7 Mean square convergence

Let us begin with a simple, but crucial observation:

Proposition 7.1 (Best mean square approximation)**.** *Let $f : [-\pi, \pi] \to \mathbb{C}$ be a Riemann-integrable function and $n \in \mathbb{N}$. Then for every trigonometric polynomial p of degree* $\deg(p) \le n$ *we have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f-p|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f-S_n(f)|^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi} |S_n(f)-p|^2. \tag{1}$$

Therefore the inequality

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f-p|^2 \ge \frac{1}{2\pi}\int_{-\pi}^{\pi} |f-S_n(f)|^2 \tag{2}$$

holds, and we have equality we have if and only if $p = S_n$.

In other words, S_n is the unique trigonometric polynomial which minimizes the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f-p|^2$ among all choices of trigonometric polynomials p of degree at most n.

In particular, if $m \le n$ then $\|f - S_m(f)\|_2 \ge \|f - S_n(f)\|_2$.

Proof. It is clear that (2) follows at once from (1) and that equality holds in (2) if and only if the last term in (1) vanishes; this happens if and only if $p = S_n$.

So let $p(t) = \sum\limits_{k=-n}^{n} c_k e^{ikt}$. If we set $g = f - S_n(f)$ and $q = S_n(f) - p$ we have

$$f - p = (f - S_n(f)) + (S_n(f) - p) = g + q.$$

Observe that, if $e_k(t) = e^{ikt}$, $|k| \le n$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f\overline{e_k} = \hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_n(f)\overline{e_k}$$

(from the definition of $S_n(f)$), and therefore

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g\overline{e_k} = 0, \qquad |k| \le n.$$

Since $q = \sum_{k=-n}^{n}(\hat{f}(k) - c_k)e_k$ is a linear combination of $\{e_k : |k| \le n\}$, it follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g\bar{q} = 0,$$

and so

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} |f-p|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |g+q|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (g+q)\overline{(g+q)} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g\bar{g} + \frac{1}{2\pi}\int_{-\pi}^{\pi} g\bar{q} + \frac{1}{2\pi}\int_{-\pi}^{\pi} q\bar{g} + \frac{1}{2\pi}\int_{-\pi}^{\pi} q\bar{q} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |g|^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi} |q|^2\end{aligned}$$

and (1) is proved. □

This Proposition suggests the study of the quantity

$$\|f\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt\right)^{1/2} \qquad f : [-\pi, \pi] \to \mathbb{C} \text{ integrable.}$$

If f, g are two (Riemann) integrable functions defined on $[-\pi, \pi]$ we define

$$\|f-g\|_2 := \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)-g(t)|^2 dt\right)^{1/2}$$

and
$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt\,.$$

Observe that $\|\cdot\|_2$ satisfies

$$\|f-g\|_2 \le \|f-g\|_\infty := \sup\{|f(t)-g(t)| : t\in[-\pi,\pi]\}$$

and that $\|f\|_2 = \langle f, f\rangle^{1/2}$.

Remark $\hat{f}(k) = \langle f, e_k\rangle,\ k\in\mathbb{Z}.$

Lemma 7.2. *If $f, g : [-\pi, \pi] \to \mathbb{C}$ are two (Riemann) integrable functions, we have*

$$(a)\ |\langle f, g\rangle| \le \|f\|_2 \|g\|_2$$
$$(b)\ \|f+g\|_2 \le \|f\|_2 + \|g\|_2\,.$$

Proof. (a) To show that $|\langle f, g\rangle| \le \|f\|_2 \|g\|_2$ I can assume that [1] that $\|g\|_2 = 1$. If $\lambda\in\mathbb{C}$, from the definition of $\langle\cdot,\cdot\rangle$ we have

$$\begin{aligned} 0 \le \langle f-\lambda g, f-\lambda g\rangle &= \|f\|_2^2 - \bar\lambda\langle f, g\rangle - \lambda\langle g, f\rangle + |\lambda|^2\|g\|_2^2 \\ &= \|f\|_2^2 - \bar\lambda\langle f, g\rangle - \lambda\langle g, f\rangle + |\lambda|^2 \end{aligned}$$

so, setting $\lambda = \langle f, g\rangle$, we have $0 \le \|f\|_2^2 - 2|\langle f, g\rangle|^2 + |\langle f, g\rangle|^2$ hence $|\langle f, g\rangle|^2 \le \|f\|_2^2 = \|f\|_2^2\|g\|_2^2$ and the required inequality is proved.

