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A GENERALIZED QUANTITY DISCOUNT PRICING MODEL TO INCREASE SUPPLIER'S PROFITS*

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In this paper, the joint problem of ordering and offering price discount by a supplier to his sole/major buyer is analyzed. The objective is to induce the buyer to alter his order schedule and size so that the supplier can benefit from lower set up, ordering, and inventory holding costs. We generalize the quantity discount pricing model of Monahan (1984) to: (1) explicitly incorporate constraints imposed on the amount of discount that can be offered; and (2) relax the implicit assumption of a lot-for-lot (or order-for-order) policy adopted by the supplier. An algorithm is developed to solve the supplier's joint ordering and price discount problem.
(INVENTORY/PRODUCTION—POLICIES, PRICING; INVENTORY/PRODUCTION—DETERMINISTIC MODELS; PURCHASING)

1. Introduction

Quantity discount models have been studied traditionally from the point of view of the buyer, but not the supplier. These studies focus on the problem of determining the economic order quantities for the buyer, given a quantity discount schedule set by the supplier. Examples of these studies include Ladany and Sternlieb (1974), Subramanyam and Kumaraswamy (1981), Hadley and Whitin (1963), Jucker and Rosenblatt (1985), Peterson and Silver (1979), Rubin, Dilts, and Barron (1983) and Sethi (1984).

The quantity discount problem from the point of view of the supplier, however, has not been adequately addressed. Recently, Monahan (1984) has studied the important economic implications from the supplier's point of view of offering quantity discounts to his sole or major buyer. An "optimal" quantity discount pricing schedule has been developed, and the induced economic ordering quantity of the buyer, resulting from the discount, has been shown to be a simple function of the ratio of the set up costs of the buyer and the supplier.

Monahan's result is an interesting one. He shows that, by appropriately setting the price discount, the supplier can always improve his profit. The "optimal" (to the supplier) ordering quantity of the buyer resulting from the discount arrangement is, by a factor, larger than his initial economic order quantity. The surprising result is that this factor is independent of the opportunity costs of holding inventory of both the supplier and the buyer.

There are two major issues that Monahan has overlooked in his model. First, there should be some constraints imposed on the amount of price discount that can be offered. In Monahan's model, there is no guarantee that the amount of price discount given by the supplier is always less than the selling price of the product. In fact, one can construct examples such that the amount of price discount using Monahan's method exceeds the selling price of the product. Given such a price discount schedule, it is clear that the buyer's optimal EOQ is infinity, since no constraints are imposed

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on the buyer's EOQ. The supplier would also not be in business, given this situation. Hence, without imposing some constraint on the amount of the price discount offered, it is possible to end up with a solution that is unrealistic and impractical.

Second, one of the potential benefits to the supplier in offering quantity discounts is to alter the pattern of orders placed by the buyer which may reduce the opportunity cost of holding inventory for the supplier. Such a link is missing in Monahan's model. In fact, Monahan's model assumes that the supplier always employs an order-for-order policy, and that the number of set ups for the supplier is exactly the same as that for the buyer. Such a policy may be neither optimal nor feasible. If the supplier manufactures the product himself, it may not always be advisable for him to use a lot-for-lot type of production policy, especially when the set up cost is high. Similarly, if the supplier orders the product from yet another external vendor, it may be desirable for the supplier to order a larger quantity each time he places an order and to incur higher inventory holding cost but reduce the total set up costs of ordering. Monahan's order-for-order assumption for the supplier is a restrictive one. In fact, it will be seen later that this is the assumption that leads to the simplified result obtained by Monahan that we described earlier. Namely, the factor of increase in the buyer's ordering quantity is independent of the opportunity cost of holding inventory for both the supplier and the buyer.

