1. Area and co-area formula

1.1. **Hausdorff measure.** In this section we will recall the definition of the Hausdorff measure and we will state some of its basic properties. A more detailed discussion is postponed to Section **??**.

Let $\omega_s = \pi^{s/2}/\Gamma(1+\frac{s}{2})$, $s \geq 0$. If $s = n$ is a positive integer, then ω_n is volume of the unit ball in $\mathbb{R}^n$. Let X be a metric space. For $\varepsilon > 0$ and $E \subset X$ we define[1]

$$\mathcal{H}^s_\varepsilon(E) = \inf \frac{\omega_s}{2^s} \sum_{i=1}^{\infty} (\operatorname{diam} A_i)^s$$

where the infimum is taken over all possible coverings

$$E \subset \bigcup_{i=1}^{\infty} A_i \quad \text{with } \operatorname{diam} A_i \leq \varepsilon.$$

Since the function $\varepsilon \mapsto \mathcal{H}^s_\varepsilon(E)$ is nonincreasing, the limit

$$\mathcal{H}^s(E) = \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(E)$$

exists. $\mathcal{H}^s$ is called the *Hausdorff measure*. It is easy to see that if $s = 0$, $\mathcal{H}^0$ is the counting measure.

The Hausdorff content $\mathcal{H}^s_\infty(E)$ is defined as the infimum of $\sum_{i=1}^\infty r_i^s$ over all coverings

$$E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i)$$

of E by balls of radii r_i. It is an easy exercise to show that $\mathcal{H}^s(E) = 0$ if and only if $\mathcal{H}^s_\infty(E) = 0$. Often it is easier to use the Hausdorff content to show that the Hausdorff measure of a set is zero, because one does not have to worry about the diameters of the sets in the covering. The Hausdorff content is an outer measure, but very few sets are actually measurable, and it is not countably additive on Borel sets. This is why $\mathcal{H}^s_\infty$ is called content, but not measure.

Theorem 1.1. *$\mathcal{H}^s$ is a metric outer measure i.e. $\mathcal{H}^s(E \cup F) = \mathcal{H}^s(E) + \mathcal{H}^s(F)$ whenever E and F are arbitrary sets with $\operatorname{dist}(E, F) > 0$. Hence all Borel sets are $\mathcal{H}^s$ measurable.*

It is an easy exercise to prove that $\mathcal{H}^s$ is an outer measure. The fact that it is a metric outer measure follows from the observation that if $\varepsilon < \operatorname{dist}(E, F)/2$, we can assume that

[1] If $B \subset \mathbb{R}^n$ is a ball, then $\frac{\omega_n}{2^n}(\operatorname{diam} B)^n = |B|$. This explains the choice of the coefficient $\omega_s/2^s$ in the definition of the Hausdorff measure.

sets of diameter less than ε that cover E are disjoint from the sets of diameter less than ε that cover F. We leave details as an exercise. Finally measurability of Borel sets is a general property of metric outer measures.

The next result is very important and difficult. We will prove it in Section **??**

Theorem 1.2. *$\mathcal{H}^n$ on $\mathbb{R}^n$ coincides with the outer Lebesgue measure $\mathcal{L}^n$. Hence a set is $\mathcal{H}^n$ measurable if and only if it is Lebesgue measurable and both measures are equal on the class of measurable sets.*

This result generalizes to the case of the Lebesgue measure on submanifolds of $\mathbb{R}^n$. We will discuss it in the Subsection 1.3.

In what follows we will often use the Hausdorff measure notation to denote the Lebesgue measure.

Proposition 1.3. *If $f : X \supset E \to Y$ is a Lipschitz mapping between metric spaces, then $\mathcal{H}^s(f(E)) \leq L^s\mathcal{H}^s(E)$. In particular if $\mathcal{H}^s(E) = 0$, then $\mathcal{H}^s(f(E)) = 0$.*

This is very easy. Indeed if $A \subset E$, then $f(A)$ has diameter less than or equal to $L\mathrm{diam}\, A$, where L is the Lipschitz constant of f. This observation and the definition of the Hausdorff measure easily yields the result.

In particular, if $f : \mathbb{R}^n \supset E \to \mathbb{R}^m$ is a Lipschitz mapping and $|E| = 0$, then $\mathcal{H}^n(f(E)) = 0$. We will prove a stronger result which is known as the Sard theorem. A more general version of the Sard theorem will be discussed in Section **??**.

Theorem 1.4 (Sard)**.** *Let $f : \mathbb{R}^n \supset E \to \mathbb{R}^m$ be Lipschitz continuous and let*

$$\mathrm{Crit}\,(f) = \{x \in E : \,\mathrm{rank}\,\mathrm{ap}Df(x) < n\},$$

then $\mathcal{H}^n(f(\mathrm{Crit}\,(f))) = 0$.

In the proof we will need the so called $5r$*-covering lemma*. It is also called a *Vitali type covering lemma*. Here and in what follows by σB we denote a ball concentric with the ball B and σ times the radius.

Theorem 1.5 ($5r$-covering lemma)**.** *Let $\mathcal{B}$ be a family of balls in a metric space such that $\sup\{\mathrm{diam}\, B : \, B \in \mathcal{B}\} < \infty$. Then there is a subfamily of pairwise disjoint balls $\mathcal{B}' \subset \mathcal{B}$*

such that

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{B\in\mathcal{B}'} 5B\,.$$

If the metric space is separable, then the family $\mathcal{B}'$ is countable and we can arrange it as a sequence $\mathcal{B}'=\{B_i\}_{i=1}^\infty$, so

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{i=1}^\infty 5B_i\,.$$

Remark 1.6. Here $\mathcal{B}$ can be either a family of open balls or closed balls. In both cases the proof is the same.

Proof. Let $\sup\{\operatorname{diam} B:\ B\in\mathcal{B}\}=R<\infty$. Divide the family $\mathcal{B}$ according to the diameter of the balls

$$\mathcal{F}_j=\{B\in\mathcal{B}:\frac{R}{2^j}<\operatorname{diam} B\leq\frac{R}{2^{j-1}}\}\,.$$

Clearly $\mathcal{B}=\bigcup_{j=1}^\infty \mathcal{F}_j$. Define $\mathcal{B}_1\subset\mathcal{F}_1$ to be the maximal family of pairwise disjoint balls. Suppose the families $\mathcal{B}_1,\ldots,\mathcal{B}_{j-1}$ are already defined. Then we define $\mathcal{B}_j$ to be the maximal family of pairwise disjoint balls in

$$\mathcal{F}_j\cap\{B:\ B\cap B'=\emptyset \text{ for all } B'\in\bigcup_{i=1}^{j-1}\mathcal{B}_i\}\,.$$

Next we define $\mathcal{B}'=\bigcup_{j=1}^\infty\mathcal{B}_j$. Observe that every ball $B\in\mathcal{F}_j$ intersects with a ball in $\bigcup_{i=1}^j\mathcal{B}_j$. Suppose that $B\cap B_1\neq\emptyset$, $B_1\in\bigcup_{i=1}^j\mathcal{B}_i$. Then

$$\operatorname{diam} B\leq\frac{R}{2^{j-1}}=2\cdot\frac{R}{2^j}\leq 2\operatorname{diam} B_1$$

and hence $B\subset 5B_1$. The proof is complete. □

Proof of the Sard theorem. Using the McShane extension (Theorem **??**) we can assume that f is defined on all of $\mathbb{R}^n$ and replace the approximate derivative by the classical one. Indeed, the set of points in E where the approximate derivative exists, but the extension to $\mathbb{R}^n$ is not differentiable at these points has measure zero and this set is mapped onto a set of $\mathcal{H}^n$ measure zero.

