightarrow B\left[-\frac{\mu^2}{h_1}\sin(\mu\ell) + \mu\cos(\mu\ell)\right] = -Bh_2\left[\frac{\mu}{h_1}\cos\mu\ell + \sin\mu\ell\right] \tag{28.18}$$

$$B\left\{\left(-\frac{\mu^2}{h_1} + h_2\right)\sin\mu\ell + \left(\mu + \frac{h_2}{h_1}\mu\right)\cos\mu\ell\right\} = 0. \tag{28.19}$$

Therefore

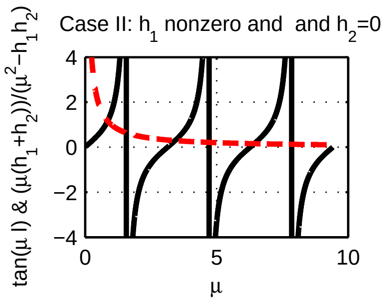
$$\tan(\mu\ell) = \left[\frac{\mu(h_1 + h_2)}{\mu^2 - h_1 h_2}\right]. \tag{28.20}$$

Case I: $\mathbf{h_1}$ and $\mathbf{h_2 \neq 0}$



$X_n = \frac{\mu_n}{h_1}\cos\mu_n x + \sin\mu_n x$, and $\mu_n \sim n\pi/\ell$ as $n \to \infty$

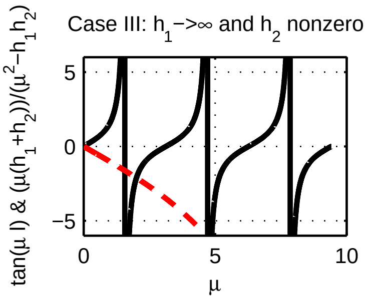
Case II: $\mathbf{h_1 \neq 0}$ and $\mathbf{h_2 = 0}$



$$X_n = \frac{\mu_n}{h_1}\cos\mu_n x + \sin\mu_n x \tag{28.21}$$

$$= \frac{\cos\mu_n(\ell - x)}{\sin\mu_n\ell} \tag{28.22}$$

Case III: $\mathbf{h_1 \to \infty \quad h_2 \neq 0}$



$$X_n = \sin(\mu_n x) \tag{28.23}$$

$$\mu_n \sim \left[\left(\frac{2n+1}{2}\right)\frac{\pi}{\ell}\right] \qquad n = 0, 1, 2, \ldots \text{ as } n \to \infty \tag{28.24}$$

28.4 Appendix: Some proofs for Sturm-Liouville Theory

28.4.1 *Lagrange's Identity:*

$$\int_0^\ell (v\mathcal{L}u - u\mathcal{L}v)\,dx = -p(x)u'v\big|_0^\ell + p(x)uv'\big|_0^\ell.$$

Proof: Let u and v be any sufficiently differentiable functions, then

$$\int_0^\ell v\mathcal{L}u\,dx = \int_0^\ell v\left\{-(pu')' + qu\right\}\,dx \tag{28.25}$$

$$= -vpu'\big|_0^\ell + \int_0^\ell u'pv'\,dx + \int_0^\ell uqv\,dx \tag{28.26}$$

$$= -vpu'\big|_0^\ell + upv'\big|_0^\ell + \int_0^\ell u\left\{-(pv')' + qv\right\}dx \tag{28.27}$$

$$\text{Therefore } \int_0^\ell v\mathcal{L}u\,dx = -pvu'\big|_0^\ell + puv'\big|_0^\ell + \int_0^\ell u\mathcal{L}v\,dx. \qquad \square \tag{28.28}$$

Now suppose that u and v both satisfy the SL boundary conditions. I.E.

$$\begin{array}{lcllcl} \alpha_1 u(0) + \alpha_2 u'(0) & = & 0 & \beta_1 u(\ell) + \beta_2 u'(\ell) & = & 0 \\ \alpha_1 v(0) + \alpha_2 v'(0) & = & 0 & \beta_1 v(\ell) + \beta_2 v'(\ell) & = & 0 \end{array} \tag{28.29}$$

then

$$\int_0^\ell v\mathcal{L}u\,dx - \int_0^\ell u\mathcal{L}v\,dx = -p(\ell)u'(\ell)v(\ell) + p(\ell)u(\ell)v'(\ell) \tag{28.30}$$

$$+p(0)u'(0)v(0) - p(0)u(0)v'(0) \tag{28.31}$$

$$= p(\ell)\left\{+\frac{\beta_1}{\beta_2}u(\ell)v(\ell) + u(\ell)\left(-\frac{\beta_1}{\beta_2}v(\ell)\right)\right\} \tag{28.32}$$

$$+p(0)\left\{-\frac{\alpha_1}{\alpha_2}u(0)v(0) - u(0)\left(-\frac{\alpha_1}{\alpha_2}v(0)\right)\right\} \tag{28.33}$$

$$= 0. \tag{28.34}$$

Thus $\int_0^\ell v\mathcal{L}u\,dx = \int_0^\ell u\mathcal{L}v\,dx$ whenever u and v satisfy the SL boundary condition.

