HOW COMBINATORICS AND ANALYSIS INTERACT

1. Loomis-Whitney Inequality

Let X be a set of unit cubes in the unit cubical lattice in $\mathbb{R}^n$, and let $|X|$ be its volume. Let Π_j be the projection onto the $x_j^\perp$ hyperplane. The motivating question is: if Π_j is small for all j, what can we say about $|X|$?

Theorem 1.1 (Loomis-Whitney 50's). *If $|\Pi_j(X)| \le A$, then $|X| \lesssim A^{\frac{n}{n-1}}$.*

Remark. The sharp constant in the $\lesssim$ is 1. The original proof is by using H older's inequality repeatedly.

Define a *column* to be the set of cubes obtained by starting at any cube and taking all cubes along a line in the x_j-direction.

Lemma 1.2 (Main lemma). *If $\sum |\Pi_j(X)| \le B$, then there exists a column of cubes with between 1 and $B^{\frac{1}{n-1}}$ cubes of X.*

Proof. Suppose not, so every column has $> B^{\frac{1}{n-1}}$ cubes. This means that there are $> B^{\frac{1}{n-1}}$ cubes in some x_1-line. Taking the x_2-lines through those, there are $> B^{\frac{2}{n-1}}$ cubes in some x_1, x_2-plane, and so on. Repeating this $n-1$ times, we get $> B$ cubes in the $x_1, \ldots, x_{n-1}$-plane, a contradiction. □

Corollary 1.3. *If $\sum_j |\Pi_j(X)| \le B$, then $|X| \le B^{\frac{n}{n-1}}$.*

Proof. Let X' be X with its smallest column removed. Then $\sum |\Pi_j(X')| \le B - 1$, so by induction we get $|X'| \le (B-1)^{\frac{n}{n-1}}$, hence $|X| \le B^{\frac{1}{n-1}} + |X'|$. □

Note that Corollary 1.3 implies Theorem 1.1.

Theorem 1.4 (more general Loomis-Whitney). *If U is an open set in $\mathbb{R}^n$ with $|\Pi_j(U)| \le A$, then $|U| \lesssim A^{\frac{n}{n-1}}$.*

Proof. Take $U_\varepsilon \subset U$ be a union of ε-cubes in ε-lattice. Then $|U_\varepsilon| \lesssim A^{\frac{n}{n-1}}$ and $|U_\varepsilon| \to |U|$. □

Corollary 1.5 (Isoperimetric inequality). *If U is a bounded open set in $\mathbb{R}^n$, then*

$$Vol_n(U) \lesssim Vol_{n-1}(\partial U)^{\frac{n}{n-1}}.$$

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Proof. By projection onto translates of each x_j-hyperplane, we see that $|\Pi_j(U)| \leq \mathrm{Vol}_{n-1}(\partial U)$, so we may apply Theorem 1.4. □

Remark. The fact that U was bounded was used to define the projection of U onto translates of each x_j-hyperplane.

2. Sobolev Inequality

Let $u \in C^1_{\mathrm{comp}}(\mathbb{R}^n)$ satisfy $\int |\nabla u| = 1$. How big can u be? We would like the find the right notion of size for u that answers this question.

Theorem 2.1 (Sobolev inequality)**.** *If $u \in C^1_{comp}(\mathbb{R}^n)$, then*

$$||u||_{L^{\frac{n}{n-1}}} \lesssim ||\nabla u||_{L^1}.$$

Here, the L^p-norm $||u||_{L^p}$ is given by

$$||u||_{L^p} = \left(\int |u|^p\right)^{1/p}$$

so that $||h \cdot \chi_A||_p = h \cdot |A|^{1/p}$. For some context about L^p-norms, for a function u, let $S(h) := \{x \in \mathbb{R}^n \mid |u(x)| > h\}$.

Proposition 2.2. *If $||u||_p \leq M$, then $|S(h)| \leq M^p h^{-p}$.*

Proof. Just estimate $M^p = \int |u|^p \geq h^p |S(h)|$. □

We now prove the Sobolev inequality. A first try is the following bound.

Lemma 2.3. *If $u \in C^1_{comp}(\mathbb{R}^n)$, $|\Pi_j(S(h))| \leq h^{-1} \cdot ||\nabla u||_{L^1}$.*

Proof. For $x \in S(h)$, take a line ℓ in the x_j-direction. It eventually reaches a point x' where $u = 0$, so $\int_\ell |\nabla U| \geq h$ by the fundamental theorem of calculus. This means that

$$||\nabla u||_{L^1} \geq \int_{\Pi_j(S(h))\times\mathbb{R}} |\nabla u| = \int_{\Pi_j(S(h))} \int_{\mathbb{R}} |\nabla u| dx_j dx_{\mathrm{other}} \geq |\Pi_j(S(h))| \cdot h.$$ □

If we apply Theorem 1.4 to the output of Lemma 2.3, we see that

$$|S(h)| \lesssim h^{-\frac{n}{n-1}} \cdot ||\nabla u||^{\frac{n}{n-1}},$$

which looks like the output of Proposition 2.2. So we would like to establish something like the converse in this case. For this, we require a more detailed analysis.

