ANOTHER PROOF FOR JACKSON THEOREM

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Jackson Theorem is an example of so called *direct theorems* of approximation thory. We will prove two variants of the Jackson theorem: for approximation with trigonometric polynomials and approximation with algebraic polynomials.

Theorem 0.1 (Jackson theorem). *If $f \in W^1_\infty(I)$ then there exists trigonometric polynomial $T(x)$ of degree n such that*

$$(0.1) \qquad\qquad E_n(f) \leq \frac{c}{n}\|f'\|_\infty.$$

Before we move to the proof of this theorem, let us first introduce some notation and state and prove some results that are needed in the proof of the theorem. Define $K_n(t)$ to be the Jackson kernel:

$$K_n(t) := \lambda_n^{-1} \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^4,$$

where λ_n is defined as

$$\lambda_n^{-1} = \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^4 dt, \quad \text{and hence} \int_{-\pi}^{\pi} K_n(t) \, dt = 1.$$

We now define for a given function $f \in L_\infty(-\pi, \pi)$:

$$\mathcal{J}_n(f; x) = \int_{-\pi}^{\pi} f(x + t) K_n(t) \, dt.$$

Below we prove that $\mathcal{J}_n(f; x)$ is in fact a trigonometric polynomial, which gives the bound stated in the theorem. In that sense, the proof here is a constructive proof, explicitly giving a trigonometric polynomial $T(x)$ for which

$$\|f - T\|_\infty \leq \frac{c}{n}\|f'\|_\infty.$$

If $g(t)$ is an integrable 2π periodic function, and a is a real number, by change of variables $t = \tau + 2\pi$ we have that

$$\int_{-\pi}^{-\pi+a} g(\tau) \, d\tau = \int_{\pi}^{\pi+a} g(t - 2\pi) \, dt = \int_{\pi}^{\pi+a} g(t) \, dt$$

Date: January 30, 2008.

Hence,

$$\int_{-\pi}^{\pi} g(t)\ dt\ =\ \int_{-\pi}^{-\pi+a} g(t)\ dt + \int_{-\pi+a}^{\pi} g(t)\ dt$$

$$=\ \int_{\pi}^{\pi+a} g(t)\ dt + \int_{-\pi+a}^{\pi} g(t)\ dt = \int_{-\pi+a}^{\pi+a} g(t)\ dt.$$

In summary,

(0.2)
$$\int_{-\pi+a}^{\pi+a} g(t)\ dt = \int_{-\pi}^{\pi} g(t)\ dt$$

Lemma 0.2. *The following idenatities hold*

(0.3)
$$\sin\frac{x}{2} + \sin\left(\frac{3x}{2}\right) + \ldots + \sin\left(\frac{2n-1}{2}x\right) = \frac{\sin^2\left(\frac{n}{2}x\right)}{\sin(\frac{x}{2})}$$

$$\sin(\frac{x}{2})[-1 + 2\sum_{k=0}^{m-1}\cos(kx)] = \sin\left(\frac{2m-1}{2}x\right)$$

Proof. Denote $b_l = \cos(lx)$, and recall that for any α and β we have

$$2\sin\alpha\sin\beta = \cos(\beta-\alpha) - \cos(\alpha+\beta).$$

Hence

$$2\sin(\frac{x}{2})\sin\left(\frac{2l-1}{2}x\right) = b_{l-1} - b_l.$$

The first identity is a consequence of the following:

$$\sin(\frac{x}{2})\sum_{l=1}^{n}\sin\left(\frac{2l-1}{2}x\right) = \sum_{l=1}^{n}\sin(\frac{x}{2})\sin\left(\frac{2l-1}{2}x\right)$$

$$=\ \frac{1}{2}\sum_{l=1}^{n}b_{l-1} - b_l = \frac{1}{2}(b_0 - b_n)$$

$$=\ \frac{1}{2}(1 - \cos(nx)) = \sin^2\left(\frac{n}{2}x\right).$$

To prove the second identity in (0.3), we set $a_l = \sin\left(\frac{2l-1}{2}x\right)$, and recall that

$$2\cos(lx)\sin(\frac{x}{2}) = \sin\left(\frac{2l+1}{2}x\right) - \sin\left(\frac{2l-1}{2}x\right) = a_{l+1} - a_l.$$

We then have

$$\sin(\frac{x}{2})\left(-1 + 2\sum_{k=0}^{m-1}\cos(kx)\right) = \sin(\frac{x}{2}) + \sum_{k=1}^{m-1} 2\sin(\frac{x}{2})\cos(kx)$$

$$= a_1 + \sum_{k=1}^{m-1}(a_{k+1} - a_k) = a_m,$$

which completes the proof. □

We now use the above identities to show that $K_n(t)$ is a trigonometric polynomial.