Let Z be the set of points in $\mathbb{R}^n$ such that $Df(x)$ exists and $\operatorname{rank} Df(x)<n$. We need to show that $\mathcal{H}^n(f(Z))=0$. By splitting Z into bounded pieces we may assume that Z is contained in the interior of the unit cube Q.[2] For $L>\varepsilon>0$ and $x\in Z$ there is $r_x>0$

[2]Indeed, if each bounded piece of Z is mapped into a set of $\mathcal{H}^n$ measure zero, then Z is mapped into a set of measure zero.

such that $B(x, r_x) \subset Q$ and

$$|f(y) - f(x) - Df(x)(y - x)| < \varepsilon r_x \quad \text{if } y \in B(x, 5r_x).$$

Hence

$$\text{dist}\,(f(y), W_x) \leq \varepsilon r_x \quad \text{for } y \in B(x, 5r_x),$$

where $W_x = f(x) + Df(x)(\mathbb{R}^n)$ is an affine space through $f(x)$. Clearly $\dim W_x \leq n - 1$. Thus

$$f(B(x, 5r_x)) \subset B(f(x), 5Lr_x) \cap \{z : \text{dist}\,(z, W_x) \leq \varepsilon r_x\}. \tag{1.1}$$

Since $\dim W_x = k \leq n - 1$ we have that

$$\mathcal{H}^n_\infty(f(B(x, 5r_x)) \leq C\varepsilon L^{n-1} r_x^n,$$

where the constant C depends on n only. Indeed, the k dimensional ball $B(f(x), 5Lr_x) \cap W_x$ can be covered by

$$C\left(\frac{Lr_x}{\varepsilon r_x}\right)^k \leq C\left(\frac{L}{\varepsilon}\right)^{n-1}$$

balls of radius εr_x. Then balls with radii[3] $2\varepsilon r_x$ and the same centers cover the right hand side of (1.1). Thus

$$\mathcal{H}^n_\infty(f(B(x, 5r_x)) \leq C\left(\frac{L}{\varepsilon}\right)^{n-1} (4\varepsilon r_x)^n = C'\varepsilon r_x^n L^{n-1}.$$

From the covering $Z \subset \bigcup_{x \in Z} B(x, r_x)$ we can select a family of pairwise disjoint balls $B(x_i, r_{x_i})$, $i = 1, 2, \ldots$ such that $Z \subset \bigcup_i B(x_i, 5r_{x_i})$. We have

$$\mathcal{H}^n_\infty(f(Z)) \leq \sum_{i=1}^\infty \mathcal{H}^n_\infty(f(B(x_i, 5r_{x_i})) \leq C\varepsilon L^{n-1} \sum_{i=1}^\infty r_{x_i}^n \leq C'\varepsilon L^{n-1},$$

because the balls $B(x_i, r_{x_i})$ are disjoint and contained in the unit cube; hence the sum of their volumes is less than one. Since ε can be arbitrarily small we conclude that $\mathcal{H}^n_\infty(f(Z)) = 0$ and thus $\mathcal{H}^n(f(Z)) = 0$. □

Exercise 1.7. *Show that if*

- $\mathcal{H}^s(E) < \infty$*, then* $\mathcal{H}^t(E) = 0$ *for all* $t > s \geq 0$*;*
- $\mathcal{H}^s(E) > 0$*, then* $\mathcal{H}^t(E) = \infty$ *for all* $0 \leq t < s$*.*

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[3] and hence diameter $4\varepsilon r_x$

Definition 1.8. The *Hausdorff dimension* is defined as follows. If $\mathcal{H}^s(E) > 0$ for all $s \geq 0$, then $\dim_H(E) = \infty$. Otherwise we define

$$\dim_H(E) = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\}.$$

It follows from the exercise that there is $s \in [0, \infty]$ such that $\mathcal{H}^t(E) = 0$ for $t > s$ and $\mathcal{H}^t(E) = \infty$ for $0 < t < s$. Hausdorff dimension of E equals s. It also easily follows from Proposition 1.3 that Lipschitz mappings do not increase the Hausdorff dimension.

1.2. **Countably rectifiable sets.**

Definition 1.9. We say that a metric space X is *countably n-rectifiable* if there is a family of Lipschitz mappings $f_i : \mathbb{R}^n \supset E_i \to X$ defined on measurable sets such that

$$\mathcal{H}^n\left(X \setminus \bigcup_{i=1}^{\infty} f(E_i)\right) = 0.$$

In particular we can talk about sets $X \subset \mathbb{R}^m$ that are countably n-rectifiable.

Clearly any Borel subset of a countably n-rectifiable set is countably n-rectifiable.

In other words X is countably n-rectifiable if it can be covered by countably many Lipschitz images of subsets of $\mathbb{R}^n$ up to a set of $\mathcal{H}^n$ measure zero. Since Lipschitz mappings map sets of finite $\mathcal{H}^n$ measure onto sets of finite $\mathcal{H}^n$ measure, the $\mathcal{H}^n$ measure on X is σ-finite and hence $\dim_H X \leq n$. We do not require the mappings f_i to be one-to-one and one can imagine that X can be very complicated. However as we will see, if X is a subset of $\mathbb{R}^m$ its structure is relatively simple.

Theorem 1.10. *A Borel set $E \subset \mathbb{R}^m$ is countably n-rectifiable, $m \geq n$, if and only if there is a sequence of n-dimensional C^1-submanifolds $\{\mathcal{M}_i\}_{i=1}^{\infty}$ of $\mathbb{R}^m$ such that*

$$\mathcal{H}^n\left(E \setminus \bigcup_{i=1}^{\infty} \mathcal{M}_i\right) = 0. \tag{1.2}$$

Proof. Clearly the condition (1.2) is sufficient for the countable n-rectifiability and we need to prove its necessity. Each mapping $f_i : E_i \to \mathbb{R}^m$ can be approximated by C^1-mappings in the sense of Theorem **??**(d). Using a sequence of such C^1 maps we can approximate f_i up to a set of measure zero. Since sets of measure zero are mapped by Lipschitz maps to sets of measure zero, we can simply assume that the mappings $f_i : \mathbb{R}^n \to \mathbb{R}^m$ are C^1 and

$$\mathcal{H}^n \left(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^n) \right) = 0.$$

A neighborhood of any point in $\mathbb{R}^n$ where $\operatorname{rank} Df_i = n$ is mapped to a C^1-submanifold of $\mathbb{R}^m$ and the remaining set of points where $\operatorname{rank} Df_i < n$ in mapped to a set of $\mathcal{H}^n$ measure zero by Theorem 1.4. □

Definition 1.11. We say that a measurable mappings $f : \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$, has the *Lusin property N* if for any measurable set $A \subset \Omega$ we have

$$|A| = 0 \quad \Rightarrow \quad |f(A)| = 0.$$

More generally we say that a measurable mapping $f : \mathbb{R}^n \supset A \to X$ to a metric space has the *Lusin property N* if for any measurable set $E \subset A$ we have

$$|A| = 0 \quad \Rightarrow \quad \mathcal{H}^n(f(E)) = 0.$$

Exercise 1.12. *Prove that a measurable mapping $f : \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$ maps Lebesgue measurable sets onto Lebesgue measurable sets if and only if it has the Lusin property N.*

For example Lipschitz mappings have the Lusin property, Proposition 1.3.