The purpose of this paper is to generalize Monahan's model to address the two issues mentioned above. Explicit constraints are imposed on the model so as to ensure that the real price of an item is nonnegative. These constraints turn out to be effective in providing lower and upper bounds for a search algorithm for the optimal pricing problem to be described below. In addition, the order-for-order assumption of the supplier is dropped, and the problem of simultaneously solving for the optimal ordering and price discount problem for the supplier is addressed. It is shown that the optimal solution now depends on both the holding cost and set up cost of the supplier and the buyer. An algorithm is presented. Monahan's results, of course, become a special case of our generalized model. We conclude our paper with possible extensions to the generalized model.

2. A Generalized Model

In this section, we present the generalized quantity discount pricing model. As far as possible, we retain the notation used by Monahan.

Let:

D_1 = total yearly number of units demanded by the buyer (equal to that demanded by his customers); the buyer's demand is assumed to be constant and uniform;

S_1 = the buyer's set up cost per order;

P_1 = the current, delivered unit price paid by the buyer;

H_1 = the buyer's yearly inventory holding cost, expressed as a percentage of the value of the item (%/year);

Q_1 = the buyer's current order size.

As described in Monahan, the current order size of the buyer without price discount is the familiar Wilson lot size formula:

$$Q_1 = \sqrt{2D_1S_1/H_1P_1}. \tag{1}$$

Hence, the buyer would place D_1/Q_1 orders each year, evenly spaced, with an interval of Q_1/D_1 throughout that time. Suppose now that the supplier requests the buyer to increase his current order size by a factor K ($K \geq 1$). Then Monahan (see equation (6) of the referenced paper) shows that the amount of price discount that the supplier must give the buyer to compensate him for his increased inventory expenses is at least

$$d(K) = \sqrt{2S_1H_1P_1/D_1}\,(K-1)^2/2K, \tag{2}$$

where $d(K)$ is the per unit dollar discount offered when the buyer orders K times his current order size. Monahan refers to this value as the "practical" break even discount, as opposed to the "exact" discount that should be given, which is slightly smaller (see equation (7) of Monahan). In practice, Monahan notes that the differences between the practical and the exact discount tend to be minor, and even when they are not, the differences could be viewed as added incentive to the buyer to go along with the larger order policy. In this paper, we also use the same "practical" break-even discount as an approximation. Henceforth, the optimal pricing and ordering policies in this paper are only "optimal" to the approximate problem, as in Monahan.

Next, we let:

S_2 = the supplier's set up cost per order;

H_2' = the supplier's yearly inventory holding cost, expressed as a percentage of the value of the item (%/year);

P_2 = the unit cost of producing or acquiring the item by the supplier.

We shall assume that the replenishment time (production lead time) for the supplier is negligible. Again, if the lead time is constant, such an assumption is equivalent to the supplier's placing an order with its external vendor (or beginning production) in advance by a period exactly equal to the lead time. In the case of the production lead time, we also assume that the productive capacity of the supplier is much greater than D_1 so that the inventory "build up" cost of the supplier need not be considered.

Since the buyer's quantity is fixed at KQ_1, the supplier is faced with a stream of demands, each with order size KQ_1 and at fixed intervals of KQ_1/D_1 year apart. Given such a stream of demands, it can be easily shown that the supplier's economic order (or production) quantity should be some integer multiple of KQ_1. Let this quantity be denoted by $k(KQ_1)$, where k is a positive integer.

We now consider the inventory holding costs to the supplier when his order (or production) size is $k(KQ_1)$ units. As illustrated in Figure 1, the total inventory held per ordering cycle of the supplier is:

$$[(k-1)(KQ_1)+(k-2)(KQ_1)+\cdots+2(KQ_1)+KQ_1+0](KQ_1/D_1)=(k-1)k(KQ_1)^2/2D_1.$$

The length of the order cycle is $k(KQ_1)/D_1$, and thus the average inventory held up per year is:

$$\frac{k(k-1)(KQ_1)(KQ_1/D_1)}{2(kKQ_1/D_1)} = \frac{(k-1)KQ_1}{2}.$$

The supplier's yearly profit function is given by his gross revenue minus his set up cost and inventory holding cost:

$$YNP = D_1P_1 - D_1d(K) - (D_1S_2/kKQ_1) - (k-1)KQ_1H_2'P_2/2,$$

where $d(K)$ is as given by (2).