Lemma 2.4 (Revised version of Lemma 2.3)**.** *Let $S_k := \{x \in \mathbb{R}^n \mid 2^{k-1} \leq |u(x)| \leq 2^k\}$. If $u \in C^1_{comp}(\mathbb{R}^n)$, then we have*

$$|\Pi_j S_k| \lesssim 2^{-k} \int_{S_{k-1}} |\nabla u|.$$

Proof. For $x \in S_k$, draw a line ℓ in the x_j-direction through x. There is a point x' on ℓ with $u(x') = 0$. Between x and x', there is some region on ℓ where $|u|$ is between 2^{k-2} and 2^{k-1}. Then we see that along each such ℓ, we have

$$\int_{S_{k-1}\cap\ell} |\nabla u| \geq \frac{1}{4}2^k.$$

Summing this along all ℓ perpendicular to a translate of the x_j-hyperplane yields the result. ☐

Corollary 2.5. $|S_k| \lesssim 2^{-k\frac{n}{n-1}} \left(\int_{S_{k-1}} |\nabla u|\right)^{\frac{n}{n-1}}$.

Proof. Put Lemma 2.4 into Theorem 1.4. ☐

Proof of Theorem 2.1. Take the estimate

$$\int |u|^{\frac{n}{n-1}} \sim \sum_{k=-\infty}^{\infty} |S_k| 2^{k\frac{n}{n-1}} \lesssim \sum_k \left(\int_{S_{k-1}} |\nabla u|\right)^{\frac{n}{n-1}} \leq \left(\int_{\mathbb{R}^n} |\nabla u|\right)^{\frac{n}{n-1}},$$

where in the last step we move the sum inside the $\frac{n}{n-1}$-power. ☐

Remark. The sharp constant in Theorem 2.1 is provided by a smooth approximation to a step function where the width of the region of smoothing is very small.

3. L^p estimates for linear operators

If $f, g : \mathbb{R}^n \to \mathbb{R}$ or $\mathbb{C}$, define the *convolution* to be

$$(f \star g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy.$$

We can explain this definition by the following story. Suppose there is a factory at 0 which generates a cloud of pollution centered at 0 described by $g(-y)$. If the density of factories at x is $f(x)$, then the final observed pollution is $f \star g$.

We would like to study linear operators like $T_\alpha f := f \star |x|^{-\alpha}$, which means explicitly that

$$T_\alpha f(x) = \int f(y)|x-y|^{-\alpha} dy.$$

We will take α in the range $0 < \alpha < n$, so that if $f \in C^0_{\text{comp}}$ then the integral converges for each x. Operators like these occur frequently in PDE. Another example is the initial value problem for the wave equation.

Example. Let us first see how T_α behaves on some examples for f.

1. χ_{B_1}, where B_r is the ball of radius r. We see that

$$|T_\alpha \chi_{B_1}(x)| \sim \begin{cases} 1 & |x| \leq 1 \\ |x|^{-\alpha} & |x| > 1. \end{cases}$$

2. χ_{B_r}. We see that

$$|T_\alpha \chi_{B_r}(x)| \sim \begin{cases} r^n \cdot r^{-\alpha} & |x| \leq r \\ r^n \cdot |x|^{-\alpha} & |x| > r. \end{cases}$$

2.1 δ, the delta function. Morally, this is given by $\lim_{n\to\infty} r^{-n}\chi_{B_r}$.

A question we would like to ask about T_α is the following. Fix α and n. For which p, q is there an inequality

$$||T_\alpha f||_q \lesssim ||f||_p \tag{1}$$

for all choices of f?

In some sense, this measures how much bigger T_α can make f. First, we determine the answer in Examples 1 and 2. For Example 1, $||\chi_{B_1}||_p \sim 1$, and

$$||T_\alpha \chi_{B_1}||_1^1 \sim \int_{\mathbb{R}^n} (1+|x|)^{-\alpha q} dx,$$

which is finite if and only if $\alpha q > n$. So (1) holds in Example 1 if and only if $\alpha q > n$. Let us assume this from now on.

For Example 2, $||\chi_{B_r}||_p \sim r^{n/p}$. For $||T_\alpha \chi_{B_r}||_q$, the value is given by two terms, one coming from the ball $|x| \leq r$ and the outside tail. The condition $\alpha q > n$ says that the contribution of the tail is finite, so we get the estimate

$$||T_\alpha \chi_{B_r}||_q \sim ||r^{n-\alpha}\chi_{B_r}||_q \sim r^{n-\alpha+n/q}.$$

Thus, we conclude that (1) holds in Example 2 if and only if $\alpha \cdot q > n$ and $r^{n/p} \lesssim r^{n-\alpha+n/q}$ for all $r > 0$. The latter condition is equivalent to $n/p = n - \alpha + n/q$.

For a general linear operator T, we would like to ask whether

$$||Tf||_q \lesssim ||f||_p$$

under the conditions that $\alpha \cdot q > n$ and $n/p = n - \alpha + n/q$. If the answer is yes, we conclude that the characteristic functions of balls are in some sense typical for the action of T; otherwise, we would like to understand which functions f this fails for.