On the Uniqueness Theorem in $\mathcal{L}^1$

Recall that, for $f, g \in C(\mathbb{T})$, the following are equivalent:

(i) $\hat{f}(k) = \hat{g}(k)$ για κάθε $k \in \mathbb{Z}$

(ii) $f = g$.

One cannot expect this equivalence to hold for $f, g \in \mathcal{L}^1(\mathbb{T})$, since if an $\mathcal{L}^1$ function is modified on a null set, then its Fourier coefficients are unchanged. In other words,

If $f, g \in \mathcal{L}^1(\mathbb{T})$ and $f = g$ almost everywhere, then $\hat{f}(k) = \hat{g}(k)$ για κάθε $k \in \mathbb{Z}$.

The converse is also true:

Theorem 1 *If $f, g \in \mathcal{L}^1(\mathbb{T})$ the folowing are equivalent:*

(i) $\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$

(ii) $f = g$ almost everywhere. That is, f and g determine the same element of $L^1(\mathbb{T})$.

The implication $(i) \Rightarrow (ii)$ was observed above. The proof of the implication $(ii) \Rightarrow (i)$ will follow from an extension of Fejér' s Theorem to the space $(L^1(\mathbb{T}), \|\cdot\|_1)$.

Recall that, for $f \in \mathcal{L}^1(\mathbb{T})$, the trigonometric polynomial $\sigma_m(f)$ is defined by

$$\sigma_m(f) = \frac{1}{m+1} \sum_{n=0}^{m} S_n(f) \quad (m \in \mathbb{N})$$

and is given by

$$\sigma_m(f)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_m(t-s) f(s) ds$$

where

$$K_m(x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{m+1}\right) e^{ikx} .$$

Proposition 1 *For every $f \in L^1(\mathbb{T})$, we have* $\lim_n \|\sigma_n(f) - f\|_1 = 0$.

Proof. Recall that, by Fejér' s Theorem, for every $g \in C(\mathbb{T})$ we have

$$\lim_n \|\sigma_n(g) - g\|_\infty = 0$$

and therefore, since $\|h\|_1 \leq \|h\|_\infty$ for $h \in C(\mathbb{T})$,

$$\lim_n \|\sigma_n(g) - g\|_1 = 0 .$$

But we know that $C(\mathbb{T})$ is dense in $(L^1(\mathbb{T}), \|\cdot\|_1)$. Thus, for every $f \in L^1(\mathbb{T})$, given $\epsilon > 0$ there exists $f \in C(\mathbb{T})$ with

$$\|f - g\|_1 < \epsilon .$$

For g we may choose $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$ we have

$$\|\sigma_n(g) - g\|_1 < \epsilon .$$

Now we have, if $n \geq n_0$

$$\begin{aligned}\|\sigma_n(f) - f\|_1 &\leq \|\sigma_n(f) - \sigma_n(g)\|_1 + \|\sigma_n(g) - g\|_1 + \|g - f\|_1 \\ &< \|\sigma_n(f - g)\|_1 + \epsilon + \epsilon\end{aligned}$$

and the proof will be complete if we can control the quantity $\|\sigma_n(f - g)\|_1$. But by Proposition 2 below, we have $\|\sigma_n(f - g)\|_1 \leq \|f - g\|_1$.

Proposition 2 *For every $f \in L^1(\mathbb{T})$, we have $\|\sigma_n(f)\|_1 \leq \|f\|_1$.*

Proof. We first claim that the inequality $\|\sigma_n(f)\|_1 \leq \|f\|_1$ holds when $f \in C(\mathbb{T})$. Indeed, we have

$$\begin{aligned}\sigma_m(f)(t) &= \int_{-\pi}^{\pi} K_m(t-s) f(s) \frac{ds}{2\pi} \\ \text{hence} \quad \|\sigma_m(f)\|_1 &= \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} K_m(t-s) f(s) \frac{ds}{2\pi} \right| \frac{dt}{2\pi} \\ &= \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} |K_m(t-s) f(s)| \frac{ds}{2\pi} \right) \frac{dt}{2\pi} \\ &\overset{(!)}{=} \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} |K_m(t-s) f(s)| \frac{dt}{2\pi} \right) \frac{ds}{2\pi} \\ &= \int_{-\pi}^{\pi} |f(s)| \left(\int_{-\pi}^{\pi} |K_m(t-s)| \frac{dt}{2\pi} \right) \frac{ds}{2\pi}\end{aligned}$$

But $\int_{-\pi}^{\pi} |K_m(t-s)| \frac{dt}{2\pi} = \int_{-\pi}^{\pi} |K_m(x)| \frac{dx}{2\pi}$ by periodicity, and we know that $\int_{-\pi}^{\pi} |K_m(x)| \frac{dx}{2\pi} = 1$. Hence the previous inequality becomes

$$\|\sigma_m(f)\|_1 \leq \int_{-\pi}^{\pi} |f(s)| \frac{ds}{2\pi} = \|f\|_1 \ .$$

Now suppose $f \in L^1(\mathbb{T})$ and let $m \in \mathbb{N}$ be fixed. Then given $\epsilon > 0$ there exists $f_\epsilon \in C(\mathbb{T})$ such that $\|f - f_\epsilon\|_1 < \frac{\epsilon}{m+1}$. Then,

$$\begin{aligned}\sigma_m(f) - \sigma_m(f_\epsilon) &= \sum_{k=-m}^{m} \left(1 - \frac{|k|}{m+1}\right) (\hat{f}(k) - \hat{f}_\epsilon(k)) e_k \\ \text{so} \quad \|\sigma_m(f) - \sigma_m(f_\epsilon)\|_1 &\leq \sum_{k=-m}^{m} \left(1 - \frac{|k|}{m+1}\right) |\hat{f}(k) - \hat{f}_\epsilon(k)| \, \|e_k\|_1\end{aligned}$$

But $\|e_k\|_1 = 1$ and $|\hat{f}(k) - \hat{f}_\epsilon(k)| \leq \left\|\hat{f} - \hat{f}_\epsilon\right\|_\infty \leq \|f - f_\epsilon\|_1$ for all k, so

$$\|\sigma_m(f) - \sigma_m(f_\epsilon)\|_1 \leq \sum_{k=-m}^{m} \left(1 - \frac{|k|}{m+1}\right) \|f - f_\epsilon\|_1 \leq (m+1) \|f - f_\epsilon\|_1 < \epsilon$$

and therefore, using the fact that $\|\sigma_m(f_\epsilon)\|_1 \leq \|f_\epsilon\|_1$ (by the claim)

$$\begin{aligned}\|\sigma_m(f)\|_1 &\leq \|\sigma_m(f) - \sigma_m(f_\epsilon)\|_1 + \|\sigma_m(f_\epsilon)\|_1 \\ &\leq \|\sigma_m(f) - \sigma_m(f_\epsilon)\|_1 + \|f_\epsilon\|_1 < \epsilon + (\|f\|_1 + \epsilon)\end{aligned}$$

so $\|\sigma_m(f)\|_1 \leq \|f\|_1$ since $\epsilon > 0$ was arbitrary. □

Proof of Theorem 1 $(ii) \Rightarrow (i)$: Suppose $\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$. Then the partial sums of the Fourier series of f and g are the same, and so $\sigma_n(f) = \sigma_n(g)$, or $\sigma_n(f-g) = 0$ for all n. Therefore, from Proposition 2,

$$\|f - g\|_1 = \lim_n \|\sigma_n(f - g)\|_1 = 0$$

and so $f = g$ almost everywhere (or $f = g$ in $L^1(\mathbb{T})$). □