JOURNAL OF FUNCTIONAL ANALYSIS 17, 227–237 (1974)

A General Rearrangement Inequality for Multiple Integrals

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Communicated by Irving Segal

Received March 21, 1974

In this paper we prove a rearrangement inequality that generalizes inequalities given in the book by Hardy, Littlewood and Pólya[1] and by Luttinger and Friedberg.[2] The inequality for an integral of a product of functions of one variable is further extended to the case of functions of several variables.

I. INTRODUCTION

Rearrangement inequalities were studied by Hardy, Littlewood and Pólya in the last chapter of their book "Inequalities." Let us start by recapitulating the definition of the symmetric decreasing rearrangement of a function, and the integral inequalities following from that definition. Our new results are contained in Theorems 1.2 and 3.4.

In the following, measure always means Lebesgue measure and is denoted by μ.

DEFINITION 1.1. Let f be a nonnegative measurable function on $\mathbf{R}$,

* Work partially supported by National Science Foundation Grant GP-16147 A#1.

[§] Work partially supported by National Science Foundation Grant GP-31674 X.

[†] Work partially supported by a grant from the National Science Foundation.

let $K_y^f = \{x \mid f(x) \geqslant y\}$ and let $M_y^f = \mu(K_y^f)$. Assume that $M_a^f < \infty$ for some $a < \infty$. If f^* is another function on $\mathbf{R}$ with the same properties as f and, additionally,

$$\text{(a)} \quad f^*(x) = f^*(-x), \ \forall x,$$

$$\text{(b)} \quad 0 < x_1 < x_2 \Rightarrow f^*(x_2) \leqslant f^*(x_1),$$

$$\text{(c)} \quad M_y^{f^*} = M_y^f, \quad \forall y > 0,$$

then f^* is called a *symmetric decreasing rearrangement* of f.

Remarks. (1) If g and h are two symmetric decreasing rearrangements of f, then

$$g(x) = h(x) \quad \text{a.e.}$$

(2) If χ is the characteristic function of a measurable set, we can define $\chi^*(x) = 1$ if $2 \mid x \mid \leqslant \int \chi$ and $\chi^*(x) = 0$, otherwise. For a general function f, define $\chi_y(x) = 1$ if $f(x) \geqslant y$ and $\chi_y(x) = 0$, otherwise. Then

$$f(x) = \int_0^\infty dy \chi_y(x),$$

and

$$f^*(x) = \int_0^\infty dy \chi_y^*(x)$$

is a symmetric decreasing rearrangement of f. The fact that $M_a^f < \infty$ implies that $f^*(x) < \infty$, $\forall x \neq 0$.

(3) In the following theorems we shall always be dealing with integrals. Consequently, by remark (1), f^* is unique for our purposes. Trivially, $f \in L^1(\mathbf{R})$ iff $f^* \in L^1(\mathbf{R})$ and $\int f = \int f^*$.

The inequalities to be found in [1] are

$$\int_{\mathbf{R}} dx f(x)\, g(x) \leqslant \int_{\mathbf{R}} dx f^*(x)\, g^*(x);$$

$$\int_{\mathbf{R}^2} dx_1 dx_2 f(x_1)\, g(x_2)\, h(x_1 - x_2) \leqslant \int_{\mathbf{R}^2} dx_1 dx_2 f^*(x_1)\, g^*(x_2)\, h^*(x_1 - x_2),$$

the latter being due to Riesz [3].

A generalization due to Luttinger and Friedberg [2] reads

$$\int_{\mathbf{R}^n} d^n x \prod_{j=1}^{n} f_j(x_j)\, h_j(x_j - x_{j+1}) \leqslant \int_{\mathbf{R}^n} d^n x \prod_{j=1}^{n} f_j^*(x_j)\, h_j^*(x_j - x_{j+1}),$$

where $x_{n+1} \equiv x_1$. This formula was derived for the purpose of physical applications (inequalities for Green's functions, Luttinger [4]).

In the present paper we give a further generalization, one which was already conjectured in [2].