Lemma 0.3. *If f is a 2π-periodic function and $f \in C(-\pi, \pi)$ then $\mathcal{J}_n(f; x)$ is a trigono-metric polynomial of order $(2n - 2)$. If f is an even function than $\mathcal{J}_n(f; x)$ is also even.*

Proof. Denote

$$Z_n(t) = \left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^2,$$

we only need to show that Z_n is a trigonometric polynomial of order $(n-1)$. Indeed, since $K_n(t)$ is proportional to $[Z_n(t)]^2$, this will imply that $K_n(t)$ is a trigonometric polynomial of order $(2n - 2)$. Applying the first identity from (0.3), for $Z_n(t)$ we get

$$Z_n(t) = \frac{\sin^2(nt/2)}{\sin^2(t/2)} = \frac{1}{\sin(t/2)}\sum_{m=1}^{n}\sin\left(\frac{2m-1}{2}t\right).$$

We now apply the second identity in (0.3) to each term in the sum, to get that

$$Z_n(t) = \frac{1}{\sin(t/2)}\sum_{m=1}^{n}\left[-\sin(t/2) + 2\sin(t/2)\sum_{k=0}^{m-1}\cos(kt)\right]$$

$$= \sum_{m=1}^{n}(-1) + 2\sum_{m=1}^{n}\sin(t/2)\sum_{k=0}^{m-1}\cos(kt)$$

$$= -n + 2\sum_{k=0}^{n-1}\cos(kt)\sum_{m=k+1}^{n}1$$

$$= -n + 2\sum_{k=0}^{n-1}(n - k)\cos(kt).$$

Clearly this shows that $Z_n(t)$ is a trigonometric polynomial of order $(n - 1)$ and hence $K_n(t)$ is a trigonometric polynomial of order $(2n - 2)$. Note also that $K_n(t)$ is even and

is a sum of cosine functions only. This is easy to see by squaring $Z_n(t)$ and using simple trigonometric identities.

(0.4) $$\mathcal{J}_n(f;x) = \int_{-\pi}^{\pi} f(x+\tau)K_n(\tau)\,d\tau = \int_{-\pi+x}^{\pi+x} f(t)K_n(t-x)\,dt.$$

As a consequence, we obtain that

$$\begin{aligned}
K_n(t-x) &= \sum_{k=0}^{2n-2} \alpha_k \cos(k(t-x)) \\
&= \sum_{k=0}^{2n-2} \alpha_k \cos(kx)\cos(kt) + \sum_{k=1}^{2n-2} \alpha_k \sin(kx)\sin(kt).
\end{aligned}$$

We then substitute the above expresion in (0.4) and use the fact that $f(\cdot)$, $\sin(kt)$ and $\cos(kt)$ are 2π-periodic functions. From (0.2) we then have:

$$\int_{-\pi+x}^{\pi+x} f(t)K_n(t-x)\,dt = \sum_{k=0}^{2n-2} \alpha_k \xi_k \cos(kx) + \sum_{k=1}^{2n-2} \alpha_k \eta_k \sin(kx),$$

where

$$\xi_k = \int_{-\pi+x}^{\pi+x} f(t)\cos(kt)\,dt = \int_{-\pi}^{\pi} f(t)\cos(kt)\,dt,$$

and

$$\eta_k = \int_{-\pi+x}^{\pi+x} f(t)\sin(kt)\,dt = \int_{-\pi}^{\pi} f(t)\sin(kt)\,dt.$$

Hence, from (0.4) it follows that $\mathcal{J}_n(f;x)$ takes the form

$$\begin{aligned}
\mathcal{J}_n(f;x) &= \int_{-\pi}^{\pi} f(t)K_n(t-x)\,dt \\
&= \sum_{k=0}^{2n-2} \alpha_k \xi_k \cos(kx) + \sum_{k=1}^{2n-2} \alpha_k \eta_k \sin(kx).
\end{aligned}$$

and clearly is a trigonometric polynomial of the same order as $K_n(t)$.

