APPLICATIONS OF FOURIER ANALYSIS TO NON-ANALYTIC FIELDS

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ABSTRACT. Fourier analysis, a classical topic in analysis, can be applied to many nonanalytic fields. This paper explores Fourier analysis in multiple domains in order to prove several nonanalytic results. Fourier analysis will be explored on a closed interval, the real line, and a finite Abelian group. We use methods from Fourier analysis to prove Weyl's equidistribution theorem and the Poisson summation formula.

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1. INTRODUCTION

It is often very useful to approximate a function by something more familiar and well behaved, and this is the mindset in which we introduce Fourier analysis. It can be helpful to compare a Fourier Series to the Taylor expansion of a function, the former being the trigonometric analogue to the polynomial Taylor series. We will begin with an introduction to the Fourier series of a function, defined on a closed interval, using the well behaved trigonometric polynomials as our building blocks for approximating functions with. We will use Fourier Series to prove the celebrated Weyl's equidistribution theorem. In section 4, we move to the real number line in the Fourier transform, proving a remarkable identity known as the Poisson summation formula. In section 5, we move into the realm of finite Abelian groups, proving that Fourier analysis analogously holds in that domain. A common theme throughout the paper will be the use of Fourier analysis to approximate complicated functions with better behaved ones.

2. DEFINITION AND PROPERTIES OF THE FOURIER SERIES

We begin by defining the Fourier series for a function on a closed interval. This definition holds for all functions that are Riemann-integrable. Essentially, a Fourier

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series approximates integrable functions using linear combinations of the family of functions $e^{2\pi ikx}$, $k \in \mathbb{Z}$.

Definition 2.1. The nth Fourier coefficient of a complex valued function defined on $[a, b]$ is defined as

$$\hat{f}(n) = \frac{1}{b-a}\int_a^b f(x)e^{\frac{-2\pi inx}{b-a}}dx.$$

Definition 2.2. The Fourier series of a complex valued function defined on $[a, b]$ with Fourier Coefficients $\hat{f}(n)$ is defined as

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{\frac{2\pi inx}{b-a}}.$$

We refer to the partial sums of the Fourier series as $S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{\frac{2\pi inx}{b-a}}$.

Definition 2.3. A trigonometric polynomial $P(x)$ is a function of the following form:

$$\sum_{n=-N}^{N} a_n e^{inx}$$

where $a_n \in \mathbb{R}$ for all $n \in [-N, N]$.

We frequently refer to functions defined on the domain of a circle, that is, on a closed interval of length 2π. It is clear that all results proved on the domain of a circle can be generalized to the domain of any closed interval $[a, b]$.

We begin with a lemma regarding the approximation of integrable functions by continuous functions.

Lemma 2.4. *Suppose f is an integrable function on the circle, and there exists B such that $|f(x)| < B$ for all $x \in [0, 2\pi]$. Then there exists a sequence of continuous functions $\{f_k\}_{k=1}^{\infty}$ such that for all $k \in \mathbb{N}$, we have*

$$\sup_k |f_k(x)| \le B \tag{2.5}$$

and

$$\lim_{k\to\infty}\int_0^{2\pi} |f(x) - f_k(x)|dx = 0. \tag{2.6}$$

Proof. Since f is integrable, it can be approximated in integral by a step function from below. Furthermore, we make this step function continuous by connecting the points of discontinuity. For any ϵ we can make the difference between this continuous approximation and f less than ϵ, and thus we can create a sequence of continuous functions by choosing $\epsilon = \frac{1}{k}$ for $k \in \mathbb{N}$. By construction, these functions satisfy (2.5) and (2.6).

□

Proposition 2.7. *Define $L^2(0, 2\pi)$ to be the space of integrable functions on $[0, 2\pi]$ with the following inner product:*

$$(f, g) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}dx.$$

Consequently,

$$||f||^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = (f, f).$$

The set $\{e^{ikx} \mid k \in \mathbb{Z}\}$ *is an orthonormal basis for* $L^2(0, 2\pi)$.

Proof. We define $e_n(x) = e^{inx}$. We first show that $\{e_n(x) \mid n \in \mathbb{Z}\}$ is orthonormal. We have:

$$\frac{1}{2\pi}\int_0^{2\pi} |e^{inx}|^2 dx = 1$$

and for all n, n', $n \neq n'$

$$\int_0^{2\pi} e^{inx} e^{in'x} dx = \int_0^{2\pi} e^{inx+in'x} dx = 0$$

because $n+n'$ is an integer and e^{ikx} is periodic when $k \in \mathbb{Z}$. Then the $\{e_n(x) \mid n \in \mathbb{Z}\}$ is orthonormal with respect to the above inner product.