THEOREM 1.2. *Let f_j, $1 \leqslant j \leqslant k$, be nonnegative measurable functions on* $\mathbf{R}$, *and let a_{jm}, $1 \leqslant j \leqslant k$, $1 \leqslant m \leqslant n$, be real numbers. Then*

$$\int_{\mathbf{R}^n} d^n x \prod_{j=1}^{k} f_j \left(\sum_{m=1}^{n} a_{jm} x_m\right) \leqslant \int_{\mathbf{R}^n} d^n x \prod_{j=1}^{k} f_j^* \left(\sum_{m=1}^{n} a_{jm} x_m\right).$$

Remark. Theorem 1.2 is nontrivial only for $k > n$. If $k < n$, both integrals diverge. If $k = n$ and $\det | a_{jm} | = 0$, both integrals diverge. If $k = n$ and $\det | a_{jm} | \neq 0$, equality holds (change variables to $y_j = \sum_{m=1}^{n} a_{jm} x_m$ and then use the fact that $\int f_j = \int f_j^*$).

A proof of Theorem 1.2 is given in Section 2. An important tool is Brunn's part of the Brunn–Minkowski theorem, which we recall here (see e.g., [5] Section 11.48). Note that every convex set in $\mathbf{R}^n$ is measurable.

LEMMA 1.3. *Let C be a convex set in $\mathbf{R}^{n+1}$, let $\varphi \in \mathbf{R}^{n+1}$, and let $V(t)$ be the family of planes $\langle \varphi, x \rangle = t$, $-\infty < t < \infty$.* Let *$S(t)$ be the n-dimensional volume of the convex set $V(t) \cap C$. Then $S(t)^{1/n}$ is a concave function of t in the interval where $S(t) > 0$.*

COROLLARY 1.4. *Let C, φ and $S(t)$ be as in Lemma* 1.3 *and, in addition, let C be balanced (i.e., $x \in C \Rightarrow -x \in C$). Then $S(t) = S(-t)$ and $S(t_2) \leqslant S(t_1)$ for $t_2 \geqslant t_1 \geqslant 0$.*

In Section 3 we generalize Theorem 1.2 to the Schwarz symmetrization (Definition 3.3) of functions of several variables. An auxiliary lemma that we need for this purpose is given in the Appendix.

II. PROOF OF THEOREM 1.2

Although in general $f \to f^*$ is not linear, by Remark (2) following Definition 1.1 it is sufficient to assume that each f_j is the characteristic function of some measurable set. By standard approximation arguments we may assume this set to be a finite union of disjoint compact intervals (cf. [1], Section 10.14).

We start by assuming that each f_j is the characteristic function of one interval.

LEMMA 2.1. *Let f_j, $1 \leqslant j \leqslant k$, be the characteristic functions of the intervals*

$$b_j - c_j \leqslant x \leqslant b_j + c_j,$$

and define

$$f_j(x \mid t) = f_j(x + b_j t).$$

Then

$$I(t) = \int_{\mathbf{R}^n} d^n x \prod_{j=1}^{k} f_j \left(\sum_{m=1}^{n} a_{jm} x_m \mid t \right)$$

is a nondecreasing function of $t \in [0, 1]$.

Remark. Note, that $f_j(x \mid 0) = f_j(x)$ and $f_j(x \mid 1) = f_j^*(x)$, so Lemma 2.1 includes a special case of Theorem 1.2.

Proof of Lemma 2.1. $I(t)$ is the volume of the intersection of the k strips

$$S_j = \left\{ x \in \mathbf{R}^n \mid b_j(1 - t) - c_j \leqslant \sum_{m=1}^{n} a_{jm} x_m \leqslant b_j(1 - t) + c_j \right\}.$$

In $\mathbf{R}^{n+1}$, consider the set

$$C = \bigcap_{1 \leqslant j \leqslant k} \left\{ x \in \mathbf{R}^{n+1} \mid -c_j \leqslant \sum_{m=1}^{n} a_{jm} x_m - b_j x_{n+1} \leqslant c_j \right\}.$$

$I(t)$ is the volume of the intersection of C with the plane $x_{n+1} = 1 - t$. Since C is convex and balanced, $I(t)$ is nondecreasing for $t \in [0, 1]$ by Corollary 1.4. Q.E.D.