To conclude the proof of the lemma, it remains to show that if $f(\cdot)$ is even function, then $\mathcal{J}_n(f;x)$ is also even. Since in such case $f(t)\sin(kt)$ is an odd function, we obtain that $\eta_k = 0$, for $k = 1, \ldots, (2n-2)$, and hence $\mathcal{J}_n(f;x)$ is even function. $\square$

We now turn our attention to the proof of several inequalities from qhich the Jackson theorem will follow.

Lemma 0.4. *The following inequalities hold:*

(0.5) $$t/\pi \leq \sin(t/2) \leq t/2, \quad \text{for all} \quad t \in [0, \pi].$$

(0.6) $$\lambda_n \geq \frac{32}{\pi^3} n^3.$$

Proof. (1) To prove that $\sin(t/2) \geq t/\pi$ for $t \in [0, \pi]$, we set $g(t) = \sin(t/2) - t/\pi$. Clearly we have $g'' = -\frac{1}{4}\sin(t/2) \leq 0$. Hence $g(t)$ can only have maximum inside the interval $[0, \pi]$ and the minimum value is achieved at one of the end points. Direct calculation shows that $g(0) = g(\pi) = 0$ and hence $g(t) \geq 0$ for all $t \in [0, \pi]$.

(2) To prove the left side inequality in (0.5), we now set $g(t) = t/2 - \sin(t/2)$. It is easy to see that $g'(t) \geq 0$ for all $t \in [0, \pi]$ and hence $g(t) \geq g(0) = 0$, which concludes the proof of (0.5).

(3) The inequality (0.6) is shown as follows:

$$\begin{aligned}
\lambda_n &= 2\int_0^\pi \left[\frac{\sin(nt/2)}{\sin(t/2)}\right]^4 dt \geq 2\int_0^{\frac{\pi}{n}} \left[\frac{\sin(nt/2)}{\sin(t/2)}\right]^4 dt \\
&\geq 2\int_0^{\frac{\pi}{n}} \left[\frac{(nt/\pi)}{(t/2)}\right]^4 dt = 2\frac{\pi}{n}\frac{2^4 n^4}{\pi^4} = \frac{32}{\pi^3}n^3.
\end{aligned}$$

$\square$

We will also need the following result.

Lemma 0.5. *There exists an absolute constant c, such that:*

$$\int_0^\pi t K_n(t)\, dt \leq \frac{c}{n}$$

Proof. Using Lemma 0.4, we obtain

$$\begin{aligned}
\int_0^\pi t K_n(t)\, dt &= \sum_{k=0}^{n-1}\int_{k\pi/n}^{(k+1)\pi/n} t K_n(t)\, dt \\
&\leq \lambda_n^{-1}\left[\int_0^{\pi/n} t\left(\frac{nt/2}{t/\pi}\right)^4 dt + \sum_{k=1}^{n-1}\int_{k\pi/n}^{(k+1)\pi/n} \frac{(k+1)\pi}{n}\left(\frac{n}{k}\right)^4 dt\right] \\
&= \frac{n^2}{\lambda_n}\left[\frac{\pi^6}{32} + \sum_{k=1}^{n-1}\pi^2\frac{k+1}{k^4}\right] \\
&\leq \frac{n^2}{\lambda_n}\left[\frac{\pi^6}{32} + \pi^2\sum_{k=1}^{\infty}\left(\frac{1}{k^3} + \frac{1}{k^4}\right)\right] \leq cn^{-1}
\end{aligned}$$

$\square$

We are now ready to prove Theorem 0.1.

Proof of Theorem 0.1. Recall that $\int_{-\pi}^{\pi} K_n(t) \, dt = 1$, and hence, for all $x \in [-\pi, \pi]$ we have

$$\mathcal{J}_n(f; x) - f(x) = \int_{-\pi}^{\pi} (f(x+t) - f(x)) K_n(t) \, dt$$

Note that $K_n(t)$ is non-negative and even and by Lemma 0.5 we have:

$$
\begin{aligned}
|\mathcal{J}_n(f; x) - f(x)| &= \left| \int_{-\pi}^{\pi} (f(x+t) - f(x)) K_n(t) \, dt \right| \\
&\leq \int_{-\pi}^{\pi} |f(x+t) - f(x)| K_n(t) \, dt = \int_{-\pi}^{\pi} \left| \int_{x}^{x+t} f'(\tau) \, d\tau \right| K_n(t) \, dt \\
&\leq \int_{-\pi}^{\pi} \left| \int_{x}^{x+t} |f'(\tau)| \, d\tau \right| K_n(t) \, dt \leq \|f'\|_\infty \int_{-\pi}^{\pi} |t| K_n(t) \, dt \\
&= 2\|f'\|_\infty \int_{0}^{\pi} t K_n(t) \, dt \leq 2cn^{-1} \|f'\|_\infty.
\end{aligned}
$$