By Weierstrass's approximation theorem, if f is a continuous function on $[0, 2\pi]$, then for any $\epsilon > 0$ we can find a trigonometric polynomial $P(x)$, a linear combination of $\{e_n \mid n \in \mathbb{Z}\}$, such that $\sup_x |P(x) - f(x)| < \epsilon$. If $f \in L^2(0, 2\pi)$, because f is integrable, we can use lemma 2.4 to approximate f in integral by a continuous step function f^*. Then we have $||P - f|| < ||P - f^*|| + ||f^* - f||$, and thus

$$||P - f|| < \frac{1}{2\pi}\int_0^{2\pi} (|P(x) - f^*(x)|^2 + |f^*(x) - f(x)|^2) dx < \frac{1}{2\pi}(2\epsilon^2) < \epsilon.$$

Thus the span of $\{e_n \mid n \in \mathbb{Z}\}$ is dense in $L^2(0, 2\pi)$. We just need to show that $e_n(x) \in L^2(0, 2\pi) \quad \forall n \in \mathbb{Z}$ to complete the proof.

We have $S_N(f) = \sum_{n=-N}^{N} a_n e_n$. Since $\{e_n \mid n \in \mathbb{Z}\}$ is orthonormal, $f - S_N(f)$ is orthogonal to the set $\{e_n \mid n \in [-N, N]\}$, because $S_N(f)$ is the orthogonal projection of f onto the subspace spanned by $\{e_n \mid n \in [-N, N]\}$. For any set of complex numbers $b_{-N}, b_{-N+1}, \cdots, b_{N-1}, b_N$ we have that $(f - \sum_{n=-N}^{N} a_n e_n)$ and $\sum_{n=-N}^{N} b_n e_n$ are orthogonal. This holds for the set of complex numbers $a_{-N}, a_{-N+1}, \cdots, a_{N-1}, a_N$, and thus the Pythagorean theorem for the decomposition $f = f - S_N(f) + S_N(f)$ yields $||f||^2 = ||f - S_N(f)||^2 + ||S_N(f)||^2$. Since each e_n is normal, we have that $||S_N(f)||^2 = ||\sum_{n=-N}^{N} a_n e_n||^2 = \sum_{n=-N}^{N} |a_n|^2$, and therefore $||f||^2 = ||f - S_N(f)||^2 + \sum_{n=-N}^{N} |a_n|^2$, which is finite. Thus $\{e^{ikx} \mid k \in \mathbb{Z}\}$ is a basis of $L^2(0, 2\pi)$.

□

Remark 2.8. If $f \in L^2$, then the Fourier series of f is the decomposition of f over the infinite orthonormal basis of $L^2(0, 2\pi)$, $\{e^{ikx} \mid k \in \mathbb{Z}\}$.

With this groundwork, we can show that up to points of continuity, the Fourier coefficients of an integrable function uniquely define that function.

Theorem 2.9. *If* f *is an integrable function on the circle and* $\hat{f}(n) = 0 \quad \forall n \in \mathbb{Z}$, *then* $f(x) = 0$ *for all* $x \in [0, 2\pi]$ *whenever* x *is a point of continuity.*

Proof. Let p be a point of continuity for f, and suppose $f(p) > 0$ for the purpose of contradiction. By translating, it suffices to consider the case when $p = 0$. Since f is continuous at 0 there exists some $\delta > 0$ such that if $|x| < \delta$, then $f(x) > \frac{f(0)}{2}$. Let $t(x) = \epsilon + \cos(x)$, where $\epsilon > 0$ satisfies $|t(x)| < 1 - \frac{\epsilon}{2}$ if $\delta \leq |x|$. Then let $\gamma > 0$ be chosen such that $\gamma < \delta$ and $t(x) \geq 1 + \frac{\epsilon}{2}$ if $|x| < \gamma$. Let $t_k(x) = [t(x)]^k$. Clearly we have that $t_k(x)$ is a trigonometric polynomial for all k, and thus we can say $t_k(x) = \sum_{-N}^{N} c_n e^{inx}$. Since $\hat{f}(n) = 0$ for all n, we have that $\int_{-\pi}^{\pi} f(x)e^{-inx}dx = 0$ for all n, and by taking linear combinations we have

$$\int_{-\pi}^{\pi} f(x)t_k(x)dx = 0 \quad \forall k. \tag{2.10}$$

We observe the following:

$$\int_{-\pi}^{\pi} f(x)t_k(x)dx = \int_{\delta \leq |x|} f(x)t_k(x)dx + \int_{\gamma \leq |x| < \delta} f(x)t_k(x)dx + \int_{|x|<\gamma} f(x)t_k(x)dx.$$

