Questions 605

II. 2. Every integrable function $f : [-\pi, \pi] \to \mathbb{C}$ can be written uniquely as $f = f_a + f_p$ where f_a is even and f_p is odd. Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_a|^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_p|^2.$$

$$\underset{\text{even}}{f = f_a} + \underset{\text{odd}}{f_p}$$

$$\langle f_a, f_p \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\underset{\text{even}}{f_a(t)}\,\underset{\text{odd}}{\overline{f_p(t)}}}_{\text{odd}}\, dt = 0$$

Pythagoras:

$$\| f_a + f_p \|_2^2 = \| f_a \|_2^2 + \| f_p \|_2^2$$

II. 3. Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous 2π-periodic function. Suppose that $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=-n}^{n} |k\hat{f}(k)| = 0$. Show that then $S_n(f) \to f$ uniformly.

Proof. $S_n(f) = \sum_{k=-n}^{n} \hat{f}(k) e_k$.

Later $\longrightarrow$ $\sigma_n(f) := \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \hat{f}(k) e_k$

$$\Downarrow$$

$$S_n(f) - \sigma_n(f) = \sum_{|k| \le n} \frac{|k|}{n+1} \hat{f}(k) e_k$$

$$\Downarrow$$

$$\|S_n(f) - \sigma_n(f)\|_\infty \le \frac{1}{n+1} \sum_{|k| \le n} \| \, |k| \hat{f}(k) \underbrace{e_k}_{=1} \|_\infty$$

$$= \frac{1}{n+1} \sum_{|k| \le n} |k \hat{f}(k)| \xrightarrow[n \to \infty]{\text{ass.}} 0$$

f cts: Fejér:

$\sigma_n(f) \to f$ uniformly i.e

$$\|\sigma_n(f) - f\|_\infty \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow \|S_n(f) - f\|_\infty \xrightarrow[n\to\infty]{} 0 \qquad \blacksquare$$

II. 4. If $f : \mathbb{R} \to \mathbb{C}$ is a ~~2π periodic and~~ integrable function, show that

(NB. Same proof for periodic fns, when intrgral is over [-π,π].)

$$\lim_{x \to 0} \int |f(t-x) - f(t)|\, dt = 0.$$

Hint: Consider first the case when f is continuous.

Suppose first f is continuous with compact support. i.e. $\exists\, [-M, M] \subseteq \mathbb{R}$ st $f(t) = 0 \ \forall t \notin [-M, M]$

f is uniformly continuous: $\forall \epsilon > 0 \ \exists \delta > 0$ st $|s-t| < \delta \Rightarrow |f(s) - f(t)| < \epsilon$

or $|x| < \delta \Longrightarrow |f(t-x) - f(t)| < \epsilon \quad \forall t \in \mathbb{R}$

$$\Longrightarrow \int_{\mathbb{R}} |f(t-x) - f(t)|\, dt$$

$$= \int_{-M-\delta}^{M+\delta} |f(t-x) - f(t)|\, dt \leq \epsilon(M+\delta+M+\delta)$$

It follows that

$$\int_{\mathbb{R}} |f(t-x) - f(t)|\, dt \longrightarrow 0 \quad x \to 0$$

General case: $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$: $\forall f \in L^1(\mathbb{R})$ $\forall \epsilon > 0 \ \exists g \in C_c(\mathbb{R})$ st $\|f - g\|_1 < \epsilon/3$

SEE ALSO p. 13

$$\int_{\mathbb{R}} |f(t) - g(t)|\, dt < \epsilon/3$$

$\forall x \in \mathbb{R}$ $\Longrightarrow$

$$\int |f(t-x) - g(t-x)|\, dt = \|f - g\|_1 < \epsilon/3$$

$$\Longrightarrow \int_{\mathbb{R}} |f(t-x) - f(t)|\, dt = \|f_x - f\|_1 \leq \underbrace{\|f_x - g_x\|_1}_{<\epsilon/3} + \underbrace{\|g_x - g\|_1}_{<\epsilon/3} + \underbrace{\|g - f\|_1}_{<\epsilon/3}.$$