Define $H_2 = H_2'P_2/P_1$, we can then rewrite YNP as

$$YNP = D_1P_1 - D_1d(K) - (D_1S_2/kKQ_1) - (k-1)KQ_1H_2P_1/2. \tag{3}$$

Evidently, as it is unrealistic for the price discount offered by the supplier to exceed the current price of the product, we need to impose the constraint $P_1 \geq d(K)$. In fact, a more reasonable constraint that should be imposed on $d(K)$ would look like:

$$P_1 - P_2 - d(K) \geq \Delta, \tag{4}$$

where Δ is some profit margin (per unit), $\Delta \geq 0$, that the supplier desires (obviously, $P_1 - P_2 \geq \Delta$).

The supplier's problem is thus to determine the optimal k and K simultaneously so as to maximize YNP of (3) subject to (2) and (4). Finally, as we are considering the

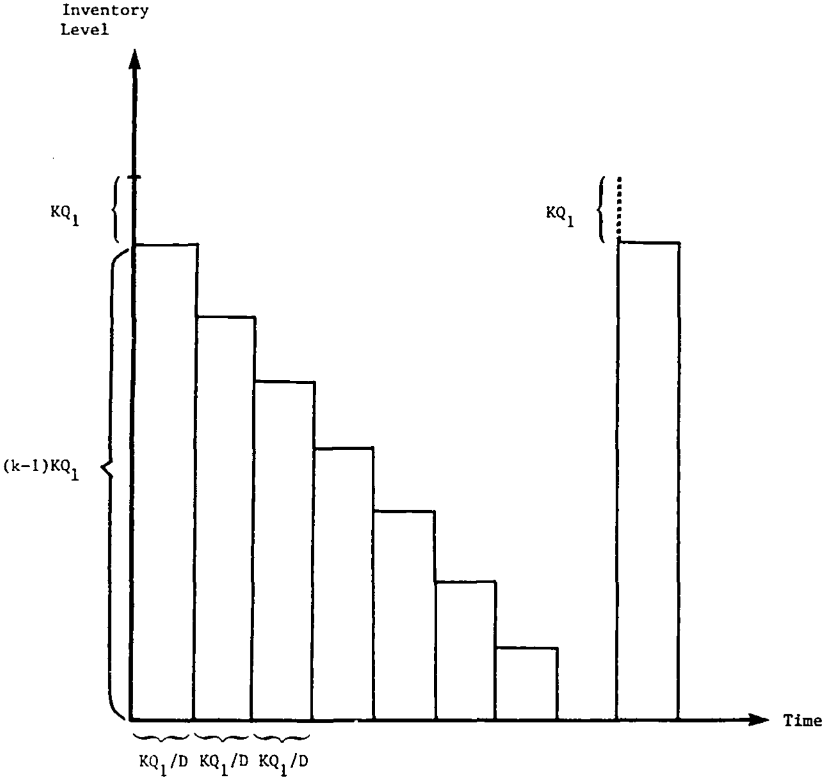

FIGURE 1. Inventory Level at the Supplier.

supplier's price discount schedule which will induce a larger ordering quantity from the buyer, we also have $K \geq 1$.

Before we proceed to solve the supplier's problem, we note that constraint (4) can be reduced to a linear constraint on K. Substituting (2) into (4) and rearranging terms, constraint (4) can be written as the following quadratic relationship in K:

$$K^2 - 2[1 + (P_1 - P_2 - \Delta)\sqrt{D_1/2S_1H_1P_1}]K + 1 \leq 0. \tag{5}$$

Define:

$$\underline{K} = 1 + (P_1 - P_2 - \Delta)\sqrt{D_1/2S_1H_1P_1} - \{[1 + (P_1 - P_2 - \Delta)\sqrt{D_1/2S_1H_1P_1}]^2 - 1\}^{1/2},$$

and

$$\bar{K} = 1 + (P_1 - P_2 - \Delta)\sqrt{D_1/2S_1H_1P_1} + \{[1 + (P_1 - P_2 - \Delta)\sqrt{D_1/2S_1H_1P_1}]^2 - 1\}^{1/2}. \tag{6}$$

Then, (5) is equivalent to: $\underline{K} \leq K \leq \bar{K}$.