Theorem 1.13. *Let $f : \mathbb{R}^n \supset E \to \mathbb{R}^m$ be an a.e. approximately differentiable mapping with the Lusin property N, and let*

$$\mathrm{Crit}\,(f) = \{x \in E : \operatorname{rank} \mathrm{ap}Df(x) < n\},$$

then $\mathcal{H}^n(f(\mathrm{Crit}\,(f))) = 0$.

Indeed, this result is a direct consequence of Lemma **??**, the Sard theorem, and the Lusin property of f. A similar argument yields

Proposition 1.14. *$X \subset \mathbb{R}^m$, $m \geq n$ is countably n-rectifiable if and only if there are a.e. approximately differentiable mappings $f_i : \mathbb{R}^n \supset E_i \to \mathbb{R}^m$ with the Lusin property N such that*

$$\mathcal{H}^n \left(X \setminus \bigcup_{i=1}^{\infty} f(E_i) \right) = 0.$$

Proposition 1.15. *$E \subset \mathbb{R}^m$, $m \geq n$ is countably n-rectifiable if and only if there is a locally Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $\mathcal{H}^n(E \setminus f(\mathbb{R}^n)) = 0$.*

Indeed, we can assume in the definition of a countably n-rectifiable set that the sets E_i are contained in a unit cube. We can place such sets in disjoint unit cubes in $\mathbb{R}^n$ that are separated by a positive distance. On each cube we apply the McShane extension to the mapping $f : E_i \to \mathbb{R}^m$. Then we glue the mappings to form a locally Lipschitz mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ by multiplying the extension of f_i by a cut-off function[4] that equals 1 on the unit cube that contains E_i. Note that the result is not true for countably rectifiable subsets of metric spaces, because in such a general setting the McShane theorem is not available.

1.3. **The area formula.** Recall the classical change of variable s formula.

Theorem 1.16. *Let $\Phi : \Omega \to \mathbb{R}^n$ be a C^1 diffeomorphism between domains $\Omega \subset \mathbb{R}^n$ and $\Phi(\Omega) \subset \mathbb{R}^n$. If $f : \Omega \to [0, \infty]$ is a nonnegative measurable function or if $f|J_\Phi| \in L^1(\Omega)$ is integrable, then*

$$\int_{\Phi(\Omega)} f(\Phi^{-1}(y))\, dy = \int_\Omega f(x)|J_\Phi(x)|\, dx,$$

where $J_\Phi(x) = \det D\Phi(x)$ is the Jacobian of the diffeomorphism Φ.

In the case in which the function f is defined on $\Phi(\Omega)$ we have

$$\int_{\Phi(\Omega)} f(y)\, dy = \int_\Omega (f \circ \Phi)(x)|J_\Phi(x)|\, dx,$$

where we assume that $f \geq 0$ or that $f \in L^1(\Omega)$. Theorem 1.16 generalizes to the case of integration over an n-dimensional submanifold $\mathcal{M}$ of $\mathbb{R}^m$, $m \geq n$. A neighborhood of any point in $\mathcal{M}$ can be represented as the image of a parametrization. Recall that a *parametrization* of $\mathcal{M}$ is a one-to-one mapping

$$\Phi : \mathbb{R}^n \supset \Omega \to \mathbb{R}^m, \quad \Phi(\Omega) \subset \mathcal{M}$$

of class C^1 such that $\operatorname{rank} Dg(x) = n$ for all $x \in \Omega$.

Observe that $\det(D\Phi)^T D\Phi$ is the Gramm determinant of vectors $\partial\Phi/\partial x_i$ and hence $\sqrt{\det(D\Phi)^T D\Phi(x)}$ is the n-dimensional volume of the parallelepiped with edges $\partial\Phi(x)/\partial x_i$. Thus it is natural to define

$$|J_\Phi(x)| = \sqrt{\det(D\Phi)^T D\Phi(x)},$$

even if $m > n$. Note that this definition is consistent with the standard definition of the absolute value of the Jacobian when $m = n$.

[4] We multiply each component of the function f_i by a cut-off function.

In the case when $m = n$ and Φ is a diffeomerphism, the change of variables formula implies that the Lebesgue measure $|E|$ of a set $E \subset \Phi(\Omega)$ equals

$$|E| = \int_{\Phi^{-1}(E)} |J_\Phi(x)|\, dx.$$

If $\Phi : \mathbb{R}^n \supset \Omega \to \mathcal{M} \subset \mathbb{R}^m$ is a parametrization, we define the Lebesgue measure (surface measure) $\sigma(E)$ of a set $E \subset \Phi(\Omega)$ by the formula (1.3), i.e.

$$\sigma(E) = \int_{\Phi^{-1}(E)} |J_\Phi(x)|\, dx. \tag{1.3}$$

If $E \subset \mathcal{M}$ is not necessarily contained in the image of a single parametrization, we divide it into small pieces that are contained in the images of parametrizations and we add the measures. One only needs to observe that the measure of a set does not depend on the choice of a parametrization. Indeed, suppose that $E \subset \Phi_1(\Omega_1) \cap \Phi_2(\Omega_2)$. By taking smaller domains we can assume that $\Phi_1(\Omega_1) = \Phi_2(\Omega_2)$. Then $\Phi_1^{-1} \circ \Phi_2 : \Omega_2 \to \Omega_1$ is a diffeomorphism and the change of variables formula easily implies that

$$\int_{\Phi_1^{-1}(E)} |J_{\Phi_1}(x)|\, dx = \int_{\Phi_2^{-1}(E)} |J_{\Phi_2}(x)|\, dx.$$

Note that the formula (1.3) can be written as

$$\int_{\Phi(\Omega)} f(y)\, d\sigma(y) = \int_\Omega (f \circ \Phi)(x) |J_\Phi(x)|\, dx,$$

where f is the characteristic function of the set E. Since measurable functions can be approximated by simple functions which are linear combinations of characteristic functions, standard limiting procedure yields

Theorem 1.17. *Let $\Phi : \mathbb{R}^n \supset \Omega \to \mathcal{M} \subset \mathbb{R}^m$, $m \geq n$ be a parametrization of an n dimensional submanifold $\mathcal{M} \subset \mathbb{R}^m$. If $f : \Phi(\Omega) \to [0, \infty]$ is a nonnegative measurable function or if $f \in L^1(\Phi(\Omega))$ is integrable, then*

$$\int_{\Phi(\Omega)} f(y)\, d\sigma(y) = \int_\Omega (f \circ \Phi)(x) |J_\Phi(x)|\, dx.$$

For $f \geq 0$ on Ω and for $f|J_\Phi| \in L^1(\Omega)$ the change of variables formula takes the form

$$\int_{\Phi(\Omega)} (f \circ \Phi^{-1})(y)\, d\sigma(y) = \int_\Omega f(x) |J_\Phi(x)|\, dx. \tag{1.4}$$

According to Theorem 1.2 the Lebesgue measure in $\mathbb{R}^n$ coincides with the Hausdorff measure $\mathcal{H}^n$. One can prove that the surface measure on $\mathcal{M}$ also coincides with the Hausdorff measure $\mathcal{H}^n$ defined either with respect to the Euclidean metric of $\mathbb{R}^m$ restricted to $\mathcal{M}$ or with respect to the natural Riemannian metric on $\mathcal{M}$. We will not prove this fact, but this result should not be surprising; $\mathcal{M}$ is locally very well approximated by tangent spaces and this approximation allows one to deduce the result from Theorem 1.2. In particular in both theorems Theorem 1.16 and 1.17 we can replace dy and $d\sigma(y)$ by $d\mathcal{H}^n(y)$.