We now conclude the proof of Theorem 1.2 with the following lemma.

LEMMA 2.2. *Theorem 1.2 holds under the restriction, that each f_j is the characteristic function of a finite union of disjoint compact intervals.*

Proof. Let f_j be the characteristic function of n_j intervals. We prove the lemma by induction on $N = \{n_1, n_2, \ldots, n_k\}$, with fixed k.

We say that $M \prec N$ if $m_j \leqslant n_j$, $1 \leqslant j \leqslant k$, and $m_i < n_i$ for some i.

Lemma 2.2 is true for $N = \{1, 1, ..., 1\}$ by Lemma 2.1. Now assume that Lemma 2.2 is true for all $M \prec N$.

Let $f_j(x)$ be the characteristic function of

$$\bigcup_{1 \leqslant p \leqslant n_j} \{x \in \mathbf{R} \mid b_{jp} - c_{jp} \leqslant x \leqslant b_{jp} + c_{jp}\},$$

with

$$b_{jp} + c_{jp} < b_{j,p+1} - c_{j,p+1}, \; 1 \leqslant p \leqslant n_j - 1, \; 1 \leqslant j \leqslant k.$$

Further define $f_j(x \mid t)$ to be the characteristic function of

$$\bigcup_{1 \leqslant p \leqslant n_j} \{x \in \mathbf{R} \mid b_{jp}(1 - t) - c_{jp} \leqslant x \leqslant b_{jp}(1 - t) + c_{jp}\}$$

for $0 \leqslant t \leqslant \tau$, where

$$\tau = \min_{j,p}[1 - (b_{j,p+1} - b_{jp})^{-1} (c_{j,p+1} + c_{jp})] > 0.$$

For $0 \leqslant t < \tau$, the intervals belonging to each function f_j remain disjoint; at $t = \tau$ at least two intervals coalesce for some j.

Since each f_j is a positive sum of characteristic functions of single intervals of the type stated in the hypothesis of Lemma 2.1, we can apply that lemma interval by interval and find

$$\int_{\mathbf{R}^n} d^n x \prod_{j=1}^{k} f_j \left(\sum_{m=1}^{n} a_{jm} x_m \right) \leqslant \int_{\mathbf{R}^n} d^n x \prod_{j=1}^{k} f_j \left(\sum_{m=1}^{n} a_{jm} x_m \mid \tau \right).$$

At $t = \tau$, the family of functions $\{f_j(x \mid \tau\}$ satisfies the hypothesis of Lemma 2.2, except that N has been reduced to some $M \prec N$. Therefore, by assumption

$$\int_{\mathbf{R}^n} d^n x \prod_{j=1}^{k} f_j \left(\sum_{m=1}^{n} a_{jm} x_m \mid \tau \right) \leqslant \int_{\mathbf{R}^n} d^n x \prod_{j=1}^{k} f_j^* \left(\sum_{m=1}^{n} a_{jm} x_m \right),$$

because $f_j(\cdot \mid \tau)$ and $f_j(\cdot)$ have the same symmetric decreasing rearrangement. This proves Lemma 2.2 and at the same time Theorem 1.2

III. Generalization to Functions of Several Variables

In this section we indicate how to generalize Theorem 1.2 to functions of several variables (Lemma 3.2 and Theorem 3.4). The intuitive idea was given in [4], *p.* 1450.

Let f be a nonnegative, measurable function on $\mathbf{R}^p$, and let V be a $p-1$ dimensional plane through the origin of $\mathbf{R}^p$. Choose an orthogonal coordinate system in $\mathbf{R}^p$ such that the x^1-axis is perpendicular to V.

DEFINITION 3.1. A nonnegative, measurable function $f^*(x \mid V)$ on $\mathbf{R}^p$ is called a Steiner-symmetrization with respect to V of the function $f(x)$, if $f^*(x^1, x^2,..., x^p)$ is a symmetric decreasing rearrangement with respect to x^1 of $f(x^1, x^2,..., x^p)$ for each fixed $x^2,..., x^p$.