We analyze each term separately. Let B be a bound for $|f(x)|$, which exists because f is integrable. We have that on the domain of the first term, $|f(x)|$ is bounded by B and since $|t(x)|$ is bounded by $(1 - \frac{\epsilon}{2})$, $|t_k(x)|$ is bounded by $(1 - \frac{\epsilon}{2})^k$. This term is finite as $k \to \infty$, so we have

$$\left| \int_{\delta \leq |x|} f(x)t_k(x)dx \right| < \infty \quad \forall k.$$

Furthermore, we have that $t(x)$, $t_k(x)$, and $f(x)$ are nonnegative when $|x| < \delta$. Thus we have

$$\int_{\gamma \leq |x| < \delta} f(x)t_k(x)dx > 0.$$

Finally, by our choice of γ we have that for $x \in [-\gamma, \gamma]$, $f(x) > \frac{f(0)}{2}$ and $t(x) > (1 + \frac{\epsilon}{2})$, which gives us $t_k(x) > (1 + \frac{\epsilon}{2})^k$. Therefore, we have

$$\int_{|x|<\gamma} f(x)t_k(x)dx \geq 2\gamma \frac{f(0)}{2} \left(1 + \frac{\epsilon}{2}\right)^k.$$

As $k \to \infty$, this expression is clearly unbounded and positive, as once again we have $t(x), t_k(x), f(x) \geq 0$. Putting this all together, we are summing a bounded term, a positive term, and a positive unbounded term, which tells us

$$\lim_{k \to \infty} \int_{-\pi}^{\pi} f(x)t_k(x)dx = \infty.$$

This contradicts (2.10), and hence $f(p) = 0$.

□

Using the theorem, we see that the Fourier coefficients determine the function up to points of discontinuity. An implicit corollary of this result is the following, further evidence that the Fourier series is a good representation of a function.

Corollary 2.11. *If f is a continuous function on the circle whose Fourier series is absolutely convergent on $[0, 1]$, then the Fourier series converges uniformly to f.*

Proof. If we let $g(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx} = \lim_{n\to\infty} S_n(x)$, then the Fourier coefficients are simply that of f. Note that we can swap integration and summing due to absolute convergence.

$$\int_0^1 \sum_{n=-\infty}^{\infty} \hat{f}e^{2\pi inx}e^{-2\pi imx} = \int_0^1 \sum_{n=-\infty}^{\infty} \hat{f}e^{2\pi ix(n-m)}dx \tag{2.12}$$

$$= \sum_{n=-\infty}^{\infty} \int_0^1 \hat{f}e^{2\pi ix(n-m)}dx \tag{2.13}$$

$$= \hat{f}(m) \tag{2.14}$$

We have (2.14) because the integral will cancel everywhere where $n \neq m$ since $n - m \in \mathbb{Z}\backslash\{0\}$, and when $n = m$ we just have $\hat{f}(m)$.

Furthermore, since $g = \lim_{N\to\infty} S_N(f)$ by uniform convergence, g is also continuous. Then by theorem 2.9 we have that by the continuity of f and g, $f = g$. □

We now examine other ways in which the Fourier series of a function f converges to f. We show that the Fourier series of f approximates f in the mean-square sense.

Theorem 2.15. *If f is integrable on the circle, then $\sum_{n=-\infty}^{\infty} \hat{f}(n)$ converges to f in the mean-square sense, that is,*

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_0^{2\pi} |f(x) - S_N(f)(x)|^2 dx = 0.$$

Proof. We use basis representation of the Fourier series, using the logic outlined in proposition (2.7). Recall that we have

$$||f||^2 = ||f - S_n(f)||^2 + \sum_{n=-N}^{N} |a_n|^2.$$

For any complex numbers $c_{-N}, c_{-N+1}, \cdots, c_{N-1}, c_N$ we note that the Pythagorean theorem yields

$$||f - \sum_{n=-N}^{N} c_n e_n||^2 = ||f - S_N(f)||^2 + ||\sum_{n=-N}^{N}(a_n - c_n e_n)||^2.$$