(b) For each every f, g we have

$$\begin{aligned} \|f+g\|_2^2 &= \langle f+g, f+g\rangle = \langle f, f\rangle + \langle f, g\rangle + \langle g, f\rangle + \langle g, g\rangle \\ &= \langle f, f\rangle + 2\,\mathrm{Re}\langle f, g\rangle + \langle g, g\rangle \\ &\le \langle f, f\rangle + 2|\langle f, g\rangle| + \langle g, g\rangle \\ &\le \|f\|_2^2 + 2\|f\|_2\|g\|_2 + \|g\|_2^2 = (\|f\|_2 + \|g\|_2)^2 \end{aligned}$$

by (a), hence $\|f+g\|_2 \le \|f\|_2 + \|g\|_2$. □

Corollary 7.3. *The map $(f, g)\to\langle f, g\rangle$ is an inner product and the map $(f, g)\to d_2(f, g) := \|f-g\|_2$ is a metric on the linear space $C([-\pi, \pi])$.* [2] *That is, they satisfy*

	$\langle f, g\rangle \in \mathbb{C}$	$d_2(f, g)\in\mathbb{R}_+$
(i)	$\langle f+\lambda g, h\rangle = \langle f, h\rangle + \lambda\langle g, h\rangle$	$(a)\ d_2(f, g) = d_2(g, f)$
(ii)	$\langle g, f\rangle = \overline{\langle f, g\rangle}$	$(b)\ d_2(f, g) \le d_2(f, h) + d_2(h, g)$
(iii)	$\langle f, f\rangle \ge 0$	$(c)\ d_2(f, g) = 0 \iff f = g.$
(iv)	$\langle f, f\rangle = 0 \iff f = 0.$	

[1] If $\|g\|_2 = 0$ the inequality holds trivially and if $\|g\|_2 \ne 0$, replace g by $\frac{g}{\|g\|_2}$.

[2] However it is not a metric on the space of integrable functions, since the equality $\|f-g\|_2 = 0$ does not imply that $f(t) = g(t)$ for every $t\in[-\pi, \pi]$. It could happen for example that $f-g$ is $\ne 0$ at a single point of the interval only. We will see later that the only conclusion one can draw is that the equality $f = g$ is valid "almost everywhere" - a concept we will define then.

Proof. Relations (i), (ii) and (iii) are immediate consequences of the *linearity of the integral*.

To prove that d_2 is indeed a metric on $C[-\pi,\pi]$, we observe directly from its definition that

$$d_2(f,g) = d_2(g,f) \quad \text{and} \quad d_2(f,g) \geq 0$$

for every f,g. Also, if f,g are *continuous* and unequal, then there exists $\delta > 0$ and an open neighbourhood $V \subseteq [-\pi,\pi]$ (of the form $(a,b)\cap[-\pi,\pi]$) so that $|f(t)-g(t)| \geq \delta$ for every $t \in V$; therefore

$$d_2(f,g)^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)-g(t)|^2 dt \geq \frac{1}{2\pi}\int_V |f(t)-g(t)|^2 dt \geq \frac{1}{2\pi}\delta^2 m(V) > 0$$

(where $m(V)$ denotes the length of V) and therefore $d_2(f,g)=0$ if and only if $f=g$ (thus we have also proved (iv)). It remains to prove the triangle inequality: if f,g,h are continuous, we have

$$d_2(f,g) = \|(f-h)+(h-g)\|_2 \leq \|f-h\|_2 + \|h-g\|_2 = d_2(f,h)+d_2(h,g)$$

using the previous Lemma. □

Remarks 7.4. *(a) The elementary, but crucial remark that the expression* $\langle f,g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f\bar{g}$ *has analogous properties to those of the inner product of Euclidean space, allows the introduction of geometric methods and notions, such as orthogonality.*

(β) Equality (1) in Lemma 7.1 can be written

$$\|f-p\|_2^2 = \|f-S_n(f)\|_2^2 + \|S_n(f)-p\|_2^2$$

and its proof only uses properties $(i),(ii)$ *and* (iii)*: it is an applications of the Pythagorean Theorem:* $\langle f,g\rangle = 0 \;\Rightarrow\; \|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$*, if one observes that* $\langle f-S_n(f), S_n(f)-p\rangle = 0$*.*

As we will show later, the next Theorem also holds for integrable functions.