Remark. The notion of Steiner symmetrization is usually reserved for sets; for any $y > 0$, the set $\{x \in \mathbf{R}^p \mid f^*(x \mid V) \geqslant y\}$ is a Steiner symmetrization with respect to V of the set $\{x \in \mathbf{R}^p \mid f(x) \geqslant y\}$ (see e.g., Pólya and Szegö [6], Note A).

LEMMA 3.2. *Let $f_j(x)$, $1 \leqslant j \leqslant k$, be nonnegative measurable functions on $\mathbf{R}^p$, let a_{jm}, $1 \leqslant j \leqslant k$, $1 \leqslant m \leqslant n$, be real numbers, and let V be any plane through the origin of $\mathbf{R}^p$. Then*

$$\int_{\mathbf{R}^{np}} d^{np}x \prod_{j=1}^{k} f_j\left(\sum_{m=1}^{n} a_{jm}x_m\right) \leqslant \int_{\mathbf{R}^{np}} d^{np}x \prod_{j=1}^{k} f_j^*\left(\sum_{m=1}^{n} a_{jm}x_m \mid V\right),$$

where $\mathbf{R}^{np} \ni x = (x_1,..., x_n)$, $x_m \in \mathbf{R}^p$.

Proof. Choose appropriate orthogonal coordinates in $\mathbf{R}^p \ni x = (x^1,..., x^p)$ as above, so that the x^1-axis is orthogonal to V. Then, by Theorem 2.1, the inequality already holds for the integration over $x_1^1, x_2^1,..., x_n^1$ for any fixed x_m^q, $1 \leqslant m \leqslant n$, $2 \leqslant q \leqslant p$. Q.E.D.

DEFINITION 3.3. Let f be a nonnegative measurable function on $\mathbf{R}^p$, let $K_y^f = \{x \mid f(x) \geqslant y\}$ and let $M_y^f = \mu(K_y^f)$. Assume that $M_a^f < \infty$ for some $a < \infty$. If f^{**} is another function on $\mathbf{R}^p$ with the same properties as f and, additionally,

(a) $f^{**}(x_1) = f^{**}(x_2)$ when $|x_1| = |x_2|$,

(b) $0 < |x_1| < |x_2| \Rightarrow f^{**}(x_2) \leqslant f^{**}(x_1)$,

(c) $M_y^{f^{**}} = M_y^f$, $\forall y > 0$,

then f^{**} is called a Schwarz symmetrization of f.

Remarks. (1) The remarks after Definition 1.1 apply, mutatis mutandis, to Schwarz symmetrization.

(2) The notion of Schwarz symmetrization is usually reserved for sets; the set in $\mathbf{R}^{p+1}$ under the graph of $y = f^{**}(x)$ is the Schwarz symmetrization with respect to the y-axis of the set under the graph of $y = f(x)$ (see [5], Note A).

It is intuitively clear, that the Schwarz symmetrization can be obtained as the $L^1(\mathbf{R}^p)$ limit of a sequence of Steiner symmetrizations with respect to different planes. That fact will be proved in the Appendix for the characteristic function of a bounded measurable set (Lemma A1). For the moment we use it, together with Lemma 3.2 and the remarks at the beginning of Section 2, to conclude our main theorem, which is the following.

THEOREM 3.4. *Under the assumptions of Lemma* 3.2,

$$\int_{\mathbf{R}^{np}} d^{np}x \prod_{j=1}^{k} f_j \left(\sum_{m=1}^{n} a_{jm} x_m \right) \leqslant \int_{\mathbf{R}^{np}} d^{np}x \prod_{j=1}^{k} f_j^{**} \left(\sum_{m=1}^{n} a_{jm} x_m \right).$$

APPENDIX

We give the lemma that suffices to establish Theorem 3.4. For two sets A and B, $A \Delta B \equiv (A \cup B) \backslash (A \cap B)$. μ_p denotes Lebesgue measure in $\mathbf{R}^p$.