Since all quantities above are positive, clearly $||f - S_N(f)|| \leq ||f - \sum_{-N}^{N} c_n e_n||$. Note that geometrically this is simply because the orthogonal projection of f onto the subspace spanned by $\{e_n \mid n \in [-N, N]\}$ is precisely $S_N(f)$. Then by lemma 2.4 we can find a continuous function g to approximate f such that their norms with respect to L^2 are arbitrarily close. Then we can approximate g by a trigonometric polynomial P, similarly with norms arbitrarily close, and so f and P are arbitrarily close. Then if P has degree M, we have by the triangle inequality that $||f - S_N(f)|| < ||f - P|| < \epsilon$ for $N \geq M$ and arbitrarily small $\epsilon > 0$. And thus we have

$$\lim_{N\to\infty} ||f - S_N(f)|| = 0.$$

□

3. Weyl's Equidistribution Theorem

The motivation for this section comes from irrational rotations about a circle. What happens when we consecutively take rotations by arclength γ along a circle with circumference 1? One way to express this is as follows:

Definition 3.1. We define the integer part of x to be $[x]$, the floor function of x, and we define the fractional part of x as $\langle x\rangle = x - [x]$. Furthermore we can extend the definition of $\langle x\rangle$ to define an equivalence relation on $\mathbb{R}$, where two reals are defined to be equivalent if their fractional parts are equal.

This process is equivalent to looking at rotations on the circle. We first examine some weaker and simpler claims that can be made about the sequence $\langle\gamma\rangle, \langle 2\gamma\rangle, \langle 3\gamma\rangle \cdots$.

Remark 3.2. If γ is rational, then the set $\{\langle n\gamma\rangle \mid n \in \mathbb{N}\}$ is finite. This follows by letting $\gamma = \frac{p}{q}$ and realizing that the set of terms $\langle\gamma\rangle, \langle 2\gamma\rangle \cdots \langle q\gamma\rangle$ is repeated every q terms.

Remark 3.3. If γ is irrational, then the list $\{\langle n\gamma\rangle \mid n \in \mathbb{N}\}$ is infinite and has no two terms that are equivalent. This follows because if two elements $\langle a\gamma\rangle = \langle b\gamma\rangle$ are equivalent, then $(b-a)\gamma \in \mathbb{Z}$, which is impossible given γ is irrational.

We are interested in proving Kronecker's theorem, which states that $\{\langle n\gamma\rangle \mid n \in \mathbb{N}\}$ is dense in $[0,1)$. To prove this we will prove an even stronger result, known as Weyl's equidistribution theorem.

Definition 3.4. We define a sequence of numbers $a_1, a_2, \cdots \in [0,1)$ to be equidistributed if for all intervals (a,b) the number of terms in that interval is proportional to the size of that interval, that is,

$$\lim_{N\to\infty} \frac{|\{n \in \mathbb{N} \mid 1 \le n \le N, a_n \in (a,b)\}|}{N} = b - a. \tag{3.5}$$

Weyl's equidistribution theorem states the following:

Theorem 3.6. *If γ is irrational, then the sequence $\langle\gamma\rangle, \langle 2\gamma\rangle, \cdots$ is equidistributed.*

Proof. Let $(a,b) \subset [0,1)$ and let $\chi_{(a,b)}(x)$ be defined as follows:

$$\chi_{(a,b)}(x) = \begin{cases} 1 & x \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

Then we can reformulate (3.5) as follows:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(n\gamma) = b - a.$$

Now that we have formulated this in terms of functions, we notice that a difficulty arises in that χ is discontinuous. Therefore, our approach will be to approximate $\chi_{(a,b)}$ with continuous functions, and use the following lemma about continuous functions to finish the proof.

Lemma 3.7. *If f is continuous and of period 1, and γ is irrational, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(n\gamma) = \int_0^1 \chi_{(a,b)}(x)dx.$$

If we assume lemma 3.7, then we can choose continuous periodic functions g_ϵ and h_ϵ of period 1 to approximate $\chi_{(a,b)}$. We define g and h as follows on $[0,1)$:

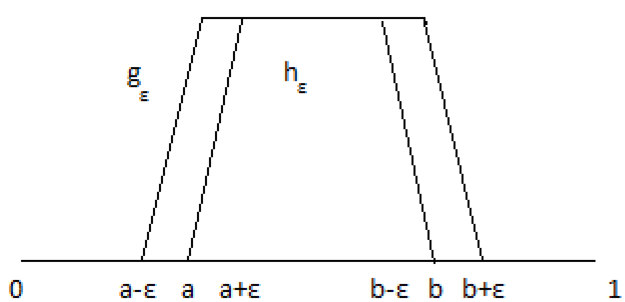
$$g_\epsilon(x) = \begin{cases} 0 & x < a-\epsilon \text{ or } x > b+\epsilon \\ 1 & a < x < b \\ \frac{x}{\epsilon} - \frac{a}{\epsilon} + 1 & a-\epsilon < x < a \\ -\frac{x}{\epsilon} + \frac{b}{\epsilon} - 1 & b < x < b+\epsilon \end{cases}$$