Although the sequence $(S_n(f))$ for a continuous f may fail to converge, even pointwise, it does converge to f with respect to the metric d_2:

Theorem 7.5. *If* $f:[-\pi,\pi]\to\mathbb{C}$ *is are continuous and* 2π*-periodic, then*

$$S_n(f) \xrightarrow{\|\cdot\|_2} f$$

that is

$$\lim_n \frac{1}{2\pi}\int_{-\pi}^{\pi}|S_n(f)-f|^2 = 0.$$

Proof. Since f is continuous, from Fejér's Theorem we know that $\sigma_n(f)\to f$ uniformly. Therefore

$$\|\sigma_n(f)-f\|_2 \leq \|\sigma_n(f)-f\|_\infty \to 0\,.$$

But $\sigma_n(f)$ is a trigonometric polynomial of degree at most n, hence by the best approximation Lemma 7.1 we have $\|f-S_n(f)\|_2 \leq \|f-\sigma_n(f)\|_2$ and so $\|f-S_n(f)\|_2 \to 0$. □

Our next target is to relate $\|f\|_2$ with the Fourier coefficients of f.

Remark 7.6. *If* $p(t) = \sum_{k=-n}^{n} c_k e^{ikt}$ *is a trigonometric polynomial, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|p|^2 = \sum_{k=-n}^{n}|c_k|^2 = \sum_{k=-n}^{n}|\hat{p}(k)|^2.$$

Proof. Since $\widehat{p}(k) = c_k = \langle p, e_k \rangle$ for $|k| \leq n$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |p|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} p\bar{p} = \frac{1}{2\pi}\int_{-\pi}^{\pi} p \sum_{k=-n}^{n} \bar{c}_k \bar{e}_k$$
$$= \sum_{k=-n}^{n} \bar{c}_k \frac{1}{2\pi}\int_{-\pi}^{\pi} p\bar{e}_k = \sum_{k=-n}^{n} \bar{c}_k c_k = \sum_{k=-n}^{n} |c_k|^2.$$

Proposition 7.7 (Bessel's Inequality)**.** *Let $f : [-\pi, \pi] \to \mathbb{C}$ be integrable. Then*

$$\sum_{k=-\infty}^{+\infty} |\hat{f}(k)|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2.$$

Proof. Let $n \in \mathbb{N}$. Applying (1) for $p = 0$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f - S_n(f)|^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi} |S_n(f)|^2 \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} |S_n(f)|^2 \tag{3}$$

But $S_n(f)$ is a trigonometric polynomial whose coefficients are $\hat{f}(k)$ for $|k| \leq n$ and 0 for $|k| > n$, hence by the previous Remark we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} |S_n(f)|^2 = \sum_{k=-n}^{n} |\hat{f}(k)|^2.$$

Since this inequality holds for every $n \in \mathbb{N}$, the conclusion follows. □

We will show later that in fact equality holds.

An immediate corollary of Bessel's Inequality is the fundamental

Theorem 7.8 (Riemann - Lebesgue)**.** *If $f : [-\pi, \pi] \to \mathbb{C}$ is an integrable function, then*

$$\lim_{k\to+\infty} \hat{f}(k) = \lim_{k\to\infty} \hat{f}(-k) = 0$$
$$\textit{equivalently} \quad \lim_{n\to+\infty} a_n(f) = \lim_{n\to\infty} b_n(f) = 0.$$

Corollary 7.9 (Parseval's equality)**.** *If $f : [-\pi, \pi] \to \mathbb{C}$ is a continuous function, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2.$$

