Fourier Series and Lebesgue Integration (605)

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Brief Description

J. Fourier (1768-1830) believed that every 2π-periodic function f could be "represented" as a "trigonometric series" of the form

$$f \sim \frac{a_0}{2} + \sum a_n \cos nt + \sum b_n \sin nt$$

the "Fourier series of f". In other words, he believed that every periodic function is a "combination" of "pure" sinusoidal functions. He used these "representations" for the solution of (partial) differential equations that described physical phenomena.

Since the beginnings of the subject, questions regarding convergence of these series arose. In order to answer these questions, many of the tools of Mathematical Analysis had to be developed; in fact, the very concept of a function had to be clarified.

At the same time, Fourier series became, and still are, an indispensable tool in very diverse fields of Mathematics (as we will see), but also in its applications to the Sciences and Technology.

We shall attempt a brief and elementary introduction to this tool, by examining some centrally important questions:

• Which functions f possess a Fourier series, and how is a function "decomposed/ analyzed" as a Fourier series (*spectral analysis*)?

• If we know the coefficients of the Fourier series of f, is it possible to "synthesize" f (*spectral synthesis*)?

• Does the Fourier series of f converge? And if so, in what mode of convergence (pointwise, uniform, mean square)?

• Is every (convergent) *trigonometric series* the *Fourier series* of a function? Is this function unique, if it exists?

As the coefficients in the Fourier series of a function f are given by integral formulae, for example

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt$$

the notion of integral will have to be reconsidered. We will see that the Riemann integral is not an adequate tool to handle the various problems that arise. This will be lead us to a particularly fruitful generalisation, the *Lebesgue integral*.

The basic idea for the definition of the Lebesgue integral can be described as follows;

For the Riemann integral, one partitions *the domain* of a function f and considers the limiting behaviour of the sequence of partial sums $\sum f(t_i)\Delta t_i$, where Δt_i is the length of the corresponding interval. Therefore, if the behaviour of f in small subintervals of its domain is very irregular, the function may fail to be Riemann-integrable. The classic example is the Dirichlet function, defined to be 0 on the irrationals and 1 on the rationals (in $[0, 1]$).

By contrast, for the Lebesgue integral, we partition *the range* of f in subintervals $I_1, I_2, \dots, I_n$ and we consider partial sums of the form $\sum f(t_i)\mu(A_i)$, where now $A_i =$

$f^{-1}(I_i)$ and $\mu(A_i)$ is the "length" of A_i. Thus, provided that the inverse images A_i have well defined "length", one is able to examine the convergence of the sequence of these partial sums.

The Lebesgue integral enjoys two basic advantages:

1) It allows the integration of a much more general family of functions than the Riemann integral.

2) It behaves much better than the Riemann integral with respect to limiting procedures.

We will be concerned with a careful definition of the Lebesgue integral for functions of a real variable, with the basic convergence theorems, as well as with the basic applications to Fourier series.

Prerequisites.

Infinitesimal calculus of one real variable, particularly the Riemann integral and convergence of series.

Sequences and series of functions: pointwise and uniform convergence.

(There will be a brief reminder of the Riemann integral and of the basic results for series of functions.)

Bibliography

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