and

$$h_\epsilon(x) = \begin{cases} 0 & x < a \text{ or } x > b \\ 1 & a+\epsilon < x < b-\epsilon \\ \frac{x}{\epsilon} - \frac{a}{\epsilon} & a < x < a+\epsilon \\ -\frac{x}{\epsilon} + \frac{b}{\epsilon} & b-\epsilon < x < b \end{cases}$$

and extend these definitions periodically across the reals. We note that it is geometrically apparent that

$$b - a = \int_0^1 g_\epsilon(x)dx - \epsilon = \int_0^1 h_\epsilon(x)dx + \epsilon.$$



Further we have

$$\frac{1}{N}\sum_{n=1}^{N} h_\epsilon(n\gamma) \le \frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(n\gamma) \le \frac{1}{N}\sum_{n=1}^{N} g_\epsilon(n\gamma).$$

We take the limit to infinity and apply lemma 3.7 and find that for all $\epsilon > 0$ we have

$$b - a - \epsilon \le \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(n\gamma) \le b - a + \epsilon.$$

□

We now prove the lemma assumed.

Proof of Lemma 3.7. We first prove the lemma for when $f(x) = e^{2\pi i k x}$ for some $k \in \mathbb{Z}$. If $f(x) = 1$ the proof is trivial. If $k \neq 0$, then since γ is irrational, $e^{2\pi i k\gamma} \neq 1$ and thus we can apply the formula for a geometric series as follows:

$$\frac{1}{N}\sum_{n=1}^{N} f(n\gamma) = \frac{e^{2\pi i k\gamma}(1 - e^{2\pi i k N\gamma})}{N(1 - e^{2\pi i k\gamma})}.$$

As $N \to \infty$, we have $\frac{1}{N}\sum_{n=1}^{N} f(n\gamma) \to 0$, and we also have that $\int_0^1 f(x)dx = 0$, and thus the lemma is proved for this family of functions. This in fact proves the lemma for all trigonometric polynomials, because we can take linear combinations of functions in the above family to form all trigonometric polynomials.

We then recall that by Weistrass's theorem, we can approximate any continuous function by a trigonometric polynomial P such that $\sup|f(x) - P(x)| < \epsilon$. By the triangle inequality we have

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(n\gamma) - \int_0^1 f(x)dx\right| \leq \frac{1}{N}\sum_{n=1}^{N} |f(n\gamma) - P(n\gamma)| + \left|\frac{1}{N}\sum_{n=1}^{N} P(n\gamma) - \int_0^1 P(x)dx\right| + \int_0^1 |P(x) - f(x)|\, dx < 3\epsilon$$

which proves the lemma.

□

Remark 3.8. We note that as a corollary we obtain Kronecker's theorem, because every interval of $[0, 1)$ will contain an infinite number of terms by applying Weyl's equidistribution theorem.

Remark 3.9. We can extend this lemma to all Riemann integrable functions by approximating with step functions and proceeding as we have done before.

We examine the following geometric interpretations of what we have just proven. Consider the domain of a circle, a rotation funtion r_θ, and a Riemann integrable function f on the circle. Then when θ is an irrational multiple of π we have that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(r_\theta(x)) = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx.$$

by applying a change of variables to the above lemma.

We can generalize this concept with Weyl's criterion, which states the following:

Theorem 3.10. *For a sequence of numbers (a_n), the following two conditions are equivalent:*

(1) The sequence (a_n) is equidistributed in $[0, 1)$
(2) For all nonzero integers k,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k a_n} = 0$$

We note that $(2) \to (1)$ is essentially proven above in the proof of Weyl's equidistribution theorem and lemma 3.7. To prove the other direction, we note that this direction holds for characteristic functions, and through linear combinations of the characteristic functions, step functions in general. We can then take a step function that approximates $e^{2\pi i k a_n}$ to conclude the proof of $(1) \to (2)$.

4. Fourier Analysis on the Reals

We have thus far performed Fourier analysis in the realm of an interval (mostly the circle). We now extend our analysis to the entire real line. We begin with some preliminary definitions.