Furthermore, it is easy to see that $\underline{K} \leq 1$. Thus, constraints (4) and $K \geq 1$ combined become:

$$1 \leq K \leq \bar{K}. \tag{7}$$

We can thus rewrite the supplier's joint (k, K) problem from (3) as:

$$\text{Maximize} \quad YNP = D_1P_1 - [\sqrt{2S_1H_1P_1D_1}(K-1)^2/2K] - (D_1S_2/kKQ_1) - (k-1)KQ_1H_2P_1/2 \quad (8)$$

$$\text{s.t.} \quad 1 \le K \le \bar{K} \quad \text{and} \quad k = 1, 2, 3, \ldots.$$

The current model of price discount offered by the supplier is similar to that of Monahan's. It is, however, different from the work of Goyal (1976) in several aspects. Goyal presents an integrated inventory model for a single supplier-single customer system where the total inventory costs of both levels are jointly minimized. Such a model is appropriate if collaborative arrangement can be enforced by some contractual agreement between the buyer and the supplier. Usually, it is only reasonable for cases when the buyer and the supplier belong to the same organization. When the buyer and the supplier are two separate entities, the problem is not that simple. In Goyal's model, the inventory holding costs are assumed to be *independent* of the price of the item. This is only appropriate if prices are completely fixed. Our model and that of Monahan, of course, assume that price is a decision variable for the supplier to change the behavior of the buyer. By assuming that inventory holding costs are constant, the pricing problem evidently does not exist. Moreover, as illustrated by the numerical example provided by Goyal, the buyer can be worse off in the solution. Indeed, Goyal suggested that one of the methods to induce the buyer to increase the order quantity to the desired level is to "offer a carefully chosen quantity discount scheme." Of course, once discounts are introduced, the inventory holding costs are no longer constant. Our model thus exactly addresses this issue of discounts offered by the supplier so as to compensate the buyer.

When the supplier and the buyer indeed belong to the same organization, the problem of determining the "optimal" price is not appropriate, since the meaning of a price charged to the buyer by the supplier does not exist. We can illustrate this point in the following manner. Suppose that price is the decision variable for the joint two-level system where the buyer continues to use (1) as his order size. The total inventory and set up costs are then given by:

$$(S_2/k)\sqrt{H_1P_1D_1/2S_1} + (k-1)H_2P_2\sqrt{D_1S_1/2H_1P_1} + \sqrt{2D_1S_1H_1P_1}.$$

It is easy to see that the optimal price for the joint system is to set $k = 1$, and $P_1 \rightarrow 0$, so that the total set up and inventory holding costs $\rightarrow 0$!

Summarizing, when both the supplier and the buyer belong to the same organization, the pricing problem is meaningless, and Goyal's model should be used for determining the ordering policies of the joint system. Our model and that of Monahan are appropriate when both the supplier and the buyer are separate entities, and that price discount is the instrument for the supplier to change the buyer's behavior in the form of a compensation.

3. A Solution Procedure

In this section, we first examine the properties of the supplier's joint (k, K) problem. Based on this analysis, an algorithm is presented to solve the problem.