The purpose of this section is to prove a far reaching generalization the change of variables formula.

If $\Phi : \mathbb{R}^n \supset E \to \mathbb{R}^m$, $m \geq n$ is approximately differentiable a.e., then we can formally define the Jacobian of Φ at almost every point of E by

$$|J_\Phi(x)| = \sqrt{\det(D\Phi)^T D\Phi(x)}$$

Theorem 1.18 (Area formula). *Let $\Phi : \mathbb{R}^n \supset E \to \mathbb{R}^m$, $m \geq n$ be approximately differentiable a.e. Then we can redefine it on a set of measure zero in such a way that the new mapping satisfies the Lusin property N. If Φ is approximately differentiable a.e., satisfies the Lusin property N and $f : E \to [0, \infty]$ is measurable or $f|J_\Phi| \in L^1(E)$, then*

$$\int_E f(x)|J_\Phi(x)|\, dx = \int_{\Phi(E)} \left(\sum_{x \in \Phi^{-1}(y)} f(x) \right) d\mathcal{H}^m(y). \tag{1.5}$$

Here we do not assume that the mapping Φ is one-to-one and this is why we have the sum on the right hand side, just to compensate the fact that the point y is the image of every point x in the set $\Phi^{-1}(y)$. Note that since $\mathcal{H}^0$ is the counting measure formula (1.5) can be rewritten as

$$\int_E f(x)|J_\Phi(x)|\, d\mathcal{H}^n(x) = \int_{\Phi(E)} \left(\int_{\Phi^{-1}(y)} f(x)\, d\mathcal{H}^0(x) \right) d\mathcal{H}^m(y)\,.$$

The reason why we want to write it this way will be apparent when we will discuss the co-area formula.

Proof. Lemma **??** shows that away from a set Z of measure zero Φ has the Lusin property since it consists of Lipschitz pieces. Now if we modify Φ on the set Z and send the set to a single point, a new mapping $\tilde{\Phi}$ will have the Lusin property and it will be equal to Φ almost everywhere. This proves the first part of the theorem. Assume now that Φ has the Lusin property. Note that if we remove from E a subset of measure zero both sides of (1.5) will not change its value. It is obvious for the left hand side, but regarding the right hand side it follows from the Lusin property of Φ. We can also remove the subset of E where $J_\Phi = 0$. According to Theorem 1.13, Φ maps this set onto a set of measure zero and hence both sides of (1.5) will not change its value after such a removal. This combined with Theorem **??** allows us to assume that there are disjoint subsets $E_i \subset E$ such that $\bigcup_i E_i = E$ and C^1 mappings $\Phi_i : \mathbb{R}^n \to \mathbb{R}^m$ such that $\Phi_i = \Phi$ on E_i, $D\Phi_i = \mathrm{ap}D\Phi$ on E_i, $\operatorname{rank} D\Phi_i = n$ on E_i. Dividing the sets into small pieces, if necessary, we can also assume that Φ_i is one-to-one in an open set containing E_i, i.e. it is a parametrization of an n-dimensional submanifold of $\mathbb{R}^m$ on that open set. According to the classical change of variables formula (1.4) we have[5]

$$\int_{E_i} f(x)|J_{\Phi_i}(x)|\, dx = \int_{\Phi_i(E_i)} f(\Phi_i^{-1}(y))\, d\mathcal{H}^n(y)$$

which yields

$$\begin{aligned} \int_E f(x)|J_\Phi(x)|\, dx &= \sum_{i=1}^{\infty} \int_{E_i} f(x)|J_{\Phi_i}(x)|\, dx = \sum_{i=1}^{\infty} \int_{\Phi_i(E_i)} f(\Phi_i^{-1}(y))\, d\mathcal{H}^n(y) \\ &= \int_{\Phi(E)} \left(\sum_{x\in\Phi^{-1}(y)} f(x) \right) d\mathcal{H}^n(y). \end{aligned}$$

Indeed, if $f \geq 0$ we can change the order of integration ans summation by the monotone convergence theorem. In the case of $f \in L^1$ we consider separately the positive and negative parts of f. □

Remark 1.19. It is necessary to require that Φ has the Lusin property. Indeed, if Φ maps a set of measure zero onto a set of positive measure, and $f = 1$, then the left hand side of the formula in Theorem 1.18 equals zero, but the right hand side is positive.

If f is a measurable function on $\mathbb{R}^m$, and $\Phi : E \to \mathbb{R}^m$ is approximately differentiable a.e. and has the Lusin property N, then Theorem 1.18 applies to the function $f \circ \Phi$ which is defined on E. Note that $f \circ \Phi$ is constant on the set $\Phi^{-1}(y)$ and hence the area formula takes the form

$$\int_E (f\circ\Phi)(x)|J_\Phi(x)|\, dx = \int_{\Phi(E)} f(y) N_\Phi(y,E)\, d\mathcal{H}^n(y),$$

where

$$N_\Phi(y,E) = \#(\Phi^{-1}(y)\cap E)$$

is the cardinality of the set $\Phi^{-1}(y)\cap E$. The function $N_\Phi(\cdot, E)$ is called the *Banach indicatrix* of Φ. This formula is true under the assumption that $f \geq 0$ or under the integrability assumption of $(f\circ\Phi)|J_\Phi|$. More precisely if $(f\circ\Phi)(x)|J_\Phi(x)|$ is integrable on E or if

[5] We replace f in (1.4) by $f\chi_{E_i}$.

$f(y)N_\Phi(y,E)$ is integrable on $\Phi(E)$, then the other function is integrable too and the formula is true.

Remark 1.20. Suppose that $\Phi : Q \to Q$ is a homeomorphism of the unit cube $Q = [0,1]^n$ that is identity on the boundary of the cube. Assume also that Φ is approximately differentiable a.e. and has the Lusin property. In this case the change of variables formula shows that $\int_Q |J_\Phi| = 1$. Since Φ is an orientation preserving homeomorphism, is it true that $J_\Phi \geq 0$ a.e.? Surprisingly, one can find such a homeomorphism with the property that $J_\Phi = -1$ a.e. or that it is positive on a subset of a cube and negative on another subset and it is reasonable to conjecture that the only real constrain for a construction of a mapping Φ with prescribed Jacobian is the condition that integral of $|J_\Phi|$ over the cube equals one.

The area formula generalizes to the case of mappings between Riemannian manifolds. Submanifolds of Euclidean spaces are examples of Riemannian manifolds.