Definition 4.1. We define a function f on $\mathbb{R}$ to be of rapid decrease if f is infinitely differentiable and every derivative f^l satisfies

$$\sup |x|^k |f^l(x)| < \infty$$

for all $k, l \geq 0$. We denote the vector space of such functions as $\mathcal{S}(\mathbb{R})$.

Definition 4.2. We now define the Fourier transform for $f \in \mathcal{S}(\mathbb{R})$ as

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$$

We note that the Fourier transform of any function in $\mathcal{S}(\mathbb{R})$ is also in $\mathcal{S}(\mathbb{R})$.

Proposition 4.3. *If $f \in \mathcal{S}(\mathbb{R})$, then $\hat{f} \in \mathcal{S}(\mathbb{R})$.*

Proof. We rewrite the definition of the Fourier transform as follows:

$$\hat{f}(\xi) = \lim_{n \to \infty} \left(\int_{-n}^{0} f(x) e^{-2\pi i x \xi} dx + \int_{0}^{n} f(x) e^{-2\pi i x \xi} dx \right).$$

Then we can take the derivative of both sides, noting that the derivative of a function in $\mathcal{S}(\mathbb{R})$ must stay in that space. Further, since f is of rapid decrease, the entire right side is well-behaved, and the derivatives can pass through to prove that $(\hat{f})' \in \mathcal{S}(\mathbb{R})$. It follows easily that $\hat{f} \in \mathcal{S}(\mathbb{R})$.

□

Thus we can perform a Fourier transform on $\hat{f}$, which remarkably enough brings us back to the original function f.

Theorem 4.4 (Fourier Inversion)**.** *If f is a function of rapid decrease, then*

$$f(x) = \int_{-\infty}^{\infty} f(\xi) e^{2\pi i \xi x} d\xi.$$

Proof. We prove the theorem for $x = 0$ and generalize with translation.

We first note that Fubini's theorem implies that if $f, g \in \mathcal{S}(\mathbb{R})$, then $\int_{-\infty}^{\infty} f(x)\widehat{g(x)}dx = \int_{-\infty}^{\infty} \widehat{f(x)} g(x) dx$.

We define $G_\delta(x) = e^{-\pi \delta x^2}$ and define $K_\delta(x) = \hat{G_\delta}$. By Fubini's theorem we have

$$\int_{-\infty}^{\infty} f(x) K_\delta(x) dx = \int_{-\infty}^{\infty} \widehat{f}(\xi) G_\delta(\xi) d\xi.$$

The second integral clearly converges to $\int_{-\infty}^{\infty} f(\xi) d\xi$ as $\delta \to 0$. We note that $K_\delta(x) = \delta^{-1/2} e^{-\pi x^2/\delta}$. Then note that we have the following:

- $\int_{-\infty}^{\infty} K_\delta(x) dx = 1$
- $\exists M \quad$ such that $\int_{-\infty}^{\infty} |K_\delta(x)| dx \leq M$
- As $\delta \to 0$, for every $\eta > 0$, $\int_{|x|>\eta} |K_\delta(x)| \to 0$.

And thus $\int_{-\infty}^{\infty} f(x) K_\delta(x) dx = f(0)$ as $\delta \to 0$. This proves the Fourier Inversion theorem for $x = 0$, which we can then generalize with translation.

□

We now examine a beautiful result that comes from summing the Fourier coefficients of a function, the Poisson Summation Formula.

Definition 4.5. If f is a function of rapid decrease, then the periodization of f is defined by

$$F_1(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

The Poisson summation formula claims that this periodization of f is equivalent to the discrete analogue of the Fourier inversion.

Theorem 4.6 (Poisson Summation Formula)**.** *If f is a function of rapid decrease, then the following identity holds:*

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}.$$

Proof. We will use a result we have proved previously. Since f is a continuous function, both sides of the equation are continuous, and thus it suffices to show that both sides have the same Fourier coefficients. By periodicity it sufficies to examine the coefficients on $[0,1]$.

For the left side, we find that the mth coefficient is $\hat{f}(m)$:

$$\begin{aligned}
\int_0^1 \left(\sum_{n=-\infty}^{\infty} f(x+n) \right) e^{-2\pi imx} dx &= \sum_{-\infty}^{\infty} \int_0^1 f(x+n)e^{-2\pi imx} dx \\
&= \sum_{n=-\infty}^{\infty} \int_n^{n+1} f(x)e^{-2\pi imx} dx \\
&= \int_{-\infty}^{\infty} f(x)e^{-2\pi imx} dx \\
&= \hat{f}(m)
\end{aligned}$$

where we justify swapping the sum and integral since $f \in \mathcal{S}(\mathbb{R})$. For the right side, we find that the mth coefficient is similarly $\hat{f}(m)$:

$$\begin{aligned}
\int_0^1 \sum_{n=-\infty}^{\infty} \hat{f} e^{2\pi inx} e^{-2\pi imx} &= \int_0^1 \sum_{n=-\infty}^{\infty} \hat{f} e^{2\pi ix(n-m)} dx \\
&= \sum_{n=-\infty}^{\infty} \int_0^1 \hat{f} e^{2\pi ix(n-m)} dx \\
&= \hat{f}(m).
\end{aligned}$$