For a given k, we observe that:

$$dYNP/dK = -\sqrt{S_1H_1P_1D_1/2}[1 - (1/K^2)] + (D_1S_2/kQ_1K^2) - (k-1)Q_1H_2P_1/2, \quad \text{and} \quad (9)$$

$$d^2YNP/dK^2 = -(\sqrt{2S_1H_1P_1D_1}/K^3) - (2D_1S_2/kQ_1K^3) < 0. \quad (10)$$

Hence, given k, YNP is a concave function in K, $K \geq 0$. The optimal value of K, for a given k, is obtained by setting (9) equal to zero, which gives:

$$K^*(k) = \left[\left(\sqrt{\frac{S_1H_1P_1D_1}{2}} + \frac{D_1S_2}{kQ_1}\right) \Big/ \left(\sqrt{\frac{S_1H_1P_1D_1}{2}} + \frac{(k-1)Q_1H_2P_1}{2}\right)\right]^{1/2}.$$

Substituting in Q_1 from (1) and simplifying, we have:

$$K^*(k) = \{[1 + S_2/kS_1]/[1 + (k-1)H_2/H_1]\}^{1/2}. \tag{11}$$

From (11), we note that the optimal value of K, for a given k, is now a function of both the set up and holding cost of both the buyer and the supplier. In fact, when $k = 1$, i.e., an order-for-order policy is used by the supplier, $K^*(k)$ is reduced to (2), which is the simple relationship that Monahan has obtained.

Now to solve the supplier's joint (k, K) problem, one can start with $k = 1$, find $K^*(1)$, determine if $K^*(1)$ is in the range of $(1, \bar{K})$, and consequently obtain the optimal value of K for this range. The process is repeated for $k = 2, 3, \ldots$, and so on. The optimal solution is the combination (k, K) that gives the highest YNP. Such a procedure is, of course, computationally cumbersome. Furthermore, there is no stopping rule for k to guarantee optimality. We now explore additional properties of the solution to (8) which result in a (typically) more efficient algorithm for solving the problem.

Consider a given K in (8). The optimal k^* for this K is given by the k^* that satisfies:

$$YNP(k^*|K) \geq YNP(k^* + 1|K) \quad \text{and} \quad YNP(k^*|K) > YNP(k^* - 1|K). \tag{12}$$

Substituting (8) into (12) and appropriately rearranging terms, (12) is equivalent to

$$K^2k^*(k^* + 1) \geq (S_2/S_1)(H_1/H_2) \quad \text{and} \quad K^2k^*(k^* - 1) < (S_2/S_1)(H_1/H_2), \quad \text{or}$$

$$\sqrt{\frac{(S_2/S_1)(H_1/H_2)}{k^*(k^* + 1)}} \leq K < \sqrt{\frac{(S_2/S_1)(H_1/H_2)}{k^*(k^* - 1)}}. \tag{13}$$

Define:

$$K_k = \sqrt{\frac{(S_2/S_1)(H_1/H_2)}{k(k+1)}} \qquad k = 1, 2, \ldots. \tag{14}$$

Then our result shows that k^* is the optimal k to (8), if K lies in the range as defined by (13). That is, $k = 1$ is optimal if $K_1 \leq K < \infty$, $k = 2$ is optimal if $K_2 \leq K < K_1$, and so on.

Using the requirement that $K \leq \bar{K}$, we immediately establish that a lower bound for the optimal value of k is given by:

$$\bar{K} < \sqrt{\frac{(S_2/S_1)(H_1/H_2)}{k(k-1)}}, \qquad \text{or}$$

$$k(k-1) < (S_2/S_1)(H_1/H_2)/\bar{K}^2. \tag{15}$$

The lower bound for the optimal value of k, $\underline{k}$, is then the largest integer k that satisifies (15).

Similarly, using the fact that $K \geq 1$, we can derive an upper bound for the optimal value of k, $\bar{k}$, as the smallest integer k that satisfies:

$$(S_2/S_1)(H_1/H_2) \leq k(k+1). \tag{16}$$

The result of (15) and (16) effectively limit our search for the optimal value of k to be between $\underline{k}$ and $\bar{k}$.

We now present three propositions which are useful for the development of an algorithm to solve the supplier's problem. The first proposition states that, for any k, the optimal K^* should always be smaller than the upper bound value for which this k is appropriate.