Theorem 1.21. *The statement of Theorem 1.18 remains true if we replace $\mathbb{R}^n$ and $\mathbb{R}^m$ by n-dimensional and m-dimansional Riemannian manifolds respectively.*

1.4. **The co-area formula.** The area formula is a generalization of the change of variable formula to the case of mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$, where $m \geq n$. Surprisingly, it is also possible to generalize to change of variables formula to the case when $m \leq n$; this is so called the *co-area formula*. First we need to generalize the Jacobian to the case of mappings $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, $m \leq n$. Suppose that Φ is differentiable at $x \in \mathbb{R}^n$. If $m < n$, then

$$\sqrt{\det(D\Phi)^T(D\Phi)(x)} = 0 \tag{1.6}$$

because this is a formula for the n-dimensional volume of a parallelepiped which in our situation has the dimension $\leq m < n$. That means (1.6) is not a good notion of the Jacobian when$m < n$. Assume that rank of $D\Phi(x)$ is maximal, i.e. $\operatorname{rank} D\Phi(x) = m \leq n$. If B is a ball in the tangent space $T_x\mathbb{R}^n$ centered at the origin, then $D\Phi(x)(B)$ is a non-degenerate m-dimensional ellipsoid in $T_{\Phi(x)}\mathbb{R}^m$. The kernel $\ker D\Phi(x)$ is an $n-m$ dimensional linear subspace of $T_x\mathbb{R}^n$ and $D\Phi(x)$ is a composition of two mappings; first we take the orthogonal projection of $T_x\mathbb{R}^n$ onto the m-dimensional space $(\ker D\Phi(x))^\perp$ and then we compose it with the linear isomorphism of m-dimensional spaces

$$D\Phi(x)|_{(\ker D\Phi(x))^\perp} : (\ker D\Phi(x))^\perp \to T_{\Phi(x)}\mathbb{R}^m. \tag{1.7}$$

Now we define $|J_\Phi(x)|$ as the absolute value of the Jacobain of the mapping (1.7), i.e. $|J_\Phi(x)|$ is factor by which the linear mapping (1.7) changes volume. Geometrically speaking the ellipsoid $D\Phi(x)(B)$ is the image of the m-dimensional ball $B \cap (\ker D\Phi(x))^\perp$ and hence

$$|J_\Phi(x)| = \frac{\mathcal{H}^m(D\Phi(x)(B))}{\mathcal{H}^m(B \cap (\ker D\Phi(x))^\perp)}\, .$$

If $\operatorname{rank} D\Phi(x) < n$ we set $|J_\Phi(x)| = 0$. Although we defined the Jacobian in geometric terms, there is a simple algebraic formula for $|J_\Phi(x)|$ which follows from the polar decomposition of the linear mapping $D\Phi(x)$.

$$|J_\Phi(x)| = \sqrt{\det (D\Phi)(D\Phi)^T(x)}.$$

Note that this is not the same formula as (1.6). However, the two formulas give the same value when $m = n$.

Exercise 1.22. *Prove this formula using the polar decomposition of $D\Phi(x)$.*

There is one more geometric interpretation of $|J_\Phi(x)|$ when $m \le n$ which easily follows from our geometric definition. Namely $|J_\Phi(x)|$ equals the supremum of m-dimensional measures of all ellipsoids $D\Phi(x)(B)$, where the supremum is over all m-dimensional balls B in $T_x\mathbb{R}^n$ of volume 1. This reminds us of the the geometric interpretation of the length of the gradient of a real valued function as the maximal rate of of change of a function. The function has maximal growth in the direction the gradient which is orthogonal to $\det D\Phi$. In our case the maximal growth of the m-dimensional measure of m-dimensional balls in $T_x\mathbb{R}^n$ is also in the direction orthogonal to $\ker D\Phi(x)$, see (1.7). This is the right intuition. If $\Phi : \mathbb{R}^n \to \mathbb{R}$, i.e. $m = 1$, one can easily see that

$$|J_\Phi(x)| = \sqrt{\det (D\Phi)(D\Phi)^T(x)} = |\nabla \Phi(x)|.$$

Now we can state the co-area formula. We will actually state both area and co-area formula in one theorem, because it will help to see similarities and differences between the two formulas.

Theorem 1.23 (The area and the co-area formulas)**.** *Let $\Phi : \mathbb{R}^n \supset E \to \mathbb{R}^m$ be a Lipschitz mapping defined on a measurable set $E \subset \mathbb{R}^n$. Let $f \ge 0$ be a measurable function on E or let $f|J_\Phi| \in L^1(E)$. Then*

- *(Area formula) If $n \le m$, then*

$$\int_E f(x)|J_\Phi(x)|\, d\mathcal{H}^n(x) = \int_{\mathbb{R}^m} \left(\int_{\Phi^{-1}(y)} f(x)\, d\mathcal{H}^0(x) \right) d\mathcal{H}^m(y).$$

- *(Co-area formula) If $n \geq m$, then*

$$\int_E f(x)|J_\Phi(x)|\, d\mathcal{H}^n(x) = \int_{\mathbb{R}^m} \left(\int_{\Phi^{-1}(y)} f(x)\, d\mathcal{H}^{n-m}(x) \right) d\mathcal{H}^m(y).$$

We will not prove the co-area formula here, but we will show that it contains results like integration in the spherical coordinates and the Fubini theorem as special cases!

Recall that if $\Phi : \mathbb{R}^n \to \mathbb{R}$, then $|J_\Phi(x)| = |\nabla \Phi(x)|$. Taking $\Phi(x) = |x|$ we have $|J_\Phi(x)| = 1$ everywhere except at the origin. Since the image of Φ is $[0, \infty)$, the co-area formula reads as

$$\int_{\mathbb{R}^n} f(x)\, d\mathcal{H}^n(x) = \int_0^\infty \left(\int_{\partial B(0,r)} f(x)\, d\mathcal{H}^{n-1}(x) \right) dr$$

which is the formula for the integration in the spherical coordinates.

Let now $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, $m < n$ be the projection on the first m coordinates $\Phi(x_1, \ldots, x_n) = (x_1, \ldots, x_m)$. Then $|J_\Phi(x)| = 1$ and we have

$$\int_{\mathbb{R}^n} f(x)\, d\mathcal{H}^n(x) = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^{n-m}} f(x_1, \ldots, x_n)\, dH^{n-m}(x_{m+1}, \ldots, x_n) \right) d\mathcal{H}^m(x_1, \ldots, x_m)$$

which is the Fubini theorem.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary Lipschitz function. Taking $\Phi = f$ we have $J_\Phi(x)| = |\nabla f(x)|$; taking the function f in the co-area formula to be equal[6] 1 we have

$$\int_{\mathbb{R}^n} |\nabla f(x)|\, dx = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{f = t\})\, dx.$$

As an application of the co-area formula we will prove

Theorem 1.24. *If $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous, then for a.e. $y \in \mathbb{R}^m$, $\Phi^{-1}(y)$ is countably $(n-m)$-rectifiable.*

Proof. If $m > n$, then $\mathcal{H}^m(f(\mathbb{R}^n)) = 0$, so $\Phi^{-1}(y) = \emptyset$ for a.e. $y \in \mathbb{R}^n$ and the empty set is countably rectifiable. Thus we can assume that $m \leq n$. Assume for a moment that $\Phi \in C^1$. Then according to the implicit function theorem

$$\Phi^{-1}(y) \cap \{\mathrm{rank}\, D\Phi = m\}$$

is a C^1, $(n-m)$-dimensional submanifold of $\mathbb{R}^n$ and it follows from the co-area formula that

$$\mathcal{H}^{n-m}(\Phi^{-1}(y) \cap \{\mathrm{rank}\, D\Phi < m\}) = 0 \quad \text{for } \mathcal{H}^m\text{-a.e. } y \in \mathbb{R}^m.$$

[6]This might be slightly confusing since we have a double meaning of f.

