An L^p differentiable non-differentiable function

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Abstract. There is a a set E of positive Lebesgue measure and a function nowhere differentiable on E which is differentible in the L^p sense for every positive p at each point of E. For every $p \in (0, \infty]$ and every positive integer k there is a set $E = E(k, p)$ of positive measure and a function which for every $q < p$ has k L^q Peano derivatives at every point of E despite not having an L^p kth derivative at any point of E.

A real-valued function f of a real variable is *differentiable at x* if there is a real number $f'(x)$ such that

$$|f(x+h) - f(x) - f'(x)h| = o(h) \qquad \text{as } h \to 0.$$

Fix $p \in (0, \infty)$. A function is *differentiable in the L^p sense at x* if there is a real number $f'_p(x)$ such that

$$\left\| f(x+h) - f(x) - f'_p(x)h \right\|_p = o(h) \qquad \text{as } h \to 0,$$

where $\|g(h)\|_p = \left(\frac{1}{h} \int_{-h}^{h} |g(t)|^p \, dt \right)^{1/p}$.

We have an infinite family of generalized first derivatives indexed by the parameter p. Most generalized derivatives are not equivalent to the ordinary derivative at a single point, but many are equivalent on an almost everywhere basis. For example, the symmetric derivative, defined by $f'_s(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$, is zero for the absolute value function at $x = 0$ even though that function is not differentiable at $x = 0$, but this phenomenon which occurs at the single point $x = 0$ never occurs on a set of positive measure: there cannot exist a set of positive measure E and a function g so that $g'_s(x)$ exists at all points of E and $g'(x)$ exists at no points of E.[**K**, page 217] In this sense the symmetric derivative is equivalent to ordinary differentiation. So a natural question to ask here is whether in this sense the various L^p derivatives are different from ordinary differentiation and from one another. The point of this paper is to answer "yes" to this question.

If $p_1 < p_2$ and f is L^{p_2} differentiable at x, then f is L^{p_1} differentiable at x; since by Holder's inequality,

$$\left\| f(x+h) - f(x) - f'_{p_2}(x)h \right\|_{p_1} \le 2^{\frac{1}{p_1} - \frac{1}{p_2}} \left\| f(x+h) - f(x) - f'_{p_2}(x)h \right\|_{p_2} = o(h)$$

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so that $f'_{p_1}(x)$ exists and equals $f'_{p_2}(x)$. It may be useful to think of a scale of derivatives indexed by p, the higher the value of p, the better the behavior. The best behavior, ordinary differentiability, occurs when $p=\infty$. Sometimes the scale is extended by placing the approximate derivative at $p=0$.

A function f has a *kth Peano derivative at* x if there are real numbers $f^i(x), i = 0,1,2,\dots,k$, such that

$$\left| f(x+h) - f^0(x) - f^1(x)h - \cdots - f^k(x)\frac{h^k}{k!} \right| = o\left(h^k\right) \qquad \text{as } h \to 0.$$

Fix $p \in (0,\infty)$. A function f has a *kth Peano derivative in the* L^p *sense at* x if there are real numbers $f_p^i(x), i = 0,1,2,\dots,k$, such that

$$\left\| f(x+h) - f_p^0(x) - f_p^1(x)h - \cdots - f_p^k(x)\frac{h^k}{k!} \right\|_p = o\left(h^k\right) \qquad \text{as } h \to 0.$$

The same p-scale mentioned for first derivatives also holds for kth Peano ones as well. Whatever the value of k, when $p \neq q$, L^p kth order Peano differentiability is not a.e. equivalent to L^q kth order Peano differentiability; this is the content of Theorems 2 and 3 below.

The first extensive discussion of the L^p Peano derivative that I am aware of appeared in reference [**CZ**]. Differentiation in the L^p sense for the characteristic function of a set is very closely related to the concept of super density, which is discussed in reference [**LMZ**].

THEOREM 1. *There is a set E of positive Lebesgue measure and a function nowhere differentiable on E which is differentiable in the L^p sense for every positive p at each point of E.*

PROOF. Note that the characteristic function of the rational numbers provides a trivial example since it is nowhere differentiable, but is L^p differentiable to 0 at every irrational point. To avoid such a triviality, we further specify that every element of the equivalence class defining the L^p function should also fail to be differentiable on E, i.e. changing the function on a set of measure 0 should not improve the differentiability of the function.

Order the rational numbers into a sequence and for $n = 1,2,\dots$, let G_n be an open interval centered at the nth rational of length 2^{-n^2}. Let C be the complement of $\cup_i G_i$. Since $|\cup_i G_i| \le \sum 2^{-n^2} < \infty$, $|C| = \infty$. Let χ be the characteristic function of C. Let $I(x,h) = [x-h, x+h]$.