LEMMA A.1. *Let K be a bounded measurable set in $\mathbf{R}^p$, and let S be the ball centered at the origin with $\mu_p(S) = \mu_p(K)$. Then there exists a sequence of sets K_n, where $K_0 = K$ and where K_{n+1} is obtained from K_n by Steiner symmetrization with respect to some $(p-1)$-dimensional subspace of $\mathbf{R}^p$, such that*

$$\lim_{n \to \infty} \mu_p(K_n \Delta S) = 0.$$

Remark. There exist various theorems stating the convergence of K_n to S in the Hausdorff metric ([7], Section 21 for compact convex sets; [8], Section 4.5.3 and [9], Section 2.10.31 for general compact sets).

Let us first give a precise definition of the Steiner symmetrization for arbitrary measurable sets (cf. the Remark following Definition 3.1).

DEFINITION A.2. Let K be a bounded measurable set in $\mathbf{R}^p$, and let V be a $(p-1)$-dimensional subspace of $\mathbf{R}^p$. Then the set K_V^* is called a Steiner symmetrization of K with respect to V, if, for every straight line L perpendicular to V with $K \cap L$ measurable in $\mathbf{R}$, $K_V{}^* \cap L$ is a segment (open or closed) with center in V and

$$\mu_1(K_V{}^* \cap L) = \mu_1(K \cap L).$$

Remarks. (1) Let K be open (resp. closed) and take for $K_V{}^* \cap L$ in Definition A2 the open (resp. closed) segments. Then $K_V{}^*$ is open resp. closed).

To prove this, choose coordinates $x = (x^1, \ldots, x^p) \in \mathbf{R}^p$ with x^1 in the direction orthogonal to V. Let χ_K be the characteristic function of K. Then the statement is true if the function $\mathbf{R}^{p-1} \ni y \to \int dx^1\, \chi_K(x^1, y)$ is lower (resp. upper) semicontinuous. But this follows from the fact that $\chi_K(x^1, y)$ is lower (resp. upper) semicontinuous in $\mathbf{R}^p$.

(2) For arbitrary measurable K, all Steiner symmetrizations are measurable and satisfy (Fubini's theorem)

$$\mu_p(K_V{}^*) = \mu_p(K).$$

Two Steiner symmetrizations can only differ by a set of measure zero. All this is readily seen by sandwiching K between closed sets from within and open sets from without.

(3) If K and M are measurable sets, Lemma 3.2 gives that

$$\mu_p(K_V{}^* \cap M_V{}^*) \geqslant \mu_p(K \cap M),$$

and therefore

$$\mu_p(K_V{}^* \,\Delta\, M_V{}^*) \leqslant \mu_p(K \,\Delta\, M).$$

In particular, if K and M differ only by a set of measure zero, so do $K_V{}^*$ and $M_V{}^*$.

(4) In view of Remarks 2 and 3, we shall further speak of *the* Steiner symmetrization of a measurable set, which in fact associates with each equivalence class of measurable sets a unique equivalence class of measurable sets.

PROPOSITION A.3. *Let K and S be as in Lemma* A.1. *Then*

$$\mu_p(K_V{}^* \,\Delta\, S) \leqslant \mu_p(K \,\Delta\, S)$$

and the equality holds for all subspaces V iff $K = S$.

Proof. The $\leqslant$ inequality holds by Remark 3 above, since $S_V{}^* = S$.

Denote by $L(v)$ the straight line perpendicular to V through $v \in V$. Let $K(v) = K \cap L(v)$, and let $\pi_V(K)$ be the projection of K on V,

$$\pi_V(K) = \{v \in V \mid \mu_1(K(v)) > 0\}.$$

Now let $K \neq S$ so that $\mu_p(K \backslash S) = \mu_p(S \backslash K) > 0$. It can be shown by a tedious but trivial argument that there exists a subspace V such that $P = \pi_V(K \backslash S) \cap \pi_V(S \backslash K)$ has positive μ_{p-1} measure. If $v \in P$, neither $K(v) \subset S(v)$ nor $S(v) \subset K(v)$; therefore

$$\mu_1(K_V{}^*(v) \mathbin{\Delta} S(v)) = |\, \mu_1(K(v)) - \mu_1(S(v))| < \mu_1(K(v) \mathbin{\Delta} S(v)).$$

for all $v \in P$. Because, generally, for all $v \in V$

$$\mu_1(K_V{}^*(v) \mathbin{\Delta} S(v)) \leqslant \mu_1(K(v) \mathbin{\Delta} S(v)),$$

we have for the particular subspace V under consideration

$$\mu_p(K_V{}^* \mathbin{\Delta} S) < \mu_p(K \mathbin{\Delta} S).$$

This proves Proposition A.3.