PROPOSITION 1. *For a given i,* $\underline{k} \le i \le \bar{k}$, *then* $K^*(i) < K_{i-1}$, *where* $K^*(i)$ *and* K_{i-1} *are as defined in* (11) *and* (14) *respectively.*

PROOF. By definition of $\bar{k}$ as in (16), we have, for all $i < \bar{k}$,

$$(S_2/S_1)(H_1/H_2) > i(i-1). \tag{17}$$

Now, for $i \le \bar{k}$,

$$K_{i-1}^2 - [K^*(i)]^2 = \{[S_2H_1/S_1H_2i(i-1)] - 1\}/\{1 + (i-1)H_2/H_1\},$$

which is clearly positive, in light of (17). Therefore, $K_{i-1} > K^*(i)$. Q.E.D.

The implication of Proposition 1 is significant. It shows that if one wants to determine the range (K_k, K_{k-1}), where $K^*(i)$, $\underline{k} \le i \le \bar{k}$, lies, we should always look for the values of k such that $k \ge i$. This is because:

$$\cdots < K_{i+1} < K_i < \cdots < K_2 < K_1.$$

PROPOSITION 2. *For a given i,* $\underline{k} \le i \le \bar{k}$, *suppose that* $K_j \le K^*(i) < K_{j-1}$, *for some integer j,* $j > i$. *Then k,* $k = i, i+1, i+2, \ldots, j-1$, *cannot be optimal for the supplier's problem.*

PROOF. Suppose $j > i$. For all $1 \le K < \bar{K}$ and $k = i$, the profit obtained is inferior to that of $k = i$ and $K = K^*(i)$, by definition of $K^*(i)$. However, we have also shown that for all $K_j \le K < K_{j-1}$, the optimal k to be used should be j. Hence, the solution $k = i$ and $K = K^*(i)$ must be inferior to some solution with $k = j$ and $K_j < K < K_{j-1}$. Thus, $k = i$ cannot be optimal.

Next, assume $j > i + 1$, we consider $k = i + 1$. From (11), it is clear that $K^*(i+1) < K^*(i)$. This implies that $K^*(i+1)$ will lie in the range: $K_n < K^*(i+1) < K_{n-1}$, where $n \ge j$. Now all solutions with $k = i + 1$ and any positive K are inferior to the one with $k = i + 1$ and $K = K^*(i+1)$, which is inferior to some solutions with $k = n$ and $K_n \le K < K_{n-1}$. Hence, $k = i + 1$ cannot be optimal. In a similar manner, we can show that $k = i, i+1, \ldots, j-1$ cannot be optimal. Q.E.D.

PROPOSITION 3. $1 \le K^*(\bar{k}) < K_{\bar{k}-1}$.

PROOF. $K^*(\bar{k}) < K_{\bar{k}-1}$ is a consequence of Proposition 1.

Now,

$$\begin{aligned}[K^*(\bar{k})]^2 - 1 &= \{(S_2/\bar{k}S_1) - (\bar{k}-1)H_2/H_1\}/\{1 + (\bar{k}-1)H_2/H_1\} \\ &= H_2\{(S_2H_1/S_1H_2) - \bar{k}(\bar{k}-1)\}/\{H_1\bar{k}[1 + (\bar{k}-1)H_2/H_1]\},\end{aligned}$$

which is nonnegative by (17). Q.E.D.

The results of Propositions 1 to 3 can then be used to derive the following algorithm:

(1) Start with $i = \underline{k}$.

(2) Determine j, $j \ge i$, such that $K_j \le K^*(i) < K_{j-1}$.

(3) If $i = j$, then $(i, K^*(i))$ is a candidate for the solution to the supplier's problem. Set $i = i + 1$.

If $j > i$, then set $i = j$.

(4) If $i < \bar{k}$, then go to (2).

If $i = \bar{k}$, then $(\bar{k}, K^*(\bar{k}))$ is a candidate for the solution to the supplier's problem.

(5) Stop. Determine the optimal solution from the set of candidates by comparing their respective profits.