$\exists \delta$ s.t if $|x| < \delta$

II. 11. Let f, f_n ($n \in \mathbb{N}$) be 2π-periodic functions, integrable in $[-\pi, \pi]$, which satisfy

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - f_n(x)|\, dx = 0.$$

Show that

$$\widehat{f_n}(k) \to \widehat{f}(k) \quad \text{as } n \to \infty,$$

uniformly in k. That is, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that for each $n \geq n_0$ and each $k \in \mathbb{Z}$, we have

$$|\widehat{f_n}(k) - \widehat{f}(k)| < \varepsilon.$$

Proof. $\forall f_n$ g $\forall k \in \mathbb{Z}$

$$\hat{g}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, e^{-ikt}\, dt$$

$$|\hat{g}(k)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(t) \underbrace{e^{-ikt}}_{|1|}|\, dt$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(t)|\, dt = \|g\|_1$$

$$|\hat{f}_n(k) - \hat{f}(k)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_n(t) - f(t)|\, dt \xrightarrow[n\to\infty]{} 0$$

$\forall \epsilon > 0$ $\exists n_0 \in \mathbb{N}$ st $\forall n \geq n_0$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f_n(t) - f(t)|\, dt < \epsilon$$

$$\Downarrow$$

$$|\hat{f}_n(k) - \hat{f}(k)| < \epsilon \quad \underline{\underline{\forall}}\ k \in \mathbb{Z}$$

equivalents: $\|\hat{f}_n - \hat{f}\|_\infty \leq \epsilon \quad \forall n \geq n_0$

$\blacksquare$

III. 3. If $f : \mathbb{T} \to \mathbb{C}$ is integrable, show that, for each $m \in \mathbb{N}$,

$$\sigma_m(f) = \sum_{k=-m}^{m} \left(1 - \frac{|k|}{m+1}\right) \hat{f}(k) e_k \,.$$

2nd solution:

$$\sigma_m(f)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_m(t-s) f(s)\, ds$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{|k| \le m} \left(1 - \frac{|k|}{m+1}\right) e^{ik(t-s)} f(s)\, ds$$

$$= \sum_{k=-m}^{m} \left(1 - \frac{|k|}{m+1}\right) e^{ikt} \Big(\underbrace{\int f(s) e^{-iks} \frac{ds}{2\pi}}_{\hat{f}(k)} \Big)$$

$$\sigma_m(f) = \sum_{|k| \le m} \left(1 - \frac{|k|}{m+1}\right) \hat{f}(k) e_k$$

SEE ALSO SLIDES p. 34

III. 4. If $f, g : \mathbb{T} \to \mathbb{C}$ are continuous, show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)g(s)ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(t-x)dx := (f*g)(t)$$

for all t. Show that $f*g$ is continuous and find $\widehat{f*g}(k)$ for each $k \in \mathbb{Z}$.

See the file apr14.pdf, page 3

III. 5. Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function which is integrable over $[-\pi, \pi]$. Suppose that for some $x \in \mathbb{R}$ the limits

$$f(x^-) := \lim_{t \to x^-} f(t) \quad \text{and} \quad f(x^+) := \lim_{t \to x^+} f(t)$$

exist. Show that the Fourier series $S[f]$ of f is Abel summable at x: more precisely, show that

$$\lim_{r \to 1^-} f_r(x) = \frac{f(x^-) + f(x^+)}{2}.$$

You may use the fact that

$$\frac{1}{2\pi} \int_{-\pi}^{0} P_r(x)\, dx = \frac{1}{2\pi} \int_{0}^{\pi} P_r(x)\, dx.$$

(Reminder: $f_r(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) P_r(t-s) ds$.)

not required for this exam

IV. 4. (α) If $A, B \subseteq \mathbb{R}$ and $\lambda^*(B) = 0$, show that $\lambda^*(A \cup B) = \lambda^*(A)$.

(β) If $A, B \subseteq \mathbb{R}$ και $\lambda^*(A \triangle B) = 0$, show that $\lambda^*(A) = \lambda^*(B)$ (the symbol $A \triangle B$ denotes the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of A and B).

Pf (α):

$$A \subseteq A \cup B$$

$\Downarrow$ λ^* monotone

$$\lambda^*(A) \le \lambda^*(A \cup B)$$

λ^* subadditive

$$\le \lambda^*(A) + \lambda^*(B) = \lambda^*(A) \quad \blacksquare$$

(β)

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

$$A = A \setminus B \cup (A \cap B) \quad \text{but} \quad A \setminus B \subseteq A \triangle B$$

$$\Downarrow$$

$$\lambda^*(A \setminus B) \underset{\text{mon}}{\le} \lambda^*(A \triangle B) = 0$$

by (α): $\lambda^*(A) = \lambda^*(A \cap B)$ $\blacksquare$

Similarly: $\lambda^*(B) = \lambda^*(B \cap A)$

IV.6. Let $E \subseteq \mathbb{R}$ with $0 < \lambda^*(E) < +\infty$ and let $0 < \alpha < 1$. Show that *there exists* an open interval I with the property

$$\lambda^*(E \cap I) > \alpha\, \ell(I).$$

Hint: Assume the opposite and, for an arbitrary $\varepsilon > 0$, consider a sequence of intervals I_k such that $E \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} \ell(I_k) < \lambda^*(E) + \varepsilon$.