Thus for almost all y, Φ^{-1} is a manifold plus a set of measure zero. Hence it is countably $(n-m)$-rectifiable. To prove the result in the case in which Φ is Lipschitz it suffices to use Theorem **??** which reduces the problem to the C^1-case. □

1.5. **The Eilenberg inequality.**

Definition 1.25. A metric space is said to be *boundednly compact* if bounded and closed sets are compact.

An important step in the proof of the co-area formula is the following

Theorem 1.26 (Eilenberg inequality)**.** *Let $\Phi : X \to Y$ be a Lipschitz mapping between boundedly compact metric spaces. Let $0 \leq m \leq n$ be real numbers.*[7] *Assume that $E \subset X$ is $\mathcal{H}^n$-measurable with $\mathcal{H}^n(E) < \infty$. Then*

(1) *$\Phi^{-1}(y) \cap E$ is $\mathcal{H}^{n-m}$-measurable for $\mathcal{H}^m$-almost all $y \in Y$.*
(2) *$y \mapsto \mathcal{H}^{n-m}(\Phi^{-1}(y) \cap E)$ is $\mathcal{H}^m$-measurable.*

Moreover

$$\int_Y \mathcal{H}^{n-m}(\Phi^{-1}(y) \cap E)\, d\mathcal{H}^m(y) \leq (\mathrm{Lip}\,(\Phi))^m \frac{\omega_m \omega_{n-m}}{\omega_n}\, \mathcal{H}^n(E).$$

Observe that the left hand side corresponds to the right hand side in the co-area formula with $f = 1$. Observe also that $|J_\Phi|$ can be estimated by $\mathrm{Lip}\,(\Phi)^m$, and then the integral of the Jacobian over E can be estimated from above by $\mathrm{Lip}\,(\Phi)^m \mathcal{H}^n(E)$. This shows a deep connection between the co-area formula and the Eilenberg inequality. Since we used the estimate from the above we only have an inequality and one cannot expect equality in the Eilenberg inequality. What is remarkable is that the Eilenberg inequality is true in a great generality of boundedly compact metric spaces where differentiable structure is not available. We will prove Theorem 1.26 under the additional assumption that $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$.

The measurability of the function $y \mapsto \mathcal{H}^{n-m}(\Phi^{-1}(y) \cap E)$ is far from being obvious and we will want to integrate this function before proving its measurablility. To do this we will have to use the upper Lebesgue integral.

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[7] Not necessarily integers.

Definition 1.27. For a nonnegative function $f : X \to [0,\infty]$ defined μ-a.e. on a measure space (X,μ) the upper Lebesgue integral is defined as

$$\int^* f\, d\mu = \inf\left\{ \int \phi\, d\mu :\ 0 \leq f \leq \phi \text{ and } \phi \text{ is } \mu\text{-measurable}\right\}.$$

We do not assume measurability of f. Clearly if f is measurable the upper Lebesgue ineqgral equals the Lebesgue integral.

An important property of the upper integral is that if $\int^* f\, d\mu = 0$, then $f = 0$, μ-a.e. and hence it is measurable. Indeed, there is a sequence $0 \leq f \leq \phi_n$ such that $\int \phi_n\, d\mu \to 0$. That means $\phi_n \to 0$ in $L^1(\mu)$. Taking a subsequence we get $\phi_{n_k} \to 0$, μ-a.e. which proves that $f = 0$, μ-a.e.

Proof of Theorem 1.26 when $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. For ever positive integer $k > 0$ there is a covering

$$E \subset \bigcup_{i=1}^{\infty} A_{ik}, \quad A_{ik} \text{ is closed}, \quad \operatorname{diam} A_{ik} < \frac{1}{k}$$

such that

$$\frac{\omega_n}{2^n}\sum_{i=1}^{\infty}(\operatorname{diam} A_{ik})^n \leq \mathcal{H}^n(E) + \frac{1}{k}. \tag{1.8}$$

It follows directly from the definition of the Hausdorff measure that

$$\mathcal{H}^{n-m}(\Phi^{-1}(y)\cap E) \leq \frac{\omega_{n-m}}{2^{n-m}} \liminf_{k\to\infty} \sum_{i=1}^{\infty} \operatorname{diam}\,(\Phi^{-1}(y)\cap A_{ik})^{n-m}. \tag{1.9}$$

For any set $A \subset X$ we have

$$\operatorname{diam}\,(\Phi^{-1}(y)\cap A) = \operatorname{diam}\,(\Phi^{-1}(y)\cap A)\chi_{\overline{\Phi(A)}}(y) \leq (\operatorname{diam} A)^{n-m}\chi_{\overline{\Phi(A)}}(y).$$

Hence (1.9) yields

$$\mathcal{H}^{n-m}(\Phi^{-1}(y)\cap E) \leq \frac{\omega_{n-m}}{2^{n-m}} \liminf_{k\to\infty} \sum_{i=1}^{\infty} (\operatorname{diam} A_{ik})^{n-m}\chi_{\overline{\Phi(A_{ik})}}(y).$$

The function on the right hand side is measurable. Hence Fatou's lemma yields

$$\begin{aligned}\int^*_{\mathbb{R}^m} \mathcal{H}^{n-m}(\Phi^{-1}(y)\cap E)\, d\mathcal{H}^m(y) \quad &\leq \quad \frac{\omega_{n-m}}{2^{n-m}} \liminf_{k\to\infty} \int_{\mathbb{R}^m} \sum_{i=1}^{\infty} (\operatorname{diam} A_{ik})^{n-m}\chi_{\overline{\Phi(A_{ik})}}(y)\, d\mathcal{H}^m(y) \\ &= \quad \frac{\omega_{n-m}}{2^{n-m}} \liminf_{k\to\infty} \sum_{i=1}^{\infty} (\operatorname{diam} A_{ik})^{n-m}\mathcal{H}^m(\overline{\Phi(A_{ik})}).\end{aligned}$$

If $p \in A_{ik}$, then

$$\overline{\Phi(A_{ik})} \subset \overline{B}(f(p), \operatorname{Lip}(\Phi)\operatorname{diam} A_{ik})$$

and hence

$$\mathcal{H}^m(\overline{\Phi(A_{ik})}) \leq \frac{\omega_m}{2^m}(\mathrm{Lip}\,(\Phi))^m(\mathrm{diam}\,A_{ik})^m.$$