Setting $N = 2(n-1)$, and recalling that $\mathcal{J}_n$ is a trigonometric polynomial of order $2(n-1)$, for the error of best approximation we have that:

$$E_N(f) \leq \frac{4c}{N+2} \|f'\|_\infty,$$

and the proof is complete. $\qquad\square$

Remark 0.6. We remark that if f is even function than, since $J_n(f; x)$ is also even, we can write

(0.7) $$\inf_{T \in T_{n,\text{even}}} \|f - T\|_\infty \leq cn^{-1} \|f'\|_\infty.$$

The inequality just stated in the remark gives the following theorem for approximation with algebraic polynomials.

Theorem 0.7. *Let $f \in W_\infty^1(-1, 1)$. Then*

$$E_n(f) \leq cn^{-1} \|f'\|_\infty$$

Proof. Set $g(t) = f(\cos t)$, for $t \in [-\pi, \pi]$. Note that $g(t)$ is even and

$$g'(t) = -f'(\cos t) \sin t, \quad \text{and hence} \quad \|g'\|_{\infty,(-\pi,\pi)} \leq \|f'\|_{\infty,(-1,1)}.$$

From Lemma 0.3 we may conclude that $\mathcal{J}_n(g; x)$ is even. Define now

$$\tilde{\mathcal{J}}_n(f; x) := \mathcal{J}_n(g; \arccos(x)).$$

Note that $\tilde{\mathcal{J}}_n(f; x)$ is an algebraic polynomial of degree $\leq 2(n-1)$. We then have

$$
\begin{aligned}
\|f - \tilde{\mathcal{J}}_n(f)\|_{\infty,(-1,1)} &= \|g - \mathcal{J}_n(g)\|_{\infty,(0,\pi)} \\
&= \|g - \mathcal{J}_n(g)\|_{\infty,(-\pi,\pi)} \leq cn^{-1}\|g'\|_{\infty,(-\pi,\pi)} \\
&\leq cn^{-1}\|f'\|_{\infty,(-1,1)}.
\end{aligned}
$$

$\square$

If the function f is smoother, then we have the following theorem:

Theorem 0.8. *Let $f \in C^k$ is 2π periodic function. Then, the following estimate holds:*

$$
E_n(f) \leq \frac{c(k)}{n^k}\|f^{(k)}\|_\infty.
$$

Proof. We will first prove the following inequality:

(0.8)
$$
E_n(f) \leq \frac{c}{n}E_n(f').
$$

To show (0.8), we fix f and denote with q the trigonometric polynomial of order n for which we have that

$$
\|f' - q\|_\infty = E_n(f').
$$

Clearly, we may write q in the form

$$
q(x) = a_0 + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx) = a_0 + r(x).
$$

Define now

$$
s(x) = \int_{-\pi}^{x} r(t)\, dt.
$$

Note that $s(x)$ is a trigonometric polynomial of order n. Hence

(0.9)
$$
\begin{aligned}
E_n(f) = E_n(f - s) &\leq \frac{c}{n}\|f' - s'\|_\infty \\
&= \frac{c}{n}\|f' - r\|_\infty \\
&\leq \frac{c}{n}[\|f' - q\|_\infty + |a_0|] = \frac{c}{n}[E_n(f') + |a_0|].
\end{aligned}
$$

Since f is 2π-periodic, we have that

$$
\int_{-\pi}^{\pi} f'(\tau)\, d\tau = f(\pi) - f(-\pi) = 0.
$$

Direct calculation using the definition of $r(x)$ shows that $\int_{-\pi}^{\pi} r(\tau)\, d\tau = 0$, as well. Thus

$$
0 = \int_{-\pi}^{\pi} f' - r\, d\tau = \int_{-\pi}^{\pi} f' - q + a_0\, d\tau,
$$

and hence

$$2\pi a_0 = \int_{-\pi}^{\pi} q - f' \, d\tau,$$

Taking absolute values on both sides of this identity ad by some obvious inequalities, we get that

$$2\pi |a_0| \leq 2\pi \|f' - q\|_\infty = 2\pi E_n(f').$$

The proof is then concluded by estimating $|a_0|$ with $E_n(f')$ in (0.9). The statement of the theorem follows by several applications of the inequality (0.8). □

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