$\cancel{\forall \epsilon > 0}$ take ϵ st $\frac{1}{1+\epsilon} = \alpha$

Pf By def of $\lambda^*(E)$ (since $\lambda^*(E) < \infty$) $\exists$ cover of E by intervals $E \subseteq \bigcup_{n=1}^{\infty} I_n$

st $\lambda^*(E) > \sum_{n=1}^{\infty} \ell(I_n) - \epsilon \lambda^*(E)$

or $\lambda^*(E) > \frac{1}{1+\epsilon} \sum_{n=1}^{\infty} \ell(I_n)$ (1)

subadditivity of λ^*:

$\lambda^*(E) \leq \sum_{n=1}^{\infty} \lambda^*(E \cap I_n)$

$\Longrightarrow (1+\epsilon) \sum_n \lambda^*(E \cap I_n) \geq (1+\epsilon)\lambda^*(E) \overset{(1)}{>} \sum_{n=1}^{\infty} \ell(I_n)$

$\exists n_0 : (1+\epsilon)\lambda^*(E \cap I_{n_0}) > \ell(I_{n_0})$

or $\lambda^*(E \cap I_{n_0}) > \underbrace{\frac{1}{1+\epsilon}}_{\alpha} \ell(I_{n_0})$ $\blacksquare$

V. 1. (α) If $E \subseteq \mathbb{R}$ is measurable with $\lambda(E) < \infty$, show that for all $\epsilon > 0$ there exists a step function f vanishing outside a bounded interval so that $\|\chi_E - f\|_1 < \epsilon$.

Hint Recall the first of the three principles of Littlewood.
Note also that $f : \mathbb{R} \to \mathbb{R}$ is a step function vanishing outside a bounded interval if and only if there are $x_0, \ldots x_n \in \mathbb{R}$, $x_0 < x_1 < \cdots < x_n$ such that f is constant on each (x_{i-1}, x_i) and $f(t) = 0$ for all $t \notin [x_0, x_n]$.

(β) If $I \subseteq \mathbb{R}$ is a bounded interval and $\epsilon > 0$, show that there is a continuous function g *with compact support* so that $\|\chi_I - g\|_1 < \epsilon$.

(γ) using the above, show that the following linear spaces are dense in $L^1(\mathbb{R})$:

(i) The space of simple integrable functions.

(ii) The space of integrable step functions.

(iii) The space $C_c(\mathbb{R})$ of continuous functions with compact support.

(α) $\forall \epsilon > 0$ By Littlewood $\exists\, A = I_1 \cup I_2 \cup \cdots \cup I_n$ (intervals)

st $\lambda(E \Delta A) < \epsilon$

$\chi_{E\Delta A} = |\chi_E - \chi_A|$ because $|\chi_E(t) - \chi_E(t)| = 1$

$\Downarrow$

$t \in$ ~~are~~ exactly one of E, A

$t \in (E \setminus A) \cup (A \setminus E) = A \Delta E$

$\Rightarrow \|\chi_A - \chi_A\|_1 = \int_{\mathbb{R}} |\chi_A - \chi_E| \, d\lambda$

$= \int \chi_{E\Delta A} \, d\lambda = \lambda(E \Delta E) < \epsilon$

To prove that χ_A is a step function

for ex of $A = I_1 \cup I_2$

$= (I_1 \setminus I_2) \cup (I_1 \cap I_2) \cup (I_2 \setminus I_1)$: disjoint union of intervals

By induction, $\exists\, J_1 \cdots J_m$ disjoint intervals of finite length

st $A = J_1 \cup J_2 \cup \cdots \cup J_m$

$\Downarrow$ (disjointness)

$\chi_A = \chi_{J_1} + \chi_{J_2} + \cdots + \chi_{J_m}$

: a step function $\blacksquare$

Continued on p. 14

V.10. (α) Show that for all $X \in \mathcal{M}$, $L^1(X) = \{fg : f, g \in L^2(X)\}$.
(β) If $f \geq 0$, show that $f \in L^2([-\pi, \pi])$ if and only if $f^2 \in L^1([-\pi, \pi])$. Is the same true when $f([-\pi, \pi]) \subseteq \mathbb{R}$;

(α) If $f, g \in L^2(X)$ then fg is measurable and

$$\int |gf| \, d\lambda \overset{CS}{\leq} \Big(\underbrace{\int |g|^2 d\lambda}_{<+\infty} \underbrace{\int |f|^2 d\lambda}_{<+\infty} \Big)^{1/2} < +\infty$$