Proof. We have shown that $d_2(S_n(f), f) \to 0$. Since d_2 is a metric on $C([-\pi, \pi])$, by the triangle inequality we have

$$|d_2(f, 0) - d_2(S_n(f), 0)| \leq d_2(S_n(f), f)$$

hence $d_2(S_n(f), 0) \to d_2(f, 0)$, that is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |S_n(f)|^2 \to \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2.$$

But by Remark 7.6 we have $\frac{1}{2\pi}\int_{-\pi}^{\pi} |S_n(f)|^2 = \sum_{k=-n}^{n} |\hat{f}(k)|^2$, hence

$$\sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 = \lim_{n\to\infty} \sum_{k=-n}^{n} |\hat{f}(k)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2.$$

Note Let us state once again that the results of this Section will be generalised and strengthened, if one uses the Lebesgue integral instead of the Riemann integral.

8 The Poisson kernel

If $f : [-\pi, \pi] \to \mathbb{C}$ is an integrable function, for each $0 \le r < 1$, the series

$$A_r(f)(t) = f_r(t) := \sum_{k\in\mathbb{Z}} r^{|k|} \hat{f}(k) e^{ikt}, \qquad t \in [-\pi, \pi]$$

converges absolutely and uniformly, hence defines a continuous function $f_r : [-\pi, \pi] \to \mathbb{C}$ (although for $r = 1$ the series, i.e. the Fourier series of f, may fail to converge, even pointwise). Indeed, the (double) sequence $(\hat{f}(k))$ is bounded, because

$$|\hat{f}(k)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)e^{-ikt}|dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|dt := \|f\|_1$$

for all $k \in \mathbb{Z}$ and therefore

$$\sum_{k\in\mathbb{Z}} \left| r^{|k|} \hat{f}(k) e^{ikt} \right| \le \|f\|_1 \sum_{k\in\mathbb{Z}} r^{|k|} < \infty.$$

We have

$$\begin{aligned}
f_r(t) &= \sum_{n\in\mathbb{Z}} r^{|n|} \hat{f}(n) e^{int} = \sum_{n\in\mathbb{Z}} r^{|n|} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(s) e^{-ins} ds \right) e^{int} \\
&= \sum_{n\in\mathbb{Z}} r^{|n|} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s) e^{-in(t-s)} ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s) \left(\sum_{n\in\mathbb{Z}} r^{|n|} e^{-in(t-s)} \right) ds \quad \text{(uniform convergence)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s) P_r(t-s) ds
\end{aligned}$$

$$\begin{aligned}
\text{where} \quad P_r(t) &:= \sum_{n\in\mathbb{Z}} r^{|n|} e^{int} = \sum_{n=-\infty}^{-1} r^{-n} e^{int} + 1 + \sum_{n=1}^{\infty} r^n e^{int} \\
&= \sum_{k=1}^{\infty} r^k e^{-ikt} + 1 + \sum_{n=1}^{\infty} r^n e^{int} = 1 + 2\sum_{n=1}^{\infty} r^n \cos nt
\end{aligned}$$

is **the Poisson kernel**. Writing $z = re^{it}$, we have $|z| < 1$ and

$$\begin{aligned}
P_r(t) &= \sum_{n=1}^{\infty} \bar{z}^k + 1 + \sum_{n=1}^{\infty} z^n = \frac{\bar{z}}{1-\bar{z}} + 1 + \frac{z}{1-z} \\
&= \frac{\bar{z}}{1-\bar{z}} + \frac{1}{1-z} = \frac{\bar{z}(1-z) + (1-\bar{z})}{(1-\bar{z})(1-z)} = \frac{1-|z|^2}{|1-z|^2} = \frac{1-r^2}{1-2r\cos t + r^2}
\end{aligned}$$

showing that $P_r(t) \ge 0$ for all t. Also, since the series converges uniformly, for all $r \in (0,1)$ and $k \in \mathbb{Z}$ we have

$$\widehat{P_r}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t) e^{-ikt} dt = \sum_{n\in\mathbb{Z}} r^{|n|} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-k)t} dt = r^{|k|}$$

$$\text{and in particular} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t) dt = r^0 = 1.$$

Remark 8.1. *The Poisson kernel has the following properties*

(α) For each $r \in [0,1)$, the function $P_r : [-\pi, \pi] \to \mathbb{R}$ is continuous and non-negative.