Thus

$$\begin{aligned}\int^*_{\mathbb{R}^m} \mathcal{H}^{n-m}(\Phi^{-1}(y)\cap E)\,d\mathcal{H}^m(y) &\leq \frac{\omega_{n-m}}{2^{n-m}}\frac{\omega_m}{2^m}\frac{2^n}{\omega_n}\liminf_{k\to\infty}\frac{\omega_n}{2^n}\sum_{i=1}^{\infty}(\mathrm{diam}\,A_k)^n \\ &\leq (\mathrm{Lip}\,(\Phi))^m\frac{\omega_{n-m}\omega_n}{\omega_n}\,\mathcal{H}^n(E),\end{aligned}$$

by (1.8). It remains to prove the $\mathcal{H}^{n-m}$-measurability of the sets $\Phi^{-1}(y)\cap E$ for $\mathcal{H}^m$ almost all $y \in \mathbb{R}^m$ and the $\mathcal{H}^m$-measurability of the function $\varphi(y) = \mathcal{H}^{n-m}(\Phi^{-1}(y)\cap E)$. Note that if $\mathcal{H}^n(E) = 0$, then $\mathcal{H}^{n-m}(\Phi^{-1}(y)\cap E) = 0$ for $\mathcal{H}^m$ almost every $y \in \mathbb{R}^m$ by the upper integral estimate. This observations shows that we can ignore subsets of E of $\mathcal{H}^n$ measure zero. Thus we can assume that E is the union of an increasing sequence of compact sets $E\bigcup_{k=1}^{\infty} E_k$, $E_k \subset E_{k+1}$. Note that the sets $\Phi^{-1}(y)\cap E_k$ are compact for every y and hence $\Phi^{-1}(y)\cap E$ is Borel as the union of compact sets. Thus it suffices to prove that every function $y \mapsto \mathcal{H}^{n-m}(\Phi^{-1}(y)\cap E_k)$ is Borel measurable, because then the function

$$\varphi(y) = \mathcal{H}^{n-m}(\Phi^{-1}(y)\cap E) = \lim_{k\to\infty}\mathcal{H}^{n-m}(\Phi^{-1}(y)\cap E_k)$$

will also be Borel. Hence we can assume that E is compact. It remains to prove that for every $t \in \mathbb{R}$ the set

$$\big\{y \in \mathbb{R}^m :\ \mathcal{H}^{n-m}(\Phi^{-1}(y)\cap E) \leq t\big\} \tag{1.10}$$

is Borel. If $t < 0$, then the set is empty, so we can assume that $t \geq 0$. Since the set in (1.10) can be written as

$$(\mathbb{R}^m \setminus \Phi(E)) \cup \big(\Phi(E) \cap \big\{y \in \mathbb{R}^m :\ \mathcal{H}^{n-m}(\Phi^{-1}(y)\cap E) \leq t\big\}\big)$$

and $\mathbb{R}^m \setminus \Phi(E)$ is open, it remains to prove that the set

$$\Phi(E) \cap \big\{y \in \mathbb{R}^m :\ \mathcal{H}^{n-m}(\Phi^{-1}(y)\cap E) \leq t\big\}$$

is Borel. In the definition of the Hausdorff measure we may restrict to coverings by open sets. However this family of sets is uncountable and we would like to have a countable family of sets from which we would choose coverings. Let $\mathcal{F}$ be the family of all open sets in $\mathbb{R}^n$ that are finite unions of balls with rational centers and radii. The family $\mathcal{F}$ is countable and we claim that it can be used as the family of sets from which we choose coverings provided we define the Hausdorff measure of a compact set K. Indeed, first we cover the set K by open sets, $K \subset \bigcup_i U_i$ Each open set is the union of a family of balls with rational centers and radii. These balls form a covering of K and hence we can select a finite subcovering $K \subset \bigcup_{j=1}^N B_j$. Now we replace each set U_i by U_i' which is the union of all the balls B_j that are contained in U_i. We obtain a new covering $K \subset \bigcup_i U_i'$. Clearly $U_i' \subset U_i$ and $U_i' \in \mathcal{F}$.

Let $\mathcal{F}_i$ be the collection of *all* finite families $\{U_{i1}, \ldots, U_{ik}\} \subset \mathcal{F}$ such that

$$\operatorname{diam} U_{ij} < \frac{1}{i},\ j = 1, 2, \ldots, \quad \text{and} \quad \frac{\omega_{n-m}}{2^{n-m}} \sum_{j=1}^{k} (\operatorname{diam} U_{ij})^{n-m} \leq t + \frac{1}{i}.$$

Note that each of the sets U_{ij} is the union of a finite family of balls. The family $\mathcal{F}_i$ is also countable. Clearly we define this family to deal with the coverings of the set $\Phi^{-1}(y) \cap E$ that satisfies $\mathcal{H}^{n-m}(\Phi^{-1}(y) \cap E) \leq t$.

If $U \subset \mathbb{R}^n$ is open, then

$$\Phi(E) \cap \{y :\ \Phi^{-1}(y) \cap E \subset U\} = \Phi(E) \setminus \Phi(E \setminus U)$$

is Borel, because both of the sets $f(E)$ and $f(E \setminus U)$ are compact. In particular the set

$$V_i = \bigcup_{\{U_{i1}, \ldots, U_{ik}\} \in \mathcal{F}_i} \left(\Phi(E) \setminus \Phi \left(E \setminus \bigcup_{j=1}^{k} U_{ij} \right) \right)$$

is Borel as a countable union over the entire family $\mathcal{F}_i$. We will prove that

$$\Phi(E) \cap \{y \in \mathbb{R}^m :\ \mathcal{H}^{n-m}(\Phi^{-1}(y) \cap E) \leq t\} = \bigcap V_i. \tag{1.11}$$

Clearly the set on the right hand side is Borel.

If $y \in E$ and $\mathcal{H}^{n-m}(\Phi^{-1}(y) \cap E) \leq t$, then for any i we can find a covering

$$\Phi^{-1}(y) \cap E \subset U_{i1} \cup \ldots \cup U_{ik}, \quad \{U_{i1}, \ldots, U_{ik}\} \in \mathcal{F}_i.$$

Thus $y \notin \Phi(E \setminus \bigcup_{j=1}^k U_{ij})$ and hence $y \in V_i$. Since i can be chosen arbitrarily, $y \in \bigcap_{i=1}^\infty V_i$. On the other hand if $y \in \bigcap_{i=1}^\infty V_i$ then $y \in \Phi(E)$ and for all i, $y \in V_i$, i.e. there is $\{U_{i1}, \ldots, U_{ik}\} \in \mathcal{F}_i$ such that $y \notin \Phi(E \setminus \bigcup_{i=1}^k U_{ik})$, i.e. $\Phi^{-1}(y) \cap E \subset U_{i1} \cup \ldots \cup U_{ik}$, so

$$\mathcal{H}^{n-m}_{1/i}(\Phi^{-1}(y) \cap E) \leq t + \frac{1}{i}.$$

Taking the limit as $i \to \infty$ we obtain $\mathcal{H}^{n-m}(\Phi^{-1}(y) \cap E) \leq t$. The proof is complete. $\square$

1.6. **Integral geometric measure.** We say that a metric space is *purely $\mathcal{H}^m$-unrectifiable* if for any Lipschitz mapping $f : \mathbb{R}^n \supset A \to X$ we have $\mathcal{H}^m(f(A)) = 0$. It easily follows from the definition that $E \subset \mathbb{R}^n$ is purely $\mathcal{H}^m$-unrectifiable if and only if for any countably $\mathcal{H}^m$ rectifiable set $F \subset \mathbb{R}^n$, $\mathcal{H}^m(E \cap F) = 0$.