1. χ is not differentiable at almost every point of C. Let $C_1 = \{x \in C : x$ is a point of density of $C\}$. Note that $|C \backslash C_1| = 0$. Let $x \in C_1$. If h is sufficiently small, $|I(x,h) \cap C| > h/2$ so the essential lim sup of χ is 1. On the other hand, since for any $h > 0$, the interval $I(x,h)$ contains a rational number and hence a subinterval on which $\chi = 0$ so the essential lim inf of χ is 0. Thus χ has no limiting value at x and so all the more is not differentiable there.

2. χ does have a zero L^p derivative for every positive p at almost every point of C_1. This full measured subset of C_1 will be a set of positive measure and is the set promised in the statement of the theorem. Suppose that for each $p > 0$, χ is L^p differentiable on C^p, a full-measured subset of C_1. Then letting $A_p = C_1 \backslash C^p$, $|A_p| = 0$. Let $A = \cup A_n$ and $C_2 = C_1 \backslash A$. Then χ is not differentiable on C_2, but is L^p differentiable on C_2 for every $p > 0$, since by definition χ is $L^{\lceil p \rceil}$ differentiable and

Holder's inequality implies L^p differentiability since $p \le \lceil p \rceil$. Thus it is sufficient to fix p and show that A_p has measure 0.

On C^p we have

$$\left(\frac{1}{h} \int_{-h}^{h} |\chi(x+t) - \chi(x) - 0 \cdot t|^p \, dt \right)^{1/p} = o(h),$$

or, equivalently,

$$\int_{-h}^{h} |\chi(x+t) - \chi(x) - 0 \cdot t|^p \, dt = o\left(h^{p+1}\right), \tag{0.1}$$

as $h \to 0$. To show that $|A_p| = 0$, it suffices to show that for each $\epsilon > 0$, $|A_p| < \epsilon$. Fix such an ϵ and pick n so large that

$$n > p + 1 \tag{0.2}$$

and so large that $(n+1)\, 2^{-n+1} < \epsilon$. Let $B_p = \cup_{i=1}^{n} \{x \in C_1 : \text{dist}(x, G_i) < 2^{-n}\} \cup \left(\cup_{j>n} \left\{x \in C_1 : \text{dist}(x, G_j) < 2^{-j}\right\}\right)$. Then $|B_p| \le (2 \cdot 2^{-n})\, n + \sum_{j>n} 2 \cdot 2^{-j} = (n+1)\, 2^{-n+1} < \epsilon$, so it remains to show that (0.1) holds for $x \in C_1 \backslash B_p$ so that $A_p \subset B_p$. Since $x \in C$, $\chi(x) = 1$ and the absolute value of the left hand side is

$$\ell = \int_{x-h}^{x+h} |\chi(s) - 1|^p \, ds = |C^c \cap I|,$$

where $I = [x-h, x+h]$. Assume $h < 2^{-n}$. Let G_j be the first complementary interval that meets I. Since $x \notin B_p$, $j > n$. Since $2^{-(i+1)^2} \le \frac{1}{2} 2^{-i^2}$ and $1 + 2^{-1} + 2^{-2} + \cdots = 2$,

$$\begin{aligned} \ell &\le |\cup_{i \ge j} G_i| \le \sum_{i \ge j} 2^{-i^2} \le 2 \cdot 2^{-j^2} \\ &= 2\left(2^{-j}\right)^j \le 2h^j. \end{aligned}$$

The last inequality holds because $x \notin B_p$ implies $2^{-j} \le dist(x, G_j)$ and $G_j \cap I(x,h) \ne \varnothing$ implies $dist(x, G_j) \le h$. Since $j > n > p+1$, h^j is $o\left(h^{p+1}\right)$ and relation (0.1) follows. □

This example splits ordinary differentiation from all finite L^p differentiation. Given any $p > 0$, we can also create a function f_p for which there is a set E of positive measure on which f_p is differentiable in the L^q sense for every $q < p$; but f_p is not differentiable at any point of E in the L^p sense. We do this by making a "fat Cantor set" the ith stage complementary open intervals being centered at all $(2j+1)/2^n$ and having measure $2^{-i(p+1)}$. The details are slightly more complicated. Theorem 3 below does this and a little bit more.

Note that the following theorem in particular separates the kth Peano derivative from all L^p kth Peano derivatives, $0 < p < \infty$.