(β) If $\delta \in (0, \pi/2)$ and $E_\delta := [-\pi, -\delta] \cup [\delta, \pi]$, we have $P_r(t) \to 0$ uniformly for $t \in E_\delta$ as $r \nearrow 1$; hence $\lim_{r \nearrow 1} \int_{E_\delta} P_r(x)dx = 0$.

(γ) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(x)dx = 1$ for every $r \in [0,1)$.

Proof. It only remains to prove (β): If $0 < \delta < \pi/2$ then for all t with $\delta \le |t| \le \pi$ we have $\cos t \le \cos\delta$, hence

$$0 \le P_r(t) = \frac{1-r^2}{1-2r\cos t + r^2} \le \frac{1-r^2}{1-2r\cos\delta + r^2}$$

and the right hand side tends to 0 as $r \nearrow 1$. □

Therefore, if f is Riemann integrable, and hence bounded, we have

$$|f_r(t)| \le \|f\|_\infty \frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(t-s)|ds = \|f\|_\infty \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(s)ds = \|f\|_\infty$$

(since P_r is 2π-periodic and nonnegative) for each t and r.

If in addition f is *continuous* και 2π-periodic, then repeating the proof of Fejér' s Theorem (which proof relied exclusively on the corresponding properties (α), (β) and (γ) of the Fejér kernel) we arrive at the following

Theorem 8.2. *If f is Riemann integrable and 2π-periodic, then at every point t of continuity of f we have $\lim_{r \nearrow 1} f_r(t) = f(t)$.*

If f is continuous*, then* $\lim_{r \nearrow 1} f_r(x) = f(x)$ uniformly*, that is,* $\lim_{r \nearrow 1} \|f_r - f\|_\infty = 0$.

Remark 8.3. *Note that although the functions f_r are (in general) not trigonometric polynomials, they are continuous (in fact differentiable - why?) functions given by absolutely and uniformly convergent Fourier series.*

9 Pointwise convergence and the localisation principle

(*For proofs, see Stein & Shakarchi, 'Fourier Analysis', paragraphs 3.2.1 and 3.2.2.* [3])

Definition 9.1. *Complex-valued functions on the unit circle*

Denote by $\mathbb{T}$ *the unit circle*

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}.$$

If $\phi : \mathbb{T} \to \mathbb{C}$*, define* $f : \mathbb{R} \to \mathbb{C}$ *by*

$$f(\theta) = \phi(e^{i\theta}).$$

The function f *is* 2π*-periodic.*

Conversely, if $f : \mathbb{R} \to \mathbb{C}$ *is* 2π*-periodic, then the function* $\phi : \mathbb{T} \to \mathbb{C}$ *given by* $\phi(e^{i\theta}) = f(\theta)$ *is well defined.*[4] *Thus we have a* $1-1$ *correspondence between functions* $\phi : \mathbb{T} \to \mathbb{C}$ *and* 2π*-periodic functions* $f : \mathbb{R} \to \mathbb{C}$.

We say ϕ *is integrable if* f *is integrable in some interval of length* 2π *(hence in all such intervals), we say* ϕ *is continuous if* f *is continuous, we say* ϕ *is differentiable if* f *is differentiable, we say* ϕ *is continuously differentiable if* f *is continuously differentiable and so on.*

In what follows we shall make no distinction between ϕ and f.

Theorem 9.1. *Let* $f : \mathbb{T} \to \mathbb{C}$ *be an integrable function. If* f *is differentiable at* $\theta_0 \in \mathbb{T}$*, then*

$$S_n(f)(\theta_0) \to f(\theta_0).$$

Remark 9.2. If we examine the proof of Theorem 9.1 we can see that the conclusion $S_n(f)(\theta_0) \to f(\theta_0)$ still holds under the following weaker assumption for f:

'The function f is integrable and satisfies **a Lipschitz condition** at θ_0, that is, there exists $M > 0$ such that

$$|f(\theta_0 - t) - f(\theta_0)| \leq M|t|$$

for all $t \in [-\pi, \pi]$'.

One can now repeat the proof without modifications.