Observations:

- If $\mathcal{L}$ and BC are such that $\int_0^\ell v\mathcal{L}u\,dx = \int_0^\ell u\mathcal{L}v\,dx$ then $\mathcal{L}$ is said to be **self-adjoint**.
- Notation: if we define $(f,g) = \int_0^\ell f(x)g(x)\,dx$ then we may write $(v,\mathcal{L}v) = (u,\mathcal{L}v)$.

28.4.2 *Proofs using Lagrange's Identity:*

(1a) The λ_j are real: Let $\mathcal{L}y = \lambda ry$ (1) $\alpha_1 y(0) + \alpha_2 y'(0) = 0$ $\beta_1 y(\ell) + \beta_2 y'(\ell) = 0$. Take the conjugate of (1) $\mathcal{L}\bar{y} = \bar{\lambda}r\bar{y}$. By Lagrange's Identity:

$$0 = (\bar{y}, \mathcal{L}y) - (y, \mathcal{L}\bar{y}) \tag{28.35}$$

$$= (\bar{y}, r\lambda y) - (y, r\bar{\lambda}\bar{y}) \tag{28.36}$$

$$= \int_0^\ell \bar{y}(x) r\lambda y(x)\,dx - \int_0^\ell y(x)r(x)\bar{\lambda}\bar{y}(x)\,dx \tag{28.37}$$

$$= (\lambda - \bar{\lambda}) \int_0^\ell r(x)\big|y(x)\big|^2\,dx \tag{28.38}$$

Since $r(x)|y(x)|^2 \geq 0$ it follows that $\lambda = \bar{\lambda} \Rightarrow \lambda$ is real.

(1c) $\lambda_j > 0$ provided $\alpha_1/\alpha_2 < 0$ $\beta_1/\beta_2 > 0$ and $q(x) > 0$. Consider $\mathcal{L}y = -(py')' + qy = \lambda ry$ (SL) and multiply (SL) by y and integrate from 0 to ℓ:

$$(y, \mathcal{L}y) = \int_0^\ell -(py')'y + qy^2\,dx = \lambda \int_0^\ell r(x)\big[y(x)\big]^2\,dx \tag{28.39}$$

$$\text{Therefore } \lambda = \frac{\int_0^\ell -(py')'y + qy^2\,dx}{\int_0^\ell ry^2\,dx} \quad \text{this is known as Rayleigh's Quotient.}$$

$$= \frac{[-py'y]_0^\ell + \int_0^\ell p(y')^2 + qy^2\,dx}{\int_0^\ell ry^2\,dx} \tag{28.40}$$

$$= \frac{+p(\ell)\frac{\beta_1}{\beta_2}\big[y(\ell)\big]^2 - p(0)\frac{\alpha_1}{\alpha_2}\big[y(0)\big]^2 + \int_0^\ell p(y')^2 + qy^2\,dx}{\int_0^\ell ry^2\,dx}. \tag{28.41}$$

Therefore $\lambda > 0$ since the RHS is all positive.

Note: If $q(x) \equiv 0$ and $\alpha_1 = 0 = \beta_1$ then with $y'(0) = 0 = y'(\ell)$ we have nontrivial eigenfunction $y(x) = 1$ and eigenvalue $\lambda = 0$.

(2b) Eigenfunctions corresponding to different eigenvalues are orthogonal. Consider two distinct eigenvalues $\lambda_j \neq \lambda_k$ $\lambda_j : \mathcal{L}\phi_j = r\lambda_j\phi_j$ and $\lambda_k : \mathcal{L}\phi_k = r\lambda_k\phi_k$. Then

$$0 = (\phi_k, \mathcal{L}\phi_j) - (\phi_j, \mathcal{L}\phi_k) \quad \text{by Lagrange's Identity} \tag{28.42}$$

$$= (\phi_k, r\lambda_j\phi_j) - (\phi_j, r\lambda_k\phi_k) \tag{28.43}$$

$$= (\lambda_j - \lambda_k)\int_0^\ell r(x)\phi_k(x)\phi_j(x)\,dx \tag{28.44}$$

now $\lambda_j \neq \lambda_k$ implies that

$$\int_0^\ell r(x)\phi_k(x)\phi_j(x)\,dx = 0. \tag{28.45}$$

(3) The eigenfunctions form a complete set: It is difficult to prove the convergence of the eigenfunction series expansion for $f(x)$ that is piecewise smooth. However, if we assume the expansion converges then it is a simple matter to use orthogonality to determine the coefficients in the expansion: Let $f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$.

$$\int_0^\ell f(x)\phi_m(x)r(x)\,dx = \sum_{n=1}^{\infty} c_n \int_0^\ell r(x)\phi_m(x)\phi_n(x)\,dx \tag{28.46}$$

orthogonality implies

$$c_m = \frac{\int_0^\ell r(x)f(x)\phi_m(x)\,dx}{\int_0^\ell r(x)\big[\phi_m(x)\big]^2\,dx}. \tag{28.47}$$