$\Rightarrow fg \in L^1(X)$

Given $h \in L^1(X)$ factorise it as follows

$$h = u|h| \quad u \text{ is measurable and } |u| = 1 \text{ a.e.}$$

$$= u|h|^{1/2} \, |h|^{1/2}$$

measurable

$$= f \cdot g$$

$$\int |f|^2 d\lambda = \int |u|h|^{1/2}|^2 d\lambda = \int |h| \, d\lambda < +\infty$$

$f \in L^2(X)$ similarly $g \in L^2(X)$ and $fg = h$

(β) First take $f \in L^2([-\pi, \pi])$, f real-valued

Then f^2 is measurable, $f \geq 0$ so

$$\int f^2 d\lambda = \int |f|^2 d\lambda < +\infty \text{ hence } f^2 \in L^1([-\pi, \pi])$$

Conversely, if $f^2 \in L^1$ AND $f \geq 0$, then $f = (f^2)^{1/2}$ is measurable

and $|f|^2 = f^2$ and so $\int |f|^2 d\lambda = \int f^2 d\lambda < +\infty$

so $f \in L^2([-\pi, \pi])$

Counterexample: take $E \subseteq [-\pi, \pi]$ non-measurable

and let $f(t) = \begin{cases} 1, & t \in E \\ -1, & t \notin E \end{cases}$

f cannot be in L^2 (not measurable)

but $f^2 = 1$ so is " and in L^1

VI.4. Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuously differentiable function with $f(-\pi) = f(\pi)$.

(α) Show that there is a constant $C(f) > 0$ so that $|k\hat{f}(k)| \le C(f)$ for all $k \in \mathbb{Z}$.

(β) Examine whether $\lim\limits_{|k|\to\infty} |k\hat{f}(k)| = 0$.

(γ) Examine whether $\sum\limits_{k=-\infty}^{\infty} |f(k)| < +\infty$.

(α) Recall $\widehat{f'}(k) = ik\hat{f}(k)$ $\forall k \in \mathbb{Z}$

$\Downarrow$

$g = f'$

$$|ik\hat{f}(k)| = |\hat{g}(k)| \le \|g\|_1 < +\infty \text{ because } g \text{ is continuous}$$

$$|k\hat{f}(k)| \le C(f) \quad C(f) = \|f'\|_1$$

(β) Parseval:

$$\sum_{k\in\mathbb{Z}} |\hat{g}(k)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(t)|^2 dt < +\infty$$

Apply to $g = f'$

$$\sum_{k\in\mathbb{Z}} |ik\hat{f}(k)|^2 = \|f'\|_2^2 < +\infty$$

$\Downarrow$

$|k\hat{f}(k)| \longrightarrow 0$ as $k \to \pm\infty$

(γ) $\sum\limits_{k\in\mathbb{Z}} |\hat{f}(k)|$?

$$\sum_{k\neq 0} |\hat{f}(k)| = \sum_{k\neq 0} |k\hat{f}(k)|\left|\frac{1}{k}\right|$$

$$\overset{CS}{\le} \Bigg(\underbrace{\Big(\sum_{k\neq 0} |k\hat{f}(k)|^2\Big)}_{< +\infty \ (\text{by } \beta)} \underbrace{\Big(\sum_{k\neq 0} \frac{1}{k^2}\Big)}_{< +\infty} \Bigg)^{1/2}$$

$$\sum_{k\in\mathbb{Z}} |\hat{f}(k)| = |\hat{f}(0)| + \sum_{|k|\neq 0} |\hat{f}(k)| < +\infty$$

$\blacksquare$

II.4 part (b)