We obtain the last line by noting that the integral will vanish when $n \neq m$, and thus we are really only taking the sum at one value of n, when $n = m$. Here, we just have $\int_0^1 \hat{f} e^{2\pi ix(0)} = \hat{f}(m)$. Since both sides have the same Fourier coefficients, they are equal by theorem 2.9.

□

In particular, when $x = 0$ we get the following nice result:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

5. Fourier Analysis on Finite Abelian Groups

We can extend Fourier analysis to the realm of finite Abelian groups. We begin with a discussion of characters of a group.

Definition 5.1. If G is a finite abelian group with binary operation $\cdot$ and we denote the complex unit circle by S^1, then a character on G is a multiplicative complex valued function $\chi : G \to S^1$. Furthermore, the trivial character is defined as the character that sends every element of G to 1. Finally, we denote the family of all characters on G as $\hat{G}$.

Proposition 5.2. *For any finite abelian group G, $\hat{G}$ is also an abelian group with binary operator $(\chi_1 \cdot \chi_2)(a) = \chi_1(a) \cdot \chi_2(a)$, where the second $\cdot$ refers to complex valued multiplication.*

Proof. The proof of associativity and commutativity are straightforward from the associativity and commutativity of multiplication. The identity element is the trivial character χ_0 of G. Note that for all $\chi \in \hat{G}$, we define $\chi^{-1} = \overline{\chi}$. Finally, $|\chi_1\chi_2| = |\chi_1||\chi_2| = 1$, and thus under the defined binary operator the character group $\hat{G}$ is closed. Thus $\hat{G}$ satisfies the requirements of an abelian group.

□

Characters serve an important role in the Fourier analysis on finite Abelian groups because they satisfy the orthogonal property, which we will soon prove. This allows these characters to act as the algebraic analogue to the family of functions $f(x) = \{e^{2\pi ikx}\}$ in section 1, as an orthonormal basis for the function space of a finite Abelian group.

Lemma 5.3. *If G is a finite abelian group, then $\hat{G}$ is an orthonormal family. We define the inner product on V by*

$$(f, g) = \frac{1}{|G|} \sum_{a \in G} f(a)\overline{g(a)}.$$

Proof. We note that $\chi(a)\overline{\chi(a)} = |\chi(a)|^2$ and $|\chi(a)| = 1$. Thus, for any character $\chi \in \hat{G}$ $(\chi, \chi) = 1$.

We prove that for any two distinct $\chi_1, \chi_2 \in \hat{G}$, $(\chi_1, \chi_2) = 0$. First, note that if χ is a non-trivial character of G, then there exists $b \in G$ such that $\chi(b) \neq 1$, and we have

$$\chi(b) \sum_{a \in G} \chi(a) = \sum_{a \in G} \chi(ab) = \sum_{a \in G} \chi(a).$$

Since $\chi(b) \neq 1$, it must be true that $\sum_{a \in G} \chi(a) = 0$. Note that this proof is contingent on such b existing. If such a b did not exist χ were the trivial character than obviously the sum would equal the order of G.

Since χ_1, χ_2 are distinct, and since we know that inverses are unique, $\chi_1(\chi_2)^{-1} = \chi_1(\overline{\chi_2})$ is nontrivial, and thus $\sum_{a \in G} \chi_1(\overline{\chi_2}) = 0$, proving orthogonality. Thus $\hat{G}$ is an orthonomal family.

□

Lemma 5.4. *For any finite Abelian group G, $|G| = |\hat{G}|$.*

Proof. We first note that the dimension of the vector space of functions $G \to \mathbb{C}$ is $|G|$. We can easily find a basis of $|G|$ elements by having a function send one element of G to $1 \in \mathbb{C}$ and each other element to $0 \in \mathbb{C}$ for each element in G. This set of functions is linearly independent and spans. Thus we have an upper bound on $|\hat{G}|$, namely, $|G|$. It suffices to prove that $|\hat{G}| \geq |G|$.