Theorem 2. *There is a set E of positive Lebesgue measure and a function having no limit at each point of E which has a kth Peano derivative in the L^p sense for every natural number k and every positive p at each point of E.*

Proof. The function χ and the subset of C of full measure appearing in the proof of the previous theorem are sufficient for this theorem also. In fact, for $x \in C$ set $f_p^0(x) = f(x) = 1$ for $p \in (0, \infty)$; and set $f_p^i(x) = 0$, for $i = 1, 2, \ldots$ and

$p\in(0,\infty)$. The defining condition for having a kth L^p Peano derivative at such an x is

$$\left(\frac{1}{h}\int_{-h}^{h}\left|f\left(x+t\right)-1-0t-\cdots-0\frac{t^k}{k!}\right|^p dt\right)^{1/p}=o\left(h^k\right)$$

or

$$\int_{-h}^{h}\left|f\left(x+t\right)-1\right|^p dt=o\left(h^{kp+1}\right).$$

The reasoning and calculations above remain unchanged, except that n must be chosen larger than $kp+1$ instead of larger than $p+1$. $\square$

Theorem 3. *Let $p>0$ and k be a positive integer. There is a set E of positive Lebesgue measure and a bounded function nowhere Peano differentiable of order k in the L^p sense on E which is Peano differentiable of order k in the L^q sense for every positive $q<p$ at each point of E.*

Proof. The case $p=\infty$ and $k=1$ was treated first. Then followed the case $p=\infty$ and general k. The required example for p finite is the characteristic function of a "fat Cantor set" with the nth stage complementary open intervals being centered at all $(2j+1)/2^n$ and having measure $c_{kp}2^{-n(kp+1)}$, where $c_{kp}=2^{kp}-1$. The details follow.

For $N=1,2,3,\ldots$, the complementary intervals of rank N will be the open intervals G_{iN}, $i=1,2,\ldots,2^{N-1}$, where the center of G_{iN} is centered at $(2i-1)/2^N$ and $|G_{iN}|=c_{kp}2^{-N(kp+1)}$. The center to center distance between contiguous intervals of rank N is $2\cdot\frac{1}{2^N}=2^{1-N}$. It will be convenient to work on $[0,1]$ thought of as a torus so that in particular G_{1N} and $G_{(2^N-1)N}$ are contiguous.

Let $C=\left(\cup_{n=1}^{\infty}\cup_{i=1}^{2^{n-1}}G_{in}\right)^c$, $\chi=$ characteristic function of C, $x\in C$, and $h>0$. Note $|C|=1-|C^c|$ and $|C^c|\le\sum_{n=1}^{\infty}2^{n-1}c_{kp}2^{-n(kp+1)}=1/2$, so $|C|>0$.

Then for any $p>0$,

$$\begin{aligned}\int_{-h}^{h}\left|\chi\left(x+t\right)-\chi\left(x\right)-0\cdot t-0\frac{t^2}{2}-\cdots-0\frac{t^k}{k!}\right|^p dt&=\int_{-h}^{h}\left|\chi\left(x+t\right)-1\right|^p dt\\&=\left|I\cap C^c\right|,\end{aligned}\tag{0.3}$$

where $I=[x-h,x+h]$. Find m so that $2^{-m}\le h<2^{-m+1}$. We have for some j, $\dfrac{j}{2^m}\le x<\dfrac{j+1}{2^m}$. The complementary interval G centered at the element of $\left\{\dfrac{j}{2^m},\dfrac{j+1}{2^m}\right\}$ having even numerator has rank at most $m-1$ so that the half of G interior to $\left[\dfrac{j}{2^m},\dfrac{j+1}{2^m}\right]$ has measure at least $\frac{1}{2}\frac{c_{kp}}{2^{(kp+1)(m-1)}}$. Thus

$$\left|I\cap C^c\right|\ge\frac{c_{kp}}{2}\left(\frac{1}{2^{m-1}}\right)^{kp+1}\ge\frac{c_{kp}}{2}h^{kp+1}.$$

We show below that when $q<p$, the first k Peano L^q derivatives of χ are 0 at a.e. $x\in C$, so by Holder's inequality, if the L^p Peano derivatives exist at all, they must be zero. However, combining this inequality with equation (0.3) shows that

$$\left(\frac{1}{h}\int_{-h}^{h}\left|\chi\left(x+t\right)-\chi\left(x\right)-0\cdot t-0\frac{t^2}{2!}-\cdots-0\frac{t^k}{k!}\right|^p dt\right)^{\frac{1}{p}}>\left(\frac{c_{kp}}{2}\right)^{\frac{1}{p}}h^k$$

which is not $o\left(h^k\right)$ so χ does not have a kth L^p Peano derivative at a.e. $x \in C$.