$f \in L^1(\mathbb{R})$, $\forall \epsilon > 0$ $\exists g \in C_c(G)$

st $\|f - g\|_1 < \epsilon/3$

$f_x(t) = f(t-x)$

$\|f_x - g_x\|_1 = \|f - g\|_1 < \epsilon/3$

$$\|f_x - g_x\|_1 = \int |f(t-x) - g(t-x)|\, dt$$

$t - x = s$

$$= \int |f(s) - g(s)|\, ds = \|f - g\|_1$$

Now by part (a) $\exists \delta > 0$ st $|x| < \delta$

l2a

$\|g_x - g\|_1 < \epsilon/3$

$$\Rightarrow \|f_x - f\|_1 \le \|f_x - g_x\|_1 + \|g_x - g\|_1 + \|g - f\|_1$$

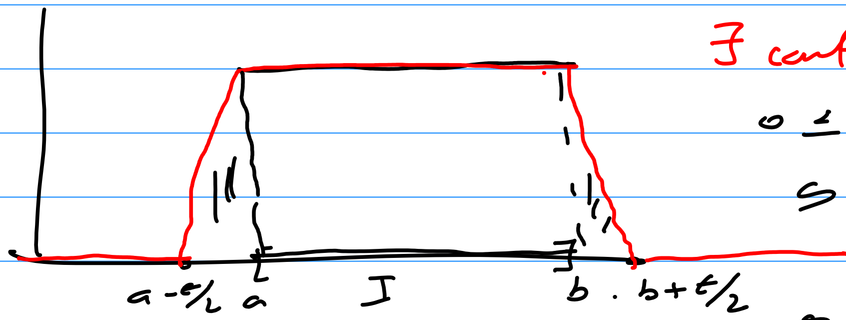
$< \epsilon$

$\blacksquare$

IV. 1. Part (β) If $I \subseteq \mathbb{R}$ is a bounded interval

$\forall \epsilon > 0 \ \exists \ g \in C_c(\mathbb{R})$ st $\|\chi_I - g\|_1 < \epsilon$

Pf



$\exists$ continuous fn g

$0 \le g(t) \le 1$

$g(t) = 0$ outside $[a-\epsilon/2, b+\epsilon/2]$

$g(t) = 1$ in $[a, b]$

$$\|\chi_I - g\|_1 = \int_{\mathbb{R}} |\chi_I - g| \, d\lambda = \int_{a-\epsilon/2}^{b+\epsilon/2} |\chi_I - g| \, d\lambda$$

$$= \left(\int_{a-\epsilon/2}^{a} + \int_{b}^{b+\epsilon/2}\right) |0 - g| \, d\lambda \le \frac{1}{2}\frac{\epsilon}{2} + \frac{1}{2}\frac{\epsilon}{2}$$

$$= \frac{\epsilon}{2} < \epsilon \quad \blacksquare$$

(γ) (i) To show integrable simple fns dense in $L^1(\mathbb{R})$

Given $f \in L^1(\mathbb{R})$ we know $\exists (s_n)$ of simple function st $s_n \longrightarrow f$ pointwise and

$|s_1| \le |s_2| \le \cdots \le |s_n| \le |f|$

$\Rightarrow \|s_n\|_1 \le \|f\|_1 < +\infty$: s_n is integrable

$|s_n - f| \longrightarrow 0$ pointwise and

$|s_n - f| \le |s_n| + |f| \le 2|f|$ in L^1

By the Dominated Convergence Theorem:

$$\int |s_n - f| \, d\lambda \longrightarrow 0$$

i.e. $\|s_n - f\|_1 \longrightarrow 0$

(ii) To show that integrable step functions are dense in $L^1(\mathbb{R})$

pf by (i), given $f \in L^1(\mathbb{R})$, $\epsilon > 0$ $\exists s$ integrable simple fn so that

$$\|f - s\|_1 < \epsilon/2$$

$s = \sum_{k=1}^{n} a_k \chi_{E_k}$ each E_k has $\lambda(E_k) < +\infty$ (because s is $\int$ble)

by (α) $\forall k = 1 \dots n$ $\exists$ step function g_k which is $\int$ble st $\|\chi_{E_k} - g_k\|_1 < \frac{\epsilon}{2 \sum |a_k|}$.

Then setting $g = \sum_{k=1}^{n} a_k g_k$: $\int$ble step function and

$$\|f - g\|_1 \leq \|f - s\|_1 + \|s - g\|_1$$
$$< \epsilon/2 + \sum_{k=1}^{n} |a_k| \, \|\chi_{E_k} - g_k\|_1 < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$\blacksquare$

(iii) $C_c(\mathbb{R})$ dense in $L^1(\mathbb{R})$

pf $\forall f \in L^1(\mathbb{R}), \epsilon > 0$ first find a step functn $g = \sum a_k \chi_{I_k}$ st

$\|f - s\|_1 < \epsilon/2$

then $\forall I_k$ by (β) $\exists h_k \in C_c(G)$ st

$$\|\chi_{I_k} - h_k\|_1 < \frac{\epsilon}{2 \sum |a_k|}$$

then set $h = \sum_{k=1}^{n} a_k h_k \in C_c(G)$

$$\|f - h\| \leq \|f - s\|_1 + \|g - h\|_1$$
$$\leq \|f - s\|_1 + \sum |a_k| \, \|\chi_{I_k} - h_k\|_1 < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$\blacksquare$