An important consequence of Theorem 9.1 is the **localisation principle of Riemann**: the convergence or divergence of the sequence $S_n(f)(\theta_0)$ depends only on the behaviour of f *in a neighbourhood of* θ_0. This is not at all obvious; indeed, the partial sums $S_n(f)(\theta_0)$ are defined in terms of the Fourier coefficients $\hat{f}(k)$, $|k| \leq n$ of f, which coefficients are given by integration on $[-\pi, \pi]$, thus taking into account the values of f *in the whole interval* $[-\pi, \pi]$.

Theorem 9.3 (Riemann's localisation principle)**.** *Let* $f, g : \mathbb{T} \to \mathbb{C}$ *be two integrable functions. Assume that, for some* $\theta_0 \in \mathbb{T}$ *and some open interval* $I \subset \mathbb{T}$ *with* $\theta_0 \in I$*, we have*

$$f(\theta) = g(\theta) \qquad \textit{for all} \ \ \theta \in I.$$

Then

$$S_n(f)(\theta_0) - S_n(g)(\theta_0) \to 0.$$

In particular, the sequence $\{S_n(f)(\theta_0)\}$ *converges if and only if* $\{S_n(g)(\theta_0)\}$ *converges.*

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[3] Από τις σημειώσεις του Απ. Γιαννόπουλου (2012) Παράγραφος 3.3

[4] Indeed, if $e^{i\theta_1} = e^{i\theta_2}$ for some $\theta_1, \theta_2 \in \mathbb{R}$ then $\theta_2 = \theta_1 + 2k\pi$ for an integer k, hence $f(\theta_1) = f(\theta_2)$ since f is 2π-periodic.

10 Complements

The goal is to prove:

Theorem 10.1. *There exists a continuous function $f : \mathbb{T} \to \mathbb{C}$ for which*

$$\limsup_{n\to\infty} |S_n(f)(0)| = +\infty.$$

Therefore $S[f](0)$ diverges.

An important role in the proof is played by the **trigonometric series**

$$\sum_{k\neq 0} \frac{e^{ikx}}{k} \qquad \text{and} \qquad \sum_{k=-\infty}^{-1} \frac{e^{ikx}}{k}.$$

Lemma 10.2. *Consider the function*

$$f(x) = \begin{cases} i(\pi - x) & \text{If } 0 < x < \pi \\ -i(\pi + x) & \text{If } -\pi < x < 0 \end{cases}$$

and extend it periodically to $\mathbb{R}$. Then, the Fourier series of f is

$$S[f](x) = \sum_{k\neq 0} \frac{e^{ikx}}{k}.$$

Proposition 10.3. *Let $f : \mathbb{T} \to \mathbb{C}$ be an integrable function. If the sequence $\{|k\hat{f}(k)|\}_{k\in\mathbb{Z}}$ is bounded, then the partial sums $S_n(f)$ of the Fourier series of f are uniformly bounded:*

$$\sup_n \|S_n(f)\|_\infty < +\infty.$$

That is, there exists $M > 0$ so that

$$|S_n(f)(x)| \leq M$$

for all $n \in \mathbb{N}$ and all $x \in \mathbb{T}$.

Lemma 10.4. *For each $n \in \mathbb{N}$ consider the trigonometric polynomial*

$$f_n(x) = \sum_{1\leq |k| \leq n} \frac{e^{ikx}}{k}.$$

There exists $M > 0$ so that $|f_n(x)| \leq M$ for all n and all x. □

Lemma 10.5. *For each $n \in \mathbb{N}$ consider the trigonometric polynomial*

$$g_n(x) = \sum_{k=-n}^{-1} \frac{e^{ikx}}{k}.$$

There exists $c > 0$ so that $|g_n(0)| \geq c \log n$ for each $n \in \mathbb{N}$. □

Corollary 10.6. *There is no Riemann integrable function $g : \mathbb{T} \to \mathbb{C}$ with*

$$S[g](x) = \sum_{k=-\infty}^{-1} \frac{e^{ikx}}{k}.$$

Comment. We will show later that there does exist a Lebesgue integrable function g with $S[g](x) = \sum_{k=-\infty}^{-1} \frac{e^{ikx}}{k}$.