We first prove this claim for cyclic groups. Let G be a cyclic group. Then it is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$. It suffices to prove the claim for $\mathbb{Z}/n\mathbb{Z}$. We note that the following family of characters is linearly independent:

$$\chi_j(x) := e^{\frac{2\pi i j x}{N}} \qquad \forall j \in \mathbb{Z}/n\mathbb{Z}.$$

Then we have a lower bound, N, for the size of $\widehat{\mathbb{Z}/n\mathbb{Z}}$, and thus we have proven the claim for cyclic groups.

Now we extend this to all finite Abelian groups. We recall a well known group theory result, that finite Abelian groups are isomorphic to a direct product of some number of cyclic groups. Thus we write $G = G_1 \times G_2 \cdots$, and let $n_1, n_2 \cdots$ be the respective sizes of the cyclic groups. If $\chi_1 \in \hat{G}_1$ and $\chi_2 \in \hat{G}_2$, then we define $\chi(n) = \chi_1(a)\chi_2(b)$, where if f is an isomorphism $G_1 \times G_2 \to G$, then $f((a, b)) = n$. Furthermore, note that this defines a character on G. We have a mapping $\widehat{G_1 \times G_2} \to \hat{G}$ and this is clearly bijective. Thus we have $|\hat{G}| = n_1 \cdot n_2 \cdot \cdots$, and thus $|\hat{G}| = |G|$ for all groups.

□

And thus we have the following important result:

Theorem 5.5. *Let G be a finite Abelian group, and let V be the space of functions on G such that $V = \{f : G \to \mathbb{C}\}$. Then $\hat{G}$ is an orthonormal basis for V.*

Armed with this tool we can use $\hat{G}$ to begin Fourier analysis on finite Abelian groups.

Definition 5.6. If G is a finite Abelian group, χ a character of G, and f is a function on G, then we define the Fourier coefficient of f with respect to χ to be $\hat{f}(e) = (f, e)$, and we define the Fourier series of f to be

$$\sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi.$$

Recall that $(f, e) = \frac{1}{|G|} \sum_{a \in G} f(a)\overline{e(a)}$.

Theorem 5.7. *For any finite Abelian group G and f a function on G, we have*

$$f = \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi.$$

Proof. Since $\hat{G}$ is a basis, we know that there exist constants c_χ such that $f = \sum_{\chi \in \hat{G}} c_\chi \chi$. But since $\hat{G}$ is orthonormal and $\chi \in \hat{G}$, this implies $(f, \chi) = c_\chi$ by our inner product definition. The conclusion is then straightforward.

□

Fourier analysis on finite Abelian groups can be used to prove Dirichlet's theorem, a result about the infinitude of primes in different residual classes. We will not prove Dirichlet's theorem in its entirety, as such a proof would far exceed the scope of this paper, but we will show where finite Fourier analysis plays a significant role in the proof.

Dirichlet's theorem claims the following:

Theorem 5.8 (Dirichlet's Theorem)**.** *If l and q are relatively prime, there are infinitely many primes of the form $l, l+q, l+2q, \cdots$.*

Recalling a often successful method we have used before, we define a characteristic function. In this case, we define a δ-function, a characteristic function, as follows:

$$\delta_l(n) = \begin{cases} 1 & n \equiv l \mod q \\ 0 & \text{otherwise} \end{cases}$$

Then we can apply finite Fourier analysis on our δ-function:

$$\hat{\delta}_l(\chi) = \frac{1}{|G|} \sum_{m \in G} \delta_l(m)\overline{\chi(m)} = \frac{1}{|G|}\overline{\chi(l)}$$

And by taking the Fourier coefficients again, we have

$$\delta_l(n) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \overline{\chi(l)}\chi(n)$$

where if $\chi_a(n)$ is a character on $(\mathbb{Z}/|G|\mathbb{Z})^\times$, we define

$$\chi(n) = \begin{cases} \chi_a(n) & n \in (\mathbb{Z}/|G|\mathbb{Z})^\times \\ 0 & \text{otherwise} \end{cases}$$

Then we have rewritten Dirichlet's theorem as the following statement to be proven:

$$\sum_{n \in \mathbb{N}} \delta_l(n) = \sum_{n \in \mathbb{N}} \frac{1}{|G|} \sum_{\chi \in \hat{G}} \overline{\chi(l)}\chi(n) > \infty$$

Stein and Shakarchi's book on Fourier analysis has a wonderful full proof of Dirichlet's theorem using Fourier analysis on finite Abelian groups, and the above is simply a brief overview of this deep result.

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References

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