By the same reasoning as in the L^∞ case above, it is enough to prove that if $q < p$ are fixed, and if $\epsilon > 0$ is fixed, then there is a set $A = A(p, q, \epsilon)$, $A \subset C$ such that $|A| < \epsilon$ and for every $x \in C \backslash A$,

$$|[x-h, x+h] \cap C^c| = o\left(h^{kq+1}\right).$$

(In the reduction to the sufficiency of this assertion, one needs to establish this estimate directly for a countable set of q's that belong to $(0, p)$ and approach p.)

Pick n such that $\frac{3}{n} < \epsilon$. Then for each positive integer i, let A_i be the points of C which are "close" to the complementary intervals of rank i; explicitly, for rank i, $i \le n$: let $A_i = \cup_{k=1}^{2^{i-1}} \{x \in C : dist(x, G_{ki}) < \dfrac{1}{n^2}\dfrac{1}{2^n}\}$; and for rank j, $j > n$: let $A_j = \cup_{k=1}^{2^{j-1}} \{x \in C : dist(x, G_{kj}) < \dfrac{1}{j^2}\dfrac{1}{2^j}\}$. Let $A = \cup_{i=1}^{\infty} A_i$, then

$$\begin{aligned}
|A| &\le \sum_{i=1}^{n} |A_i| + \sum_{i=n+1}^{\infty} |A_i| \\
&= \frac{2}{n^2}\frac{1}{2^n}\left(\sum_{j=1}^{n} 2^{j-1}\right) + \sum_{i=n+1}^{\infty} \frac{2}{i^2}\frac{1}{2^i} 2^{i-1} \\
&= \frac{2}{n^2}\frac{1}{2^n}\left(2^n - 1\right) + \sum_{i=n+1}^{\infty} \frac{1}{i^2} \\
&\le \frac{2}{n^2} + \int_n^{\infty} x^{-2} dx = \frac{2}{n^2} + \frac{1}{n} < \frac{3}{n} < \epsilon.
\end{aligned}$$

Let $x \in C \backslash A$ and fix $h > 0$ so small that $h < \dfrac{1}{n^2}\dfrac{1}{2^n}$. Let $I = [x-h, x+h]$. Let G be the first complementary interval intersecting I and let ℓ be the rank of G so that $|G| = \dfrac{c_{kp}}{2^{(kp+1)\ell}}$. Note that $\ell \ge n+1$ since h is too small to allow any G of rank $\le n$ to intersect I. Since G intersects I,

$$h > \frac{1}{\ell^2}\frac{1}{2^\ell}. \tag{0.4}$$

Let $m = \lfloor \log_2 h \rfloor$ so that $2^{-m} \le h < 2^{-m+1}$,

$$m \lesssim \log(1/h). \tag{0.5}$$

Let $a(s)$ be the number of elements of rank s that intersect I. Excluding the left-most and right-most elements, $a(s) - 2$ centers of rank s intervals are in I and each of the $a(s) - 3$ distances between these centers is $2\frac{1}{2^s}$, whence $(a(s) - 3) 2^{-s+1} \le 2h$, so

$$a(s) \le 3 + 2^s h. \tag{0.6}$$

Since $h < 2^{1-m}$, it follows that

$$\text{if } s < m, \text{ then } a(s) \le 4. \tag{0.7}$$

If $\ell < m$, use inequalities (0.7) and (0.6) to obtain

$$
\begin{aligned}
|I \cap C^c| &\leq \sum_{s=\ell}^{\infty} a(s)\, c_{kp} 2^{-(kp+1)s} \\
&\leq \sum_{s=\ell}^{m-1} 4 \cdot c_{kp} 2^{-(kp+1)s} + 3 \sum_{s=m}^{\infty} c_{kp} 2^{-(kp+1)s} + h \sum_{s=m}^{\infty} c_{kp} 2^{-kps} \qquad (0.8) \\
&\lesssim 2^{-(kp+1)\ell} + h 2^{-kpm},
\end{aligned}
$$

where $A \lesssim B$ means that for some constant $C(k,p)$, $A \leq C(k,p)\, B$. From this and inequalities (0.4) and (0.5) we have

$$
\begin{aligned}
|I \cap C^c| &\lesssim \ell^{2kp+2} \left(\frac{1}{\ell^2 2^{\ell}} \right)^{kp+1} + h \left(\frac{1}{2^{m-1}} \right)^{kp} \\
&\leq m^{2kp+2} h^{kp+1} + h^{kp+1} \\
&\lesssim \log^{2kp+2}(1/h)\, h^{kp+1} \\
&= o\left(h^{kq+1}\right).
\end{aligned}
$$

If $\ell \geq m$, the estimate is even simpler; we get

$$
|I \cap C^c| \leq \sum_{s=m}^{\infty} a(s)\, c_{kp} 2^{-(kp+1)s} \lesssim h^{kp+1} = o\left(h^{kq+1}\right).
$$

□

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