Theorem 1.28. *If $\mathcal{H}^m(X) < \infty$, then there is a Borel countably rectifiable set $E \subset X$ such that $X \setminus E$ is purely $\mathcal{H}^m$-unrectifiable. Hence X has a decomposition into a rectifiable and a nonrectifiable parts $X = E \cup (X \setminus E)$. This decomposition is unique up to sets of $\mathcal{H}^m$-measure zero.*

Proof. Let M be the supremum of $\mathcal{H}^m(E)$ over all $\mathcal{H}^m$-countably rectifiable Borel sets $E \subset X$. Hence there are Borel countably $\mathcal{H}^m$-rectifiable sets $E_i \subset X$ such that $\mathcal{H}^m(E_i) > M - 1/i$. It is easily to see that $E = \bigcup_i E_i$ satisfies the claim of the theorem. Uniqueness is easy. □

Definition 1.29. Let $E \subset \mathbb{R}^n$ be a Borel set and let $1 \leq m \leq n$ be integers. If $m < n$, the *integral geometric measure* $\mathcal{I}^m$ of E is defined as

$$\mathcal{I}^m(E) = \frac{1}{\beta(n,m)} \int_{p \in O^*(n,m)} \int_{y \in \operatorname{Im} p} N_p(y,E)\, d\mathcal{H}^m(y)\, d\vartheta^*_{n,m}(p), \tag{1.12}$$

where $O^*(n,m)$ is the space of orthogonal projections p from $\mathbb{R}^n$ onto m-dimensional linear subspaces of $\mathbb{R}^n$, $\operatorname{Im} p$ is the image of the projection and $\vartheta^*_{n,m}$ is the Haar measure on $O^*(n,m)$ invariant under the action of $O(n)$, normalized to have total mass 1. Moreover $N_p(y,E)$ is the Banach indicatrix, i.e. $N_p(y,E) = \#(p^{-1}(y) \cap E)$. The coefficient $\beta(n,m)$ will be defined later. If $m = n$ we simply define $\mathcal{I}^m(E) = \mathcal{H}^m(E)$.

Thus roughly speaking $\mathcal{I}^m(E)$ is defined as follows. We fix an m-dimensional subspace of $\mathbb{R}^n$ and denote by p the orthogonal projection from $\mathbb{R}^n$ onto that subspace. Next we compute the measure of the projection of the set E onto that subspace taking into account the multiplicity function N_p and then we average resulting measures over all possible projections $p \in O^*(n,m)$. Note that since the measure $\vartheta^*_{n,m}$ is invariant under rotations $O(n)$, $\mathcal{I}^m(E_1) = \mathcal{I}^m(E_2)$ if $E_1, E_2 \subset \mathbb{R}^n$ are isometric. We still need to define the coefficient $\beta(n,m)$. Let $[0,1]^m \subset \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n$ be the m-dimensional unit cube in $\mathbb{R}^n$. We define

$$\beta(n,m) = \int_{p \in O^*(n,m)} \int_{y \in \operatorname{Im} p} N_p(y,[0,1]^m)\, d\mathcal{H}^m(y)\, d\vartheta^*_{n,m}(p).$$

Clearly $\beta(n,m)$ is a positive constant and with its definition $\mathcal{I}^m([0,1]^m) = 1$. Note that $\mathcal{I}^m(Q) = \mathcal{H}^m(Q)$ for *any* m-dimensional cube in $\mathbb{R}^n$ regardless how the cube is positioned in the space. This follows from the $O(n)$ invariance of the measure $\vartheta^*_{n,m}$ and from the fact that both measures $\mathcal{I}^m$ and $\mathcal{H}^m$ scale in the same way under homothetic transformations. Hence $\mathcal{I}^m(E) = \mathcal{H}^m(E)$ if E is an m-dimensional polyhedron in $\mathbb{R}^n$. Indeed, up to a set of $\mathcal{H}^m$-measure zero such a polyhedron is the union of countably many m-dimensional cubes and if $\mathcal{H}^m(A) = 0$, then $\mathcal{I}^m(A) = 0$. This observation can be generalized to arbitrary countably $\mathcal{H}^m$-dectifiable sets.

Theorem 1.30 (Federer)**.** *If $E \subset \mathbb{R}^n$ is countably $\mathcal{H}^m$-rectifiable, $m \leq n$, then $\mathcal{I}^m(E) = \mathcal{H}^m(E)$.*

Proof. We can assume that $m < n$, because $\mathcal{I}^n = \mathcal{H}^n$ by the definition. It suffices to assume that E is a subset of an m-dimensional C^1-submanifold $\mathcal{M}^m \subset \mathbb{R}^n$. Indeed, the general case will follow from Theorem 1.10 and the fact that $\mathcal{H}^m(A) = 0$ implies that $\mathcal{I}^m(A) = 0$. Let p' be the restriction of $p \in O^*(n,m)$ to $\mathcal{M}^m$. the area formula yields

$$\int_E |J_{p'}(x)|\, d\mathcal{H}^m(x) = \int_{\operatorname{Im} p} N_p(y,E)\, d\mathcal{H}^m(y). \tag{1.13}$$

Let L be an m-dimensional affine subspace of $\mathbb{R}^n$. Let p'' be the restriction of p to L and let $|J_{p''}|$ be the Jacobian of the orthogonal projection p'' of L onto $\operatorname{Im} p$. Clearly $|J_p(x)| = |J_{p''}|$,where $L = T_x\mathcal{M}^m$ is regarded as an affine subspace of $\mathbb{R}^n$. Observe that

$$\int_{p\in O^*(n,m)} |J_{p''}|\, d\vartheta^*_{n,m}(p) = C(n,m)$$

is a constant that depends on n and m only. Indeed, the measure $\vartheta^*_{n,m}$ is invariant under rotations $O(n)$ and hence we can rotate L without changing the value of the integral, so that L is parallel to $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^n$. Hence (1.13) yields

$$\int_{p\in O^*(n,m)} \int_{y\in \operatorname{Im} p} N_p(y,E)\, d\mathcal{H}^m(y)\, d\vartheta^*_{n,m}(p) = C(n,m)\mathcal{H}^m(E).$$

Taking $E = [0,1]^m$ we see that $C(n,m) = \beta(n,m)$. $\square$

The next result is a celebrated structure theorem of Besicovitch-Federer which we state without proof.

Theorem 1.31 (Structure theorem)**.** *If $E \subset \mathbb{R}^n$, $\mathcal{H}^m(E) < \infty$, $m < n$ is purely $\mathcal{H}^m$-unrectifiable, then $\mathcal{I}^m(E) = 0$.*

Thus any set $E \subset \mathbb{R}^n$ with $\mathcal{H}^m(E) < \infty$ can be decomposed into a rectifiable part on which $\mathcal{H}^m = \mathcal{I}^m$ and a non-rectifiable part on which $\mathcal{I}^m = 0$. This says a lot about the structure of E, which explains the name of the theorem. This result also implies that $\mathcal{H}^m(E) \geq \mathcal{I}^m(E)$ for any Borel set $E \subset \mathbb{R}^n$.

One can construct Cantor type sets with $\mathcal{H}^m(E) > 0$, but $\mathcal{I}^m(E) = 0$. Clearly E must be purely $\mathcal{H}^m$-unrectifiable. However, the integral geometric measure can be used to detect the Hausdorff dimension of a set:

Theorem 1.32 (Mattila)**.** *If* $\dim_H E > m$*, then* $\mathcal{I}^m(E) = \infty$*. Hence* $\mathcal{I}^m(E) < \infty$ *implies that* $\dim_H E \leq m$*.*

The proof requires quite a lot of harmonic analysis and potential theory and we will not present it here.

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