Let us now specify the sequence of sets in Lemma A.1. Given K_n, choose a subspace V_1, such that

$$\mu_p(K^*_{nV_1} \mathbin{\Delta} S) < \inf_V \mu_p(K^*_{nV} \mathbin{\Delta} S) + n^{-1}$$

Then construct K_{n+1} from K_n by p consecutive Steiner symmetrizations with respect to a set of $p-1$ dimensional subspaces $V_1, V_2, \ldots, V_p$ (beginning with V_1 specified above) whose orthogonal complements are pairwise orthogonal. In that way,

$$\mu_p(K_{n+1} \mathbin{\Delta} S) < \mu_p(K^*_{nW} \mathbin{\Delta} S) + n^{-1}$$

for all n and for all subspaces W.

PROPOSITION A.4. *There exist a subsequence K_{n_j} and a measurable set M such that*

$$\lim_{j\to\infty} \mu_p(K_{n_j} \mathbin{\Delta} M) = 0$$

Proof. Express a point $x \in \mathbf{R}^p$ in coordinates $(x^1, x^2, \ldots, x^p)$ corresponding to the planes used to construct K_n. Then, it is not difficult to show that for $n > 0$ (i.e., after the first set of p orthogonal symmetrizations), $x \in K_n$ implies $y \in K_n$ if $|\, y^m \,| \leqslant |\, x^m \,|$, $m = 1, \ldots, p$. Therefore, if χ_n is the characteristic function of K_n

$$\int_{\mathbf{R}} dx^m \,|\, \chi_n(x^1, \ldots, x^m + y^m, \ldots, x^p) - \chi_n(x^1, \ldots, x^m, \ldots, x^p)| \leqslant 2 \,|\, y^m \,|.$$

Note that by assumption K is contained in some ball B of radius R centered at the origin; then also $K_n \subset B$. This implies that

$$\int_{\mathbf{R}^p} d^p x \mid \chi_n(x+y) - \chi_n(x)| \leqslant 2(2R)^{p-1} \sum_{m=1}^{p} | y^m |.$$

In other words

$$\lim_{y \to 0} \int_{\mathbf{R}^p} d^p x \mid \chi_n(x+y) - \chi_n(x)| = 0$$

uniformly in n. Hence the family of functions $\{\chi_n\}$ is conditionally compact in $L^1(\mathbf{R}^p)$ (Dunford and Schwartz [10], Theorem IV, 8.21). Q.E.D.

Propositions A.3. and A.4. immediately give the following.

COROLLARY A.5. *$\mu_p(K_n \varDelta S)$ decreases monotonously to $\mu_p(M \varDelta S)$.*

Let us now conclude the proof of Lemma A.1. Assume that $M \neq S$; we shall show that this leads to a contradiction.

Let $\mu_p(M \varDelta S) = \delta > 0$. Then there exist a $p-1$ dimensional subspace W and an $\epsilon > 0$ such that

$$\mu_p(M_W{}^* \varDelta S) = \delta - \epsilon.$$

by Proposition A.3. Also

$$\lim_{j \to \infty} \mu_p(K^*_{n_j W} \varDelta M_W{}^*) = 0,$$

so

$$\lim_{j \to \infty} \mu_p(K^*_{n_j W} \varDelta S) = \delta - \epsilon.$$

Then there exists an n_k such that $n_k > 2\epsilon^{-1}$ and

$$\mu_p(K^*_{n_k W} \varDelta S) < \delta - \epsilon/2.$$

But by the construction of the sequence K_n,

$$\mu_p(K_{n_k+1} \varDelta S) < \mu_p(K^*_{n_k W} \varDelta S) + n_k^{-1} < \delta,$$

which contradicts Corollary A.5.

Thus we find that $M = S$; then by Corollary A.5., $\mu_p(K_n \varDelta S)$ decreases monotonously to zero. This proves Lemma A.1.

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