Propositions 1 to 3 guarantee that the above procedure will derive the optimal k for the problem considered. Since $K^*(k)$ and K_k are both decreasing in k, the j's obtained via step (2) are always increasing in magnitude. Proposition 3 guarantees that there always exists a j such that $j \le \bar{k}$, using such a procedure. Hence, using the algorithm, we will never end up with j obtained in step (2) such that $j > \bar{k}$. Proposition 1 shows that such a j is also greater than or equal to i, which is, in turn, greater than or equal to $\underline{k}$. The termination step (4) is guaranteed by Proposition 3. Moreover, computational savings from using Proposition 2 can be significant. Once we find a j in step (2), all intermediate values of k that are between j and i can be discarded. As soon as j hits $\bar{k}$, the algorithm may terminate. The procedure bears a striking resemblance to the traditional models of determining economic order quantities for a buyer, given a quantity discount schedule (see Johnson and Montgomery 1974, Love 1979, and Hax and Candea 1984, and the efficient procedure suggested by Rubin et al. 1983).

Once the optimal (k, K) is obtained, the price discount $d(K)$ can be found by equation (2).

Finally, the following proposition provides the conditions under which the order-for-order policy (i.e., $k = 1$), implicitly suggested by Monahan, is optimal.

PROPOSITION 4. *If* $(S_2/H_2)/(S_1/H_1) \le 2$, *then the order-for-order policy is optimal.*

PROOF. From equation (16), it is known that the upper bound for the optimal value of k, $\bar{k}$, is the smallest integer satisfying

$$(S_2/S_1)(H_1/H_2) \le k(k+1). \qquad \text{Q.E.D.}$$

The implication of this proposition is realized in the following special case. If the set up costs and the holding costs of both the supplier and the buyer are equal, then an order-for-order (lot-for-lot) policy is optimal.

4. Extensions and Conclusion

In this paper, we have generalized the quantity discount pricing model of Monahan (1984) to: (1) explicitly incorporate constraints imposed on the amount of discount that can be offered; and (2) relax the assumption of lot-for-lot (or order-for-order) policy adopted by the supplier. These generalizations are realistic extensions to Monahan's model. The supplier would, of course, never offer a price discount so that the price of the product becomes negative or small enough so that he would incur a loss by selling the product to the buyer. In a manufacturing or external supply environment, the order-for-order or lot-for-lot policy may not be always optimal to the supplier. Our generalizations explicitly deal with these two major issues.

Analysis of the problem shows that the extent to which the supplier would offer a price discount so as to induce the buyer to order a larger quantity every time he places an order is a function of both the set up and inventory holding costs of both the supplier and the buyer. This is a more reasonable result than that of Monahan's which is independent of the holding costs of both parties. Furthermore, it is shown that Monahan's result is essentially a special case of the generalized model presented in this paper.

An algorithm based on some properties of the supplier's problem is developed to solve the supplier's joint ordering and price discount problem. Optimality of the algorithm relative to the problem studied is guaranteed by means of a series of propositions.

In this model, it is implicitly assumed that the cost for the supplier to process the buyer's orders is negligible, compared to the supplier's set up cost. However, when this is not the case, then the processing cost of the buyer's orders should be included. Let

U_2 be the cost of processing a buyer's order by the supplier, then we need to modify the supplier's net profit function, *YNP*, as

$$D_1P_1 - [\sqrt{2S_1H_1P_1D_1}(K-1)^2/2K] - (D_1S_2/kKQ_1)$$
$$- (D_1U_2/KQ_1) - (k-1)KQ_1H_2P_1/2.$$

For a given k, the $K^*(k)$ that optimizes the above expression is a slight modification of (11):

$$K^*(k) = \{[1 + (S_2/kS_1) + (U_2/S_1)]/[1 + (k-1)H_2/H_1]\}^{1/2}.$$

The search for the optimal (k, K) can proceed as before, although here the results of Propositions 1 to 3 cannot be applied directly.

Finally, we refer to Monahan (1984) for an excellent discussion on the operational and managerial issues concerned with the implementation of this kind of pricing discount in the supplier-